Linear Endomorphisms of Function Algebras and Related Functional Equations

GYÖRGY I. TARGONSKI*

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1. Introduction. The First Order Linear Functional Equation is

$$(1.1) \quad (f \circ \omega)(P) + g(P)f(P) = h(P).$$

Here, $P \in S$ where S is an arbitrary set; ω a (known) mapping of S into itself; g, h (known) and f (unknown) are mappings from S into a suitable algebra, often the reals or the complex numbers.

(1.1), which was first discussed, in a more special form, by Kordylewski and Kuczma ([1], p. 46) plays a part analogous to that of the first order linear differential equation. The operator appearing in (1.1) is

$$(1.2) \quad (\Omega f)(P) = (f \circ \omega)(P).$$

It is readily seen to be a linear endomorphism of the function algebra on which it acts (multiplication in the algebra being defined as the "pointwise" multiplication); we shall occasionally refer to Ω as a "substitution operator", or as a "scale transformation". The first name refers to the fact that Ω effects a "substitution" *i.e.* replaces the independent variable by a (known) function thereof; while the second name is used in mathematical physics. One observes that a mapping ω of the "underlying set" S into itself induces a linear endomorphism on the algebra of functions defined on S . It is one of the aims of the present paper to investigate linear endomorphisms of function algebras not induced by a mapping of the underlying set into itself.

The study of linear endomorphisms of function algebras is relevant to the study of (1.1), in particular if we replace (1.1) by the more general equation

$$(1.3) \quad (\Omega f)(P) + g(P)f(P) = h(P),$$

where Ω is now any linear endomorphism of a function algebra containing f . The reason for this generalization is the following. Let φ be an eigenfunction of Ω with eigenvalue λ :

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$$(1.4) \quad (\Omega\varphi)(P) = \lambda\varphi(P)$$

then, for any natural number n ,

$$(1.5) \quad (\Omega\varphi^n)(P) = \lambda^n\varphi^n(P),$$

i.e. we have, together with any one eigenfunction, a countable infinity of eigenfunctions which are moreover, in an appropriate setting linearly independent. This fact is of use when we attempt to solve (1.3) in the form of an infinite series by expanding known and unknown functions in power series of the form $\sum_{n=0}^{\infty} a_n\varphi^n(P)$. Here one uses only the fact that Ω is a linear endomorphism and not the "specific form" of Ω in (1.1) as given by (1.2). This leads to the "specific form property" problem, which in this case can be formulated as follows. Investigate, for a given function algebra A , whether every linear endomorphism Ω of A has the "specific form" $\Omega f = f \circ \omega$ *i.e.* whether it is induced by a mapping ω of the underlying set into itself. The corresponding question for derivations, (defined here more narrowly for an algebra $\mathfrak{B}$ of complex valued functions of one complex variable) is the following. Investigate, whether every derivation Ω on $\mathfrak{B}$ (that is, every linear mapping of $\mathfrak{B}$ into itself with the addition property $\Omega(fg) = f\Omega g + g\Omega f$ is of the "specific form"

$$(\Omega f)(z) = \omega(z) \frac{d}{dz} f(z).$$

These problems have been investigated in detail earlier in connection with what we have termed Bourlet operators. Bourlet operators are linear operators on a function algebra which possess, in a precisely defined way, a multiplication theorem

$$(1.6) \quad \Omega(fg) = F[f, g, \Omega f, \Omega g].$$

It turns out, that under some restrictions on the function algebra, a Bourlet operator is essentially either an endomorphism, or a derivation, or a multiplier. (*cf.* [2], [3]) Once the "specific form" question has been answered in the negative for a given function algebra, the following question can be asked. What is the form of endomorphisms not of the specific form (1.2), *i.e.* not induced by a mapping of the underlying set into itself.

