*The Weyl Spectrum of an Operator**

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1. Introduction. The concept of Weyl spectrum is native to the theory of perturbation of the spectrum, but recently it has found application to other themes in operator theory. The aims of this article are to give a brief exposition of Weyl spectrum for (bounded linear) operators in Hilbert space, discuss its spectral mapping properties, show its coincidence with the set of “limit points” of the spectrum (suitably defined) for some classes of not necessarily normal operators, and exhibit some applications to polynomially compact operators, commutators, and spectral theory in the Calkin algebra.

In what follows, $\mathcal{H}$ denotes an infinite-dimensional Hilbert space (the theory of Weyl spectrum is vacuous for finite-dimensional spaces). If T is an operator, we write $N(T)$ and $R(T)$ for the null space and range of T ; $\sigma(T)$ for the spectrum of T ; $\pi_0(T)$ for the set of eigenvalues of T ; $\pi_{0,r}(T)$ for the eigenvalues of finite multiplicity; $\pi_{0,i}(T)$ for the eigenvalues of infinite multiplicity; and $\pi_{0o}(T)$ for the isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity.

If $\mathcal{J}$ is the class of invertible operators, then $\sigma(T) = \{\lambda: T - \lambda I \notin \mathcal{J}\}$. Analogously, let

$$\sigma_{\mathcal{F}}(T) = \{\lambda: T - \lambda I \notin \mathcal{F}\},$$

where $\mathcal{F}$ is the class of all Fredholm operators. {An operator T is called a *Fredholm operator* if $N(T)$ is finite-dimensional, $R(T)$ is closed, and $R(T)^\perp$ is finite-dimensional.} It is a theorem of F. V. Atkinson that an operator is Fredholm iff it is invertible modulo the ideal of compact operators iff it is invertible modulo the ideal of operators of finite rank [14, Problem 142]. Thus, if $L(\mathcal{H})$ denotes the algebra of all operators, and $L_c(\mathcal{H})$ the ideal of all compact operators, and if $\hat{T} = T + L_c(\mathcal{H})$ denotes the canonical image of T in the quotient Banach algebra $L(\mathcal{H})/L_c(\mathcal{H})$ (which is a C^* -algebra [11, p. 17], called the *Calkin algebra* of $\mathcal{H}$, denoted briefly L/L_c), then by Atkinson's theorem,

$$(1) \quad \sigma_{\mathcal{F}}(T) = \{\lambda: \hat{T} - \lambda \hat{I} \text{ is not invertible in } L/L_c\} = \sigma(\hat{T});$$

in particular, $\sigma_{\mathcal{F}}(T)$ is nonempty and compact [14, Problem 73].

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To any Fredholm operator S there corresponds an integer

$$(2) \quad i(S) = \dim N(S) - \dim R(S)^\perp = \dim N(S) - \dim N(S^*),$$

called the *index* of S . For example, if $S \in \mathcal{F}$ is normal then $i(S) = 0$ because $N(S) = N(S^*)$. Writing $\mathcal{F}_0$ for the class of all Fredholm operators of index 0, we have

$$(3) \quad \mathcal{J} \subset \mathcal{F}_0 \subset \mathcal{F}.$$

The *Weyl spectrum* of T , denoted $\omega(T)$, is defined by

$$(4) \quad \omega(T) = \{\lambda: T - \lambda I \notin \mathcal{F}_0\}.$$

The concept is due to M. Schechter [19]. By (1) and (3),

$$(5) \quad \sigma(\hat{T}) \subset \omega(T) \subset \sigma(T);$$

in particular, $\omega(T)$ is nonempty. From (1) and (4),

$$(6) \quad \omega(T) = \sigma(\hat{T}) \cup \{\lambda: T - \lambda I \in \mathcal{F} \text{ \& } i(T - \lambda I) \neq 0\};$$

if T is normal, then the second term of the union is empty, thus

$$(7) \quad T \text{ normal} \Rightarrow \omega(T) = \sigma(\hat{T}).$$

In general $\sigma(\hat{T}) \subset \omega(T)$ properly, even if T is hyponormal (Example 1.2).

Proposition 1.1. *If T is a Fredholm operator of index 0, then T is invertible iff T is injective iff T is surjective.*

The proof is immediate from (2) and the fact that a bijective operator is invertible [14, Problem 41].