We first list several known results. The algebra has the "specific form property" means, as said: every linear endomorphism of the algebra is induced by a mapping of the underlying set into itself and therefore has the "specific form" (1.2).

a) The algebra of all complex polynomials has the specific form property. ([2])

b) The algebra $C(X)$ of all continuous real-valued functions on a compact Hausdorff space X has the specific form property. Moreover, the inducing mappings of X into itself are necessarily continuous, and it follows that all endomorphisms of $C(X)$ are continuous. (*cf.* [4] p. 278 and [5].)

c) The algebra of all entire complex functions has the specific form property. This result, a generalization of a), follows from the theorem of L. Bers [6], later generalized by M. Nakai [7]. See the discussion in [8], pp. 195 and 200-204. However, not every function algebra possesses the specific form property:

d) Let A be the algebra of all mappings from an arbitrary set S to the reals. Let E be a proper subset of S , and ω a mapping of S into itself. Consider the mappings

$$(1.7) \quad (\Omega f)(P) = \chi_E(P)(f \circ \omega)(P),$$

where χ_E is the characteristic function of E . Ω is clearly a linear endomorphism of A . We show that A does not possess the specific form property. Assume, contrarily, that Ω is induced by a mapping ξ of S into itself:

$$(1.8) \quad \chi_E(P)(f \circ \omega)(P) = (f \circ \xi)(P),$$

and let f be a function not vanishing for any $P \in S$, and choose some $P_0 \notin E$. Then the l.h.s. in (1.5) vanishes at P_0 , but not the r.h.s. Thus, a representation of Ω in the form $\Omega f = f \circ \xi$ does not exist. This argument however fails if we consider, instead of A , the maximal ideal A_0 of functions with the property $f(P_0) = 0$ for some fixed $P_0 \in S$. In A_0 , we can proceed as follows.

We define ξ as follows:

$$(1.9) \quad \xi(P) = \begin{cases} \omega(P), & P \in E \\ P_0 & P \notin E. \end{cases}$$

Then

$$(\Omega f)(P) = \chi(P)(f \circ \omega)(P) = \begin{cases} (f \circ \omega)(P) = (f \circ \xi)(P), & P \in E \\ 0 = f(P_0) = (f \circ \xi)(P), & P \notin E, \end{cases}$$

and Ω has been represented in the specific form.

We now proceed to show that, in a fairly general case, (1.4) is the most general form of a linear endomorphism of a function algebra.

2. Linear endomorphisms of some algebras of functions on compact Hausdorff Spaces. We call an algebra $\mathfrak{B}$ of complex-valued functions defined on a set S *self-adjoint*, if $f \in \mathfrak{B}$ implies $\bar{f} \in \mathfrak{B}$ where $\bar{f}(P)$ is defined to be $\overline{f(P)}$; *reciprocal-closed* if $f \in \mathfrak{B}$ implies $1/f \in \mathfrak{B}$ whenever $f(P) \neq 0$ for all $P \in S$. (We prefer "reciprocal-closed" to the more customary "inverse-closed" which in our context might have another meaning.); *separating* if for any $P_1 \neq P_2$ there exists a $f \in \mathfrak{B}$ such that $f(P_1) \neq f(P_2)$. We formulate

Theorem 2.1. *Every linear endomorphism of a self-adjoint, reciprocal-closed, separating Banach algebra of complex-valued functions on a compact Hausdorff space X is of the form*

$$(2.1) \quad (\Omega f)(P) = \chi_E(P)(f \circ \omega)(P),$$

where χ_E is the characteristic function of some subset $E \subseteq X$ and ω is a mapping of X into itself.

Proof. It is known ([9] p. 204) that a self-adjoint, separating, reciprocal-closed algebra $\mathfrak{B}$ of complex-valued functions on a compact Hausdorff space X has the following property. Every nonzero multiplicative linear functional L on $\mathfrak{B}$ has the form

$$(2.2) \quad Lf = f(P_0) \text{ for some } P_0 \in X, \text{ i.e. } L \text{ is a "point evaluation functional".}$$

A zero (trivial) functional is of the form

$$(2.3) \quad Lf = 0 \quad \text{for all } f \in \mathfrak{B}.$$

Consider now Ω , an arbitrary linear endomorphism of $\mathfrak{B}$. Ω gives rise to a one-parameter family of linear multiplicative functionals on $\mathfrak{B}$, with parameter values in X , in the following manner.