Example 1.2. *If U is the unilateral shift, then $\omega(U) = \sigma(U) = D$ (the closed unit disc) and $\sigma(\hat{U}) = C$ (the unit circle).*

Proof. $\omega(U) = \sigma(U) = D$ (this follows from Proposition 2.10 since U has no eigenvalues [14, Solution 67]). Since $\hat{U}$ is unitary, $\sigma(\hat{U}) \subset C$. Conversely, suppose $|\lambda| = 1$ and assume to the contrary that $\hat{U} - \lambda I$ is invertible, i.e., that $U - \lambda I$ is Fredholm. Since $N(U - \lambda I) = N(U^* - \lambda^* I) = \{0\}$, $U - \lambda I$ has index 0; by Proposition 1.1, $U - \lambda I$ is invertible, a contradiction. ■

2. Stability of the Weyl spectrum. If T is any operator, then $\omega(T + K) = \omega(T)$ for all compact operators K [19, Theorem 2.1]; we give here a brief proof, based on the following formulation of the Riesz-Schauder theory [1, Chapter 10, §2]:

Lemma 2.1. *If K is any compact operator, then $I + K$ is Fredholm of index 0.*

Lemma 2.2. *If S is invertible and K is compact, then $S + K$ is Fredholm of index 0.*

Proof. $S + K = S(I + S^{-1}K)$, where $I + S^{-1}K$ is Fredholm of index 0 by Lemma 2.1. ■

Lemma 2.3. *If T is Fredholm of index 0, then there exists an operator K of finite rank such that $T + K$ is invertible.*

Proof. By assumption, $\dim N(T) = \dim N(T^*) < \infty$ and $R(T) = N(T^*)^\perp$; let K be any operator that maps $N(T)$ onto $N(T^*)$ and vanishes on $N(T)^\perp$. (A key point is that since $R(T)$ is closed, T is bounded below on $N(T)^\perp$ [cf. 14, Solution 142].) ■

Lemma 2.4. *If T is any operator and K is compact, then $\omega(T) \subset \sigma(T + K)$.*

Proof. If $(T + K) - \lambda I$ is invertible, then $T - \lambda I = [(T + K) - \lambda I] - K$ is Fredholm of index 0 by Lemma 2.2. ■

Theorem 2.5. *For any operator T , $\omega(T) = \bigcap_K \sigma(T + K)$, where K varies over the set of all compact operators (or merely the operators of finite rank).*

Proof. $\omega(T) \subset \bigcap_K \sigma(T + K)$ by Lemma 2.4. On the other hand if $\lambda \notin \omega(T)$ then by Lemma 2.3 there exists an operator K of finite rank such that $(T - \lambda I) + K$ is invertible, i.e., $\lambda \notin \sigma(T + K)$. ■

Corollary 2.6. *For any operator T , $\omega(T)$ is a nonempty compact subset of $\sigma(T)$.*

Proof. Nonemptiness was noted in Section 1; compactness is immediate from Theorem 2.5. ■

Corollary 2.7. *If T is any operator, then $\omega(T + K) = \omega(T)$ for all compact operators K .*

Proof. Immediate from Theorem 2.5. ■

We conclude the section with miscellaneous useful results.

Corollary 2.8. *The following conditions on an operator T are equivalent: (a) T is Fredholm of index 0; (b) $T = S + F$, with S invertible and F of finite rank; (c) $T = S + K$, with S invertible and K compact.*

Proof. Immediate from Lemmas 2.2 and 2.3. ■

Let us interpret Corollary 2.8 in the context of the Calkin algebra. Let $s \in L/L_c$. By Atkinson's theorem, s is invertible iff it has a Fredholm lifting, i.e., $s = \hat{S}$ for some Fredholm operator S (in which case all liftings are Fredholm). By Corollary 2.8, s has an invertible lifting iff some (and therefore every) lifting is Fredholm of index 0.