Define the functional $L(P_0)$, $P_0 \in X$ by applying Ω to f and evaluating the image function at $P = P_0$:

$$(2.4) \quad L(P_0)f = (\Omega f)(P_0).$$

$L(P_0)$ is obviously a linear and multiplicative functional on $\mathfrak{B}$. Consider now the set $E \subseteq X$ with the property that $L(P_0)$ is a zero (trivial) functional exactly if $P_0 \in X - E$, that is

$$(2.5) \quad L(P_0)f = 0, \quad P_0 \in X - E.$$

For any $P_0 \in E$, $L(P_0)$ is a nonzero multiplicative linear functional, and according to the above quoted theorem ([9]): for every $P_0 \in E$ there exists at least one $P'_0 \in X$ such that for all $f \in \mathfrak{B}$

$$(2.6) \quad L(P_0)f = f(P'_0).$$

The evaluation point P'_0 need not equal the point P_0 labelling the functional $L(P_0)$. On the other hand, P'_0 can be considered a function of P_0 in the following sense. For every $P_0 \in E$, there exists a set $S(P_0) \subseteq X$ such that

$$(2.7) \quad L(P_0)f = f(P),$$

for every $P \in S(P_0)$. Using the axiom of choice, we can select for every P_0 , an arbitrary point from $S(P_0)$ which we design by $\omega(P_0)$, thereby creating a function defined on E ; (2.6) then becomes, writing P instead of P_0

$$(2.8) \quad L(P)f = f[\omega(P)] = (f \circ \omega)(P), \quad P \in E.$$

Combining (2.4), (2.5) and (2.8) we find

$$(2.9) \quad (\Omega f)(P) = \begin{cases} 0 & P \in X - E, \\ (f \circ \omega)(P) & P \in E. \end{cases}$$

If we now extend ω to all of X by defining ω arbitrarily for $P \in X - E$, we can

re-write (2.9) as

$$(2.10) \quad (\Omega f)(P) = \chi_E(P)(f \circ \omega)(P). \text{ This completes the proof.}$$

Lemma 2.1. *If $\mathfrak{B} \subseteq C(X)$ and X is connected then $E = X$ and the endomorphism (2.10) reduces to a scale transformation.*

Proof. Assume $E \neq \emptyset$, $E \neq X$; choose $f = e$ where e is the multiplicative unit in $C(X)$, then $\Omega f = \chi_E \cdot e$ and since $E \neq X$, $E \neq \emptyset$ there exist two points P_1, P_2 such that $(\Omega e)(P_1) = 0$, $(\Omega e)(P_2) = 1$ and $(\Omega e)(P)$ takes no other values than 0 or 1 for any $P \in X$. Therefore, $(\Omega e)(P)$ is not continuous, since it maps the connected set X onto the set $\{0, 1\}$ which is not connected in the standard topology of complex numbers. It follows that Ω is not an endomorphism of $\mathfrak{B}(X)$; the contradiction can be resolved only by assuming $E = \emptyset$ (then Ω is the trivial endomorphism mapping all of $\mathfrak{B}(X)$ into the zero function) or $E = X$ as asserted by Lemma 2.1, which is thus proven.

The Wiener algebra W of all absolutely convergent Fourier Series

$$(2.11) \quad f = \sum_{n=-\infty}^{\infty} a_n \cdot e^{-inx}; \quad \sum_{n=-\infty}^{\infty} |a_n| < \infty,$$

is a Banach algebra under the norm

$$(2.12) \quad ||f|| = \sum_{n=-\infty}^{\infty} |a_n|.$$

It is immediately seen to be self-adjoint and separating. Moreover, the Wiener Lemma (see e.g. [9], p. 228 where it is stated in a generalized form) state that W is reciprocal-closed.

Corollary 2.1. *Every non-trivial linear endomorphism of the Wiener algebra is a scale transformation.*

Proof. W satisfies the conditions of Theorem 2.1 which therefore applies. Also, the interval $[0, 2\pi]$ being connected, Lemma 2.1 applies to W .

We remark that the elements of W are strictly speaking defined on $0 \leq x < 2\pi$; by assigning to 2π the left hand limit of f , W becomes a subset of $C(I)$, where I is the interval $0 \leq x \leq 2\pi$. An alternative approach to the study of the endomorphisms of W would be to consider W as a sub-algebra of $C(I)$; one then has to remember that the norm in W is different from the norm in $C(I)$.

It is of interest to see Theorem 2.1 apply to an algebra which is not a subset (although with a different norm) of $C(X)$. We can construct such an example in the following way. Consider the family Σ of all complex-valued functions defined on the interval $[0, 1]$ and constant on every interval of the form

$$\frac{k}{n} \leq x < \frac{k+1}{n} \quad k = 0, 1, 2, \dots, n-1; \quad n \geq 2 \text{ fixed.}$$

Σ is easily seen to satisfy the conditions of Theorem 2.1 under the sup-norm.

Moreover Σ contains discontinuous functions (in fact, every non-constant function in Σ is discontinuous) thus $\Sigma \subsetneq C(0, 1)$. Σ can however be discussed more naturally as the family of all complex valued functions on a finite set with the discrete topology; we will do so in the next section.

If we drop the condition of reciprocal-closedness imposed on $\mathfrak{B}$ in Theorem 2.1, additional endomorphisms besides those of the form (2.10) appear. Consider *e.g.* the algebra Λ of all bounded complex-valued functions on the unit interval for which the right limit exists everywhere. Λ is a self-adjoint separating Banach algebra under the sup-norm, but not reciprocal-closed, since the reciprocal of a nowhere vanishing element of Λ need not be bounded. In Λ we have the linear endomorphism

$$(2.13) \quad (\Omega f)(x) = f(x + 0), \quad \text{where} \quad f(x + 0) = \lim_{\epsilon \rightarrow 0} f(x + |\epsilon|).$$

One can construct a function in Λ for which the following property holds: $f(x_0 + 0) = \alpha \neq 0$ for some x_0 and α , but $f(x) \neq \alpha$ for all x ; therefore there exists no $E \subset [0, 1]$ and no mapping ω of $[0, 1]$ into itself such that Ω could be written in the form (2.10).

3. Linear endomorphisms of the set of all n -tuples. Consider now the set $T^{(n)}$ of all complex n -tuples, n fixed. This set is a commutative Banach algebra under the addition:

$$(3.1) \quad (z_1, \dots, z_n) + (w_1, \dots, w_n) = (z_1 + w_1, \dots, z_n + w_n),$$

multiplication:

$$(3.2) \quad (z_1, \dots, z_n)(w_1, \dots, w_n) = (z_1 w_1, \dots, z_n w_n)$$

and scalar multiplication

$$(3.3) \quad \lambda(z_1, \dots, z_n) = (\lambda z_1, \dots, \lambda z_n).$$

The norm is given by

$$(3.4) \quad \|(z_1, \dots, z_n)\| = \max \{|z_1|, \dots, |z_n|\}.$$

There is a unit element $(1, \dots, 1)$ and the zero element is $(0, \dots, 0)$.

We want to find the explicit form of all linear endomorphisms

$$(3.5) \quad (z_1, \dots, z_n) \rightarrow (F_1(z_1, \dots, z_n), \dots, F_n(z_1, \dots, z_n)), \text{ i.e. a set } F_1, \dots, F_n \text{ of functions of } n \text{ variables with the properties}$$

$$(3.6) \quad F_i(z_1 + w_1, \dots, z_n + w_n) = F_i(z_1, \dots, z_n) + F_i(w_1, \dots, w_n),$$

$$i = 1, \dots, n$$

$$(3.7) \quad F_i(z_1 w_1, \dots, z_n w_n) = F_i(z_1, \dots, z_n) \cdot F_i(w_1, \dots, w_n), \quad i = 1, \dots, n$$

$$(3.8) \quad F_i(\lambda z_1, \dots, \lambda z_n) = \lambda F_i(z_1, \dots, z_n), \quad i = 1, \dots, n.$$

One notices immediately that the unknown functions do not "mix" in the system of Functional Equations (3.6)–(3.8) and therefore a solution can be found for each F_i alone. We will do so in Appendix 1. At present we apply Theorem 2.1. Let $\{P_1, \dots, P_n\}$ be a finite set; with the discrete topology it becomes a compact Hausdorff space $X^{(n)}$. We introduce now the algebra $\mathfrak{B}^{(n)}$ of complex valued functions $\{f\}$ on $X^{(n)}$. Now every n -tuple in $T^{(n)}$ corresponds in a unique way to a function in $\mathfrak{B}^{(n)}$ namely

$$(3.9) \quad (z_1, \dots, z_n) \leftrightarrow f: f(P_i) = z_i, \quad i = 1, \dots, n.$$

$$(z_1, \dots, z_n) \in T^{(n)}; f \in \mathfrak{B}^{(n)},$$

and this isomorphism is norm preserving if we introduce

$$(3.10) \quad \|f\| = \sup_{P_i \in X^{(n)}} |f(P_i)| = \max \{|z_1|, \dots, |z_n|\}.$$