Corollary 2.9. *For any operator T , $\omega(T^*) = (\omega(T))^*$.*

Proof. It is clear from Corollary 2.8 that $T - \lambda I \in \mathcal{F}_0$ iff $T^* - \lambda^* I \in \mathcal{F}_0$. ■

Proposition 2.10. *If T is any operator, then*

$$(8) \quad \sigma(T) - \omega(T) \subset \pi_{of}(T),$$

equivalently,

$$(8') \quad \sigma(T) - \pi_{of}(T) \subset \omega(T).$$

Proof. If $\lambda \in \sigma(T) - \omega(T)$ then $S = T - \lambda I$ is singular and Fredholm of index 0, therefore $N(S) \neq \{0\}$ by Proposition 1.1, thus $0 < \dim N(S) < \infty$. ■

Corollary 2.11. *If T is any operator and K is compact, then*

$$(9) \quad \sigma(T) - \pi_{of}(T) \subset \sigma(T + K),$$

equivalently,

$$(9') \quad \sigma(T) - \sigma(T + K) \subset \pi_{of}(T).$$

Proof. Immediate from (8') and Lemma 2.4. ■

Example 2.12. $\omega(K) = \{0\}$ for every compact operator K . {This is immediate from Lemma 2.1, or from Corollary 2.7 with $T = 0$.}

3. Spectral mapping theorems.

Theorem 3.1. *If T is a normal operator, then $\omega(f(T)) = f(\omega(T))$ for every continuous complex-valued function f on $\sigma(T)$.*

Proof. Since $\hat{T}$ is a normal element of the Calkin algebra L/L_c , and $\sigma(\hat{T}) \subset \sigma(T)$, by standard C^* -algebra theory $f(\hat{T})$ exists and $f(\hat{T}) = (f(T))^\wedge$ [11, p. 11, Proposition 1.5.3]. Citing (7) and the usual spectral mapping theorem [11, p. 11], we have $\omega(f(T)) = \sigma((f(T))^\wedge) = \sigma(f(\hat{T})) = f(\sigma(\hat{T})) = f(\omega(T))$. ■

Without normality, not much of Theorem 3.1 survives:

Theorem 3.2. *If T is any operator and p is any polynomial, then $\omega(p(T)) \subset p(\omega(T))$. The inclusion may be proper.*

Proof. We can suppose p nonconstant. Suppose $\mu \notin p(\omega(T))$. Writing $p(\lambda) - \mu \equiv a(\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$, we have

$$(*) \quad p(T) - \mu I = a(T - \lambda_1 I) \cdots (T - \lambda_n I).$$

For each j , $p(\lambda_j) = \mu \notin p(\omega(T))$, therefore $\lambda_j \notin \omega(T)$, i.e., $T - \lambda_j I \notin \mathfrak{F}_0$; it follows from (*) that $p(T) - \mu I \notin \mathfrak{F}_0$ (e.g., by Corollary 2.8). The final assertion of the theorem is shown by the following example. ■

Example 3.3. *Let $T = U \oplus (U^* + 2I)$, where U is the unilateral shift, and let $p(\lambda) \equiv \lambda(\lambda - 2)$. Then $0 \in p(\omega(T))$ but $0 \notin \omega(p(T))$.*

Proof. $p(T) = T(T - 2I) = [U \oplus (U^* + 2I)][(U - 2I) \oplus U^*]$; since U is Fredholm of index -1 , and since $U^* + 2I$ and $U - 2I$ are invertible, it follows that T and $T - 2I$ are Fredholm of indices -1 and $+1$, respectively, therefore $p(T) \in \mathfrak{F}_0$ [20, Theorem 2.5]. Thus $0 \notin \omega(p(T))$ whereas $0 = p(0) \in p(\omega(T))$. ■

This example shows that the notion of Weyl spectrum does not derive from the notion of invertibility in some algebra, since in the latter situation the spectral mapping theorem holds for all polynomials (indeed, for all appropriate rational functions [9, p. 278]).

Theorem 3.4. *If either $\pi_{of}(T) = \emptyset$ or $\pi_{of}(T^*) = \emptyset$, then $\omega(p(T)) = p(\omega(T))$ for every polynomial p .*

Proof. It suffices by Corollary 2.9 to consider the case that $\pi_{of}(T) = \emptyset$. It is elementary that $\pi_{of}(p(S)) \subset p(\pi_{of}(S))$ for any operator S and any polynomial p . Thus $\pi_{of}(T) = \emptyset$ implies $\pi_{of}(p(T)) = \emptyset$, therefore $\omega(T) = \sigma(T)$ and $\omega(p(T)) = \sigma(p(T))$ by Corollary 2.10, and our assertion follows from the usual polynomial spectral mapping formula $\sigma(p(T)) = p(\sigma(T))$ [14, Solution 59]. ■

4. Dividing out reducing eigenspaces. In our applications of the following decomposition theorem, we are interested in the cases $\delta = \pi_0(T)$, $\pi_{of}(T)$ and $\pi_{00}(T)$:

Proposition 4.1. *Let T be an operator, let $\delta \subset \pi_0(T)$, and suppose that T is reduced by each of the eigenspaces*

$$(*) \quad N(T - \lambda I) \quad (\lambda \in \delta).$$

Let $\mathfrak{N}$ be the closed linear span of the subspaces (*). Then (1) the subspaces (*) are mutually orthogonal and $\mathfrak{N}$ is their orthogonal direct sum, and (2) $\mathfrak{N}$ reduces T . Let $T_1 = T|_{\mathfrak{N}}$ and $T_2 = T|_{\mathfrak{N}^\perp}$. Then (3) T_1 is a normal operator with pure point spectrum, (4) T_1 has no new eigenvalues, i.e., $\pi_0(T_1) = \delta$, (5) $\sigma(T_1) = \text{Cl } \delta$, (6) $\pi_0(T_2) = \pi_0(T) - \delta$, (7) if $\lambda \in \pi_0(T_2)$ then $N(T - \lambda I) \subset \mathfrak{N}^\perp$ and therefore $N(T_2 - \lambda I) = N(T - \lambda I)$ (thus the multiplicity of λ for T_2 equals its multiplicity for T), (8) $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$, and (9) $\omega(T) = \omega(T_1) \cup \omega(T_2)$.

Proof. (1) It is elementary that $N(T)$ reduces T iff $N(T) \subset N(T^*)$, and that $N(T - \lambda I)$ reduces T iff it reduces $T - \lambda I$; thus the hypothesis reads

$$N(T - \lambda I) \subset N(T^* - \lambda^* I) \quad (\lambda \in \delta),$$

from which it is immediate that the subspaces (*) are orthogonal.

(2) and (3) are obvious. (" T_1 has pure point spectrum" means, by definition, that the space on which T_1 acts is the closed linear span of the eigenspaces of T_1 .)

(4) See [2, p. 164, Example 5].

(5) See [14, Problem 48].

(6), (7) If $\lambda \in \pi_0(T_2)$ then $T - \lambda I$ annihilates some nonzero vectors of $\mathfrak{N}^\perp$, therefore $N(T - \lambda I)$ is not contained in $\mathfrak{N}$; it follows from the definition of $\mathfrak{N}$ that $\lambda \notin \delta$. Thus $\pi_0(T_2) \subset \pi_0(T) - \delta$.

Conversely, suppose $\lambda \in \pi_0(T) - \delta$. We show that $N(T - \lambda I) \subset \mathfrak{N}^\perp$; this will establish simultaneously that $\lambda \in \pi_0(T_2)$, thus $\pi_0(T) - \delta = \pi_0(T_2)$, and that (7) holds. Let $x \in N(T - \lambda I)$. Write $x = y + z$ with $y \in \mathfrak{N}$ and $z \in \mathfrak{N}^\perp$. Then $\lambda y + \lambda z = \lambda x = Tx = Ty + Tz$; since $\mathfrak{N}$ and $\mathfrak{N}^\perp$ are invariant under T , it

follows that $Ty = \lambda y$. If y were nonzero, this would imply $\lambda \in \pi_0(T_1) = \delta$, a contradiction. Thus $y = 0$, $x = z \in \mathfrak{N}^\perp$.

(8) holds for any direct sum $T = T_1 \oplus T_2$, and (9) holds if T_1 is normal [4, Example 5]. ■

5. Weyl's theorem. Weyl's result can be formulated thus: If T is a self-adjoint operator, then

$$(10) \quad \omega(T) = \sigma(T) - \pi_{00}(T).$$

{Actually, the classical formulation is in other terms, but it is shown below to coincide with (10).} Following L. A. Coburn [8] we say that *Weyl's theorem holds for an operator T* if (10) holds, equivalently, if

$$(10') \quad \sigma(T) - \omega(T) = \pi_{00}(T).$$

It is known that Weyl's theorem holds for any hyponormal operator [8]—indeed, for any seminormal operator [4, Example 6]—and for any Toeplitz operator [8].