Evidently $\mathfrak{B}^{(n)}$ is a separating, self-adjoint, reciprocal-closed Banach algebra of functions defined on the compact Hausdorff space $X^{(n)}$. Therefore Theorem 2.1 applies and we have

Theorem 3.1. *Every linear endomorphism of the algebra of complex n -tuples is of the form*

$$(3.11) \quad (f(P_1), \dots, f(P_n))$$

$$\rightarrow (\chi_E(P_1), \dots, \chi_E(P_n))((f \circ \omega)(P_1), \dots, (f \circ \omega)(P_n))$$

$$= (\chi_E(P_1)(f \circ \omega)(P_1), \dots, \chi_E(P_n)(f \circ \omega)(P_n)),$$

where $\omega(P_i)$ may take on any of the values $P_1, \dots, P_n$ and $E \subseteq X^{(n)}$. Every linear endomorphism of $T^{(n)}$ therefore can be constructed by mapping every component into itself, or into any other component, or into zero. The number of different endomorphisms of $T^{(n)}$ is $(n+1)^n$.

It is implicit in the proof that the automorphisms of $T^{(n)}$ are characterized by $E = X^{(n)}$ i.e. $\chi_E(P_i) = 1, \dots, n$ and ω an automorphism of $X^{(n)}$, a permutation. The total number of automorphisms of $T^{(n)}$ is $n!$. As indicated, we solve the equations (3.6)–(3.8) in Appendix 1.

4. Algebras of functions on non-compact spaces. If some of the conditions in Theorem 2.1 are omitted, the form (2.10) need not be the most general form for linear endomorphisms. As seen in Section 3, omitting the condition of reciprocal closedness (or, equivalently, of continuity in some cases) we find new types of endomorphisms e.g. right limit (and left-limit) endomorphisms. The problem is related to the question of maximal ideals in the function algebra considered. If every maximal ideal is a fixed ideal i.e. of the form $\{f \mid f(P_0) = 0\}$ for some fixed P_0 , then every linear endomorphism is of the form (2.10). If however, other (free) maximal ideals appear, we may not use the reasoning we used in establishing Theorem 2.1. It is therefore of interest to know that in

some algebras of functions on a non-compact domain, the equivalent of Theorem 2.1 still holds. We saw such an example already under a) Section 1.

Consider now the Banach algebra C_L of functions continuous in $(-\infty, \infty)$ and such that $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$ both exist (but are not necessarily equal) for any $f \in C_L$. The norm in C_L is the sup-norm. One has $C_L \subset C^*(-\infty, \infty) \subset C(-\infty, \infty)$ where C , resp. C^* are the algebras of functions continuous, resp. continuous and bounded on $(-\infty, \infty)$.

Lemma 4.1. *The algebra C_L contains no free ideals; and every linear endomorphism of C_L is a scale transformation.*

Proof. The mapping

$$(4.1) \quad \psi(x) = \frac{x}{1 + |x|}, \quad \psi^{-1}(x) = \frac{x}{1 - |x|},$$

establishes a continuous one-to-one mapping between $[-1, 1]$ and $[-\infty, \infty]$. Consider now the families C_L and $C(-1, 1)$. The pair of operators

$$(4.2) \quad (\Psi f)(x) = f\left(\frac{x}{1 + |x|}\right), \quad (\Psi^{-1}f)(x) = f\left(\frac{x}{1 - |x|}\right),$$

establishes a homeomorphism between C_L and $C(-1, 1)$. Let Ω now be a linear endomorphism of C_L ; then $\Psi\Omega\Psi^{-1}$ is a linear endomorphism of $C(-1, 1)$ and therefore, according to Lemma 2.1, of the specific form H :

$$(4.3) \quad (Hf)(x) = (f \circ h)(x) \text{ but then } \Omega = \Psi^{-1}H\Psi, \text{ and } \Omega \text{ as the product of three scale transformations is itself a scale transformation. Explicitly}$$

$$(4.4) \quad (\Omega f)(x) = f\{h(x(1 + |x|)^{-1})[1 - |h(x(1 + |x|)^{-1})|]^{-1}\}.$$