An operator T is said to satisfy condition (G_1) if $(T - \lambda I)^{-1}$ is normaloid for all $\lambda \notin \sigma(T)$. If T satisfies (G_1) then every isolated point of $\sigma(T)$ is an eigenvalue of T [22, proof of Theorem C]. In general, if every isolated point of $\sigma(T)$ is an eigenvalue of T , we say that T is *isoloid*. An operator T is said to satisfy condition (α) if the restriction of T to any invariant subspace satisfies (G_1) ; (α') if every direct summand of T satisfies (G_1) ; (α'') if the restriction of T to any invariant subspace is convexoid; and (α''') if every direct summand of T is isoloid. As noted in [4], $(\alpha) \Rightarrow (\alpha') \Rightarrow (\alpha'')$, $(\alpha) \Rightarrow (\alpha'') \Rightarrow (\alpha''')$, hyponormal $\Rightarrow (\alpha)$, and seminormal $\Rightarrow (\alpha')$. For reference, we state the principal result of [4]:

Theorem 5.1. *If T satisfies (α''') and is reduced by each of its finite-dimensional eigenspaces, then Weyl's theorem holds for T .*

If Weyl's theorem holds for an operator T , then, by Corollary 2.5, the points of $\sigma(T)$ that cannot be dislodged by perturbation are the points of $\sigma(T) - \pi_{00}(T)$. It is instructive to analyze the condition $\lambda \in \sigma(T) - \pi_{00}(T)$ for an arbitrary operator T : either (i) λ is an accumulation point of $\sigma(T)$, or (ii) λ is an isolated point of $\sigma(T)$ and is an eigenvalue of infinite multiplicity, or (iii) λ is an isolated point of $\sigma(T)$ but is not an eigenvalue of T . As noted above, case (iii) cannot occur when T satisfies (G_1) or (α''') (in particular, when T is seminormal).

Weyl's classical result for self-adjoint operators is expressed in terms of "limit points" of the spectrum; we now explore the relation between this concept and $\omega(T)$.

Suppose T is an operator that is reduced by each of its eigenspaces. Let $\delta = \pi_0(T)$ and write $T = T_1 \oplus T_2$ as in Proposition 4.1. In particular, T_1 is normal with pure point spectrum, $\pi_0(T_1) = \pi_0(T)$, $\sigma(T_1) = \text{Cl } \pi_0(T)$, and $\pi_0(T_2) = \emptyset$.

Definition 5.2. With the preceding notations (in particular, T is an operator that is reduced by each of its eigenspaces), we define $\tau(T)$ to be the set of all

points $\lambda \in \sigma(T)$ such that either (1) $\lambda \in \sigma(T_2)$, or (2) λ is an accumulation point of $\pi_0(T)$ (i.e., $\lambda \in \pi_0(T)'$, the derived set of $\pi_0(T)$), or (3) λ is an eigenvalue of T of infinite multiplicity. Thus,

$$(11) \quad \tau(T) = \sigma(T_2) \cup \pi_0(T)' \cup \pi_{0i}(T).$$

Note that $\tau(T)$ is closed (the first two terms are closed, and any accumulation point of the third term belongs to the second). Also, $\tau(T) \neq \emptyset$. {Proof: Suppose $\tau(T) = \emptyset$. In particular, $\sigma(T_2) = \emptyset$, which entails $\mathfrak{N}^\perp = \{0\}$; thus $\mathfrak{N} = \mathfrak{R}$, $T = T_1$. Also $\pi_0(T)' = \emptyset$, so $\pi_0(T)$ is finite. Thus, $T = T_1$ is a finite linear combination of orthogonal projections with sum I ; since $\pi_{0i}(T) = \emptyset$ the projections are finite-dimensional, so $\mathfrak{R}$ is finite-dimensional, contrary to hypothesis.}

For T self-adjoint, the points of $\tau(T)$ are called the limit points of $\sigma(T)$ in the classical terminology [18, p. 361; 15, p. 4]. For a substantial class of operators, $\tau(T)$ coincides with the Weyl spectrum:

Theorem 5.3. *If T satisfies (α'') and is reduced by each of its eigenspaces, then $\tau(T) = \omega(T) = \sigma(T) - \pi_{00}(T)$.*

Proof. Assume the notations of Definition 5.2. Weyl's theorem holds for T by Theorem 5.1, thus the problem is to show that $\tau(T) = \sigma(T) - \pi_{00}(T)$, equivalently, that $\sigma(T) - \tau(T) = \pi_{00}(T)$.

Suppose $\lambda \in \pi_{00}(T)$; we show that λ does not belong to any of the three terms of the defining formula (11). Obviously $\lambda \notin \pi_{0i}(T)$ and $\lambda \notin \pi_0(T)'$. Assume to the contrary that $\lambda \in \sigma(T_2)$; since λ is isolated in $\sigma(T)$, it is also isolated in $\sigma(T_2)$, therefore it is an eigenvalue of T_2 by (α'') , contrary to $\pi_0(T_2) = \emptyset$.