In evaluating the limit in the argument of f we have to remember that h is a mapping of $[-1, 1]$ into itself and thus the denominator is never negative. (4.4) proves the part of the Lemma about scale transformation. The non-existence of free ideals in C_L follows from the fact that C_L is a homeomorphic image of C induced by the scale transformation (4.1); consequently, every fixed ideal in C is mapped into a fixed ideal in C_L including the fixed ideal of functions vanishing at $+\infty$, or at $-\infty$. This procedure exhausts all maximal ideals in C_L therefore, there are no free ideals. The matter, of course, can be looked at in a more simple fashion. We may consider the interval $[-\infty, \infty]$ arising from joining the elements $-\infty$ and ∞ to $(-\infty, \infty)$. As a result of this two-point compactification, the premises of Theorem 2.1 are now satisfied and the second part of Lemma 4.1 directly follows. Naturally, the mapping function ω inducing the endomorphism must be defined also for $-\infty$ and ∞ . Explicitly

$$(4.5) \quad \omega(x) = h(x(1 + |x|)^{-1})[1 - h(x(1 + |x|)^{-1})]^{-1},$$

where h is the inducing scale transformation of the corresponding endomorphism on $C(-1, 1)$.

Appendix 1. We now solve directly, without the use of Theorem 2.1, the system of functional equations (3.6)–(3.8) in the real case. It is clearly sufficient to solve for one of the unknown functions; we put $F_i = F$.

$$(A.1.1) \quad F(x_1 + y_1, \dots, x_n + y_n) = F(x_1, \dots, x_n) + F(y_1, \dots, y_n),$$

$$(A.1.2) \quad F(x_1 y_1, \dots, x_n y_n) = F(x_1, \dots, x_n) F(y_1, \dots, y_n),$$

$$(A.1.3) \quad F(\lambda x_1, \dots, \lambda x_n) = \lambda F(x_1, \dots, x_n); \quad f \text{ real.}$$

Put now in (A.1.2) $x_i = y_i = |(\xi_i)^{1/2}|$, then

$$F(\xi_1, \dots, \xi_n) = [F(|(\xi_1)^{1/2}|, \dots, |(\xi_n)^{1/2}|)]^2, \quad \xi_i \geq 0, \quad i.e.$$

$$(A.1.4) \quad F(\xi_1, \dots, \xi_n) \geq 0 \text{ for non-negative } \xi_i. \text{ Every such solution of (A.1.1), on the other hand, is known ([10]) p. 214) to be of the form}$$

$$(A.1.5) \quad F(x_1, \dots, x_n) = C_1 x_1 + \dots + C_n x_n. \text{ Substituting (A.1.5) into (A.1.2) we find}$$

$$(A.1.6) \quad \sum_{k=1}^n C_k x_k y_k = \sum_{i,i=1}^n C_i C_i x_i y_i.$$

Substituting 0 for all variables except x_k and y_k we find $C_k = C_k^2$ i.e. $C_k = 0$ or $C_k = 1$. Thus F must be of the form:

$$(A.1.7) \quad F(x_1, \dots, x_n) = \delta_1 x_1 + \dots + \delta_n x_n$$

where $\delta_i = 0$ or $\delta_i = 1$, $i = 1, \dots, n$.

We now substitute (A.1.7) into (A.1.2) and find putting $x_i = 1$, $i = 1, 2, \dots, n$

$$(A.1.8) \quad \delta_1 + \dots + \delta_n = (\delta_1 + \dots + \delta_n)^2$$

i.e. $\delta_1 + \dots + \delta_n = 0$ or 1 .

Since we already know $\delta_i = 0$ or 1 , (A.1.8) implies either all δ_i equal zero or one of them equals one and the others equal zero. We therefore find the most general solution for (A.1.1)–(A.1.3) as

$$(A.1.9) \quad \begin{aligned} F(x_1, \dots, x_n) &= 0 \quad \text{or} \\ F(x_1, \dots, x_n) &= x_j, \quad j = 1, \dots, n, \end{aligned}$$

which is readily seen to satisfy the functional equation. The result is, for the real case, identical with the one defined from Theorem 2.1. Interestingly, this straight-forward calculation cannot be carried out in the case of complex n -tuples. The crucial point is to show that the solution is continuous, or possesses a measurable majorant or minorant on a set of positive measure etc. (See Aczél [10]). The steps are the following. From (A.1.1) the solution is built up from general ("Hamel-type") solutions of the First Cauchy Equation $F(x + y) = F(x) + F(y)$. The real-case solution is then shown to be non-negative for non-negative values of the variables using (A.1.2). This approach breaks down in the complex case. The method used in Section 2, based on Theorem 2.1, however,