Conversely, suppose $\lambda \in \sigma(T)$ belongs to none of the three terms in (11). In particular $\lambda \notin \sigma(T_2)$, therefore $\lambda \in \sigma(T_1) = \text{Cl } \pi_0(T) = \pi_0(T) \cup \pi_0(T)'$; since $\lambda \notin \pi_0(T)'$ necessarily $\lambda \in \pi_0(T)$, and since $\lambda \notin \pi_{0i}(T)$ the multiplicity of λ is finite. It remains to show that λ is an isolated point of $\sigma(T)$. Assume to the contrary that there exists a sequence $\lambda_n \in \sigma(T)$ such that $\lambda_n \rightarrow \lambda$, $\lambda_n \neq \lambda$. Since λ is not an accumulation point of eigenvalues, $\lambda_n \in \text{Cl } \pi_0(T)$ for at most finitely many n ; for all n sufficiently large, $\lambda_n \notin \text{Cl } \pi_0(T) = \sigma(T_1)$, therefore $\lambda_n \in \sigma(T_2)$, and so $\lambda \in \sigma(T_2)$, a contradiction. ■

More generally, let T be an operator that is reduced by each of its finite-dimensional eigenspaces. Let $\delta = \pi_{0f}(T)$ and write $T = T_1 \oplus T_2$ as in Proposition 4.1. In particular, T_1 is normal with pure point spectrum, $\pi_0(T_1) = \pi_{0f}(T)$, $\sigma(T_1) = \text{Cl } \pi_{0f}(T)$, and $\pi_0(T_2) = \pi_0(T) - \delta = \pi_{0i}(T)$.

Definition 5.4. With the preceding notations (in particular, T is an operator that is reduced by each of its finite-dimensional eigenspaces), we define

$$(12) \quad \tau'(T) = \pi_{0f}(T)' \cup \sigma(T_2).$$

Obviously $\tau'(T)$ is closed. Also, $\tau'(T) \neq \emptyset$. {Proof: Suppose $\tau'(T) = \emptyset$. In particular, $\sigma(T_2) = \emptyset$, therefore $\mathfrak{N} = \mathfrak{R}$, $T = T_1$, $\sigma(T) = \text{Cl } \pi_{0f}(T)$. Also $\pi_{0f}(T)'$

$= \emptyset$, thus $\pi_{0f}(T)$ is finite; it follows that $\mathcal{H}$ is finite-dimensional, contrary to hypothesis.}

The analogue of Theorem 5.3 for τ' is as follows:

Theorem 5.5. *If T satisfies (α''') and is reduced by each of its finite-dimensional eigenspaces, then $\tau'(T) = \omega(T) = \sigma(T) - \pi_{00}(T)$.*

Proof. Assume the notations of Definition 5.4. Since Weyl's theorem holds for T (Theorem 5.1), it is equivalent to show that $\sigma(T) - \tau'(T) = \pi_{00}(T)$.

Suppose $\lambda \in \pi_{00}(T)$, and assume to the contrary that $\lambda \notin \tau'(T)$. Obviously $\lambda \notin \pi_{0f}(T)'$, therefore $\lambda \in \sigma(T_2)$; then λ is also isolated in $\sigma(T_2)$, thus λ is an eigenvalue of T_2 by (α''') . But this is impossible since $\lambda \in \pi_{00}(T)$ implies $N(T - \lambda I) \subset \mathfrak{N}$ by the definition of $\mathfrak{N}$.

Conversely, suppose $\lambda \in \sigma(T)$ belongs to neither term in the defining formula (12). In particular, $\lambda \notin \sigma(T_2)$; since $\sigma(T_2)$ is closed, there exists a neighborhood U of λ such that $U \cap \sigma(T_2) = \emptyset$. Also, $\lambda \notin \sigma(T_2)$ implies $\lambda \in \sigma(T_1) = \text{Cl } \pi_{0f}(T)$; by assumption $\lambda \notin \pi_{0f}(T)'$, thus $\lambda \in \text{Cl } \pi_{0f}(T) - \pi_{0f}(T)'$, therefore $\lambda \in \pi_{0f}(T)$ and λ is an isolated point of $\text{Cl } \pi_{0f}(T) = \sigma(T_1)$. Let V be a neighborhood of λ such that $V \cap \sigma(T_1) = \{\lambda\}$. Then $W = U \cap V$ is a neighborhood of λ such that $W \cap \sigma(T) = \{\lambda\}$, thus λ is an isolated point of $\sigma(T)$; since $\lambda \in \pi_{0f}(T)$ we conclude that $\lambda \in \pi_{00}(T)$. ■

Combining Theorems 5.3 and 5.5, we have:

Corollary 5.6. *If T satisfies (α''') and is reduced by each of its eigenspaces (in particular, if T is hyponormal), then $\tau(T) = \tau'(T) = \omega(T) = \sigma(T) - \pi_{00}(T)$.*

Since Weyl spectrum is perturbation-invariant, Corollary 5.6 implies at once that if S and T are hyponormal operators such that $S - T$ is compact, then $\tau(S) = \tau(T)$. A special case is Weyl's classical result:

Corollary 5.7. (Weyl) *If A is a self-adjoint operator and K is a compact self-adjoint operator, then $\tau(A + K) = \tau(A)$.*

6. Polynomially compact normal operators. An operator T is said to be *polynomially compact* if there exists a nonzero polynomial p such that $p(T)$ is compact (by infinite-dimensionality, p is even nonconstant). The structure of such operators (even in Banach spaces) has been described recently by F. Gilfeather [12]. For operators in Hilbert space showing symptoms of normality, the structure is more accessible and transparent, as we show in this section.

Theorem 6.1. *If (i) T satisfies (α''') , (ii) T is reduced by each of the eigenspaces*

$$(*) \quad N(T - \lambda I) \quad (\lambda \in \pi_{00}(T)),$$

and (iii) Weyl's theorem holds for T , then one can write $T = T_1 \oplus T_2$, where T_1 is normal with pure point spectrum, and $\sigma(T_2) \subset \omega(T)$. Namely, $T_1 = T \upharpoonright \mathfrak{N}$ and $T_2 = T \upharpoonright \mathfrak{N}^\perp$, where $\mathfrak{N}$ is the closed linear span of the subspaces $(*)$.

Proof. With $\delta = \pi_{00}(T)$, we have $T = T_1 \oplus T_2$ as in Proposition 4.1. It remains to show that $\sigma(T_2) \subset \omega(T)$. Let $\lambda \in \sigma(T_2)$ and assume to the contrary that $\lambda \notin \omega(T) = \sigma(T) - \pi_{00}(T)$. Then $\lambda \in \pi_{00}(T)$. In particular, λ is an isolated point of $\sigma(T)$, and $N(T - \lambda I) \subset \mathfrak{N}$ by the definition of $\mathfrak{N}$. All the more, λ is an isolated point of $\sigma(T_2)$; citing (α'') , λ is an eigenvalue of T_2 , contrary to $N(T - \lambda I) \subset \mathfrak{N}$. ■

Corollary 6.2. *If T satisfies (α''') and is reduced by each of its finite-dimensional eigenspaces, then the hypotheses (and therefore the conclusions) of Theorem 6.1 hold.*

Proof. Weyl's theorem holds for T by Theorem 5.1. ■

Corollary 6.3. *If T satisfies (α''') and (G_1) (in particular if T satisfies (α')), and if Weyl's Theorem holds for T , then the hypotheses (and therefore the conclusions) of Theorem 6.1 hold.*

Proof. If $\lambda \in \pi_{00}(T)$ then, in view of (G_1) , we have $N(T - \lambda I) = N(T^* - \lambda^* I)$ [4, Example 6] and in particular $N(T - \lambda I)$ reduces T . ■

Corollary 6.3. is applicable to any seminormal operator T ; if, moreover, $\omega(T)$ lies on a line or on a circle (or on any smooth arc) then $\sigma(T_2) \subset \omega(T)$ implies that T_2 —hence also T —is normal, a result due to J. G. Stampfli [22, Theorem 4]. In particular, if T is seminormal and $\omega(T) = \{0\}$ then T is compact and normal. {Proof: $\sigma(T_2) \subset \omega(T) = \{0\}$; since T_2 is normaloid, $T_2 = 0$. Thus $T = T_1 \oplus 0$ is normal. Since $\sigma(T)' \subset \sigma(T) - \pi_{00}(T) = \omega(T) = \{0\}$, $\pi_{00}(T)$ either is finite or is a null sequence; in either case T_1 is compact, hence so is T . This result is also contained in Theorem 6.8.} For our discussion of polynomially compact operators, we need only the following fragment of this: If T is normal and $\omega(T) = \{0\}$ then T is compact.