shows that the solution is the same in the complex case. (There, however, linearity (A.1.3) is used.) It is remarkable that the most general endomorphism of $T^{(n)}$ turns out to be so simple; this is due to the unsophisticated multiplication rule (A.1.2); if we consider $T^{(2)}$ but then replace the multiplication rule by

$$(A.1.10) \quad (x_1, x_2)(y_1, y_2) = (x_1y_1 - x_2y_2, x_1y_2 + x_2y_1),$$

$T^{(2)}$ is replaced by the field of complex numbers. It is known [11] that the set of endomorphisms (even the set of automorphisms) of the complex field is vastly larger than the set of endomorphisms of $T^{(2)}$.

Appendix 2. Let B be an algebra of functions from the reals to the reals. Let us call $\epsilon(x)$ a composition unit if

$$(A.2.1) \quad f \circ \epsilon = \epsilon \circ f \quad \text{for all } f \in B.$$

E.g. $\epsilon(x) = x$ is a composition unit in $C(0, 1)$. The composition unit, whenever it exists, is uniquely determined. Assume ϵ_1 and ϵ_2 are two composition units then for all $f \in B$

$$(A.2.2) \quad f \circ \epsilon_1 = \epsilon_1 \circ f = f,$$

$$(A.2.3) \quad f \circ \epsilon_2 = \epsilon_2 \circ f = f.$$

Putting $f = \epsilon_2$ in (A.2.2) and $f = \epsilon_1$ in (A.2.3) we find $\epsilon_2 \circ \epsilon_1 = \epsilon_1 \circ \epsilon_2 = \epsilon_2$; $\epsilon_1 \circ \epsilon_2 = \epsilon_2 \circ \epsilon_1 = \epsilon_1$ i.e. $\epsilon_2 = \epsilon_1$.

Moreover, if B is a separating algebra, the composition is the identity mapping $\epsilon(x) = x$. For assume there exists x_0 such that $\epsilon(x_0) = x_1 \neq x_0$; then, since B is separating, there exists an $f \in B$ such that $f(x_1) \neq f(x_0)$. But then contrarily to the definition of ϵ , $f \circ \epsilon \neq f$ since $(f \circ \epsilon)(x_0) = f[\epsilon(x_0)] = f(x_1) \neq f(x_0)$. We formulate this result as

Lemma A. 2. 1. *Let B be a separating algebra of real-valued real functions. Then B possesses a (uniquely determined) composition unit if and only if the identity mapping belongs to B , and then the composition unit is the identity mapping.*

We can use this observation to determine the multiplier part and the scale transformation part of some endomorphisms of certain function algebras.

Theorem A. 2. 1. *Let B be a separating algebra of real-valued real functions and let the identity mapping belong to B ; and let B possess a multiplicative unit. Then for every linear endomorphism of B of the form*

$$(A.2.4) \quad (\Omega f)(x) = \chi_E(x)(f \circ \omega)(x), \quad \text{we have}$$

$$(A.2.5) \quad \chi_E(x) = (\Omega e)(x),$$

$$(A.2.6) \quad \omega(x) = (\Omega e)(x),$$

where e is the multiplicative unit, and ϵ is the composition unit.

Proof. $e \circ \omega = e$ since $e(x)$ is a constant equalling 1; therefore, putting $f = e$

in (A.2.4), $(\Omega e)(x) = \chi_E(x)$. Even without assuming the form (A.2.4), we find that $(\Omega e)(x)$ is a characteristic function, since $\Omega e = \Omega e^2 = (\Omega e)^2$. Let us now restrict ourselves to $x \in E$, then $(\Omega f)(x) = (f \circ \omega)(x)$, $x \in E$, putting $f = \epsilon$ we find

$$(A.2.7) \quad (\Omega e)(x) = \omega(x).$$

Here the l.h.s. is defined for all x while the r.h.s. is defined for $x \in E$ only.

Using (A.2.7) as a defining equation, we can extend ω to all reals; at the same time, seen in conjunction with Theorem 2.1, (A.2.7) is a more constructive way to determine ω than the one described in Section 2, making use of the Axiom of Choice. We have to remember however that existence of a composition unit is a rather severe restriction, if we consider that e.g. the algebra of all real functions vanishing at infinity has no composition unit (and no multiplicative unit).

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Fordham University

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