Theorem 6.4. *If T is a normal operator, the following conditions are equivalent: (a) T is polynomially compact; (b) there exists a continuous complex-valued function f on $\sigma(T)$ such that $f(T)$ is compact and f has at most finitely many zeros on $\omega(T)$; (c) $\omega(T)$ is finite.*

Proof. (a) $\Rightarrow$ (b) trivially.

(b) $\Rightarrow$ (c): We have $f(\omega(T)) = \omega(f(T)) = \{0\}$ by Theorem 3.1 and Example 2.12; since f has only finitely many zeros on $\omega(T)$, it follows that $\omega(T)$ is finite.

(c) $\Rightarrow$ (a): Suppose $\omega(T)$ is finite. Let p be any nonzero polynomial such that $p = 0$ on $\omega(T)$. Citing Theorem 3.1 we have $\omega(p(T)) = p(\omega(T)) = \{0\}$; since $p(T)$ is normal, it follows that $p(T)$ is compact (see the remarks following Corollary 6.3), thus T is polynomially compact.

{The structure of T can be worked out using the decomposition of Proposition 4.1 with $\delta = \pi_{0f}(T)$. We suppress the details, but the outcome is that a normal operator T is polynomially compact if and only if it is the orthogonal direct sum of finitely many thin normal operators, that is,

$$T = (K_1 + \lambda_1 I) \oplus \cdots \oplus (K_n + \lambda_n I),$$

where the K_i are compact and normal [cf. 12].} ■

We now consider some extensions of Theorem 6.4.

Theorem 6.5. *Suppose T satisfies (α''') and is reduced by each of its eigenspaces. Then T is polynomially compact if and only if $\omega(T)$ is finite; in this case, T is normal.*

Proof. Let $\delta = \pi_0(T)$ and write $T = T_1 \oplus T_2$ as in Proposition 4.1; thus T_1 is normal and T_2 has no eigenvalues. Let $\mathfrak{M}$ be the closed linear span of the eigenspaces of T ; thus $T_1 = T|_{\mathfrak{M}}$ and $T_2 = T|_{\mathfrak{M}^\perp}$. It will suffice to show that if either (i) T is polynomially compact, or (ii) $\omega(T)$ is finite, then $\mathfrak{M}^\perp = \{0\}$ (therefore $T = T_1$ is normal and Theorem 6.4 applies).

(i) Suppose T is polynomially compact. Assume to the contrary that $\mathfrak{M}^\perp \neq \{0\}$. Let p be a nonzero polynomial such that $p(T)$ is compact. Then $p(T) = p(T_1) \oplus p(T_2)$ shows that $p(T_2)$ is also compact. Since $\pi_0(p(T_2)) = p(\pi_0(T_2)) = p(\emptyset) = \emptyset$, $p(T_2)$ is a compact operator with no eigenvalues; then $\sigma(p(T_2)) = \{0\}$ by the Riesz-Schauder theory, and so $p(\sigma(T_2)) = \sigma(p(T_2)) = \{0\}$ shows that $\sigma(T_2)$ is finite. Citing (α''') , every point of $\sigma(T_2)$ is an eigenvalue of T_2 , contrary to $\pi_0(T_2) = \emptyset$.

(ii) Suppose $\omega(T)$ is finite. Assume to the contrary that $\mathfrak{M}^\perp \neq \{0\}$. Since $\omega(T) = \omega(T_1) \cup \omega(T_2)$, $\omega(T_2)$ is also finite. Citing Proposition 2.10, $\sigma(T_2) - \omega(T_2) \subset \pi_{0f}(T_2) \subset \pi_0(T_2) = \emptyset$, thus $\sigma(T_2) = \omega(T_2)$. In view of (α''') , the finiteness of $\sigma(T_2)$ leads to a contradiction as in (i). ■

Corollary 6.6. *If T is a seminormal operator, then T is polynomially compact if and only if $\omega(T)$ is finite, and in this case T is normal.*

Proof. Since T is polynomially compact if and only if T^* is polynomially compact, and since $\omega(T^*) = (\omega(T))^*$, it is sufficient to consider the case that T is hyponormal—in which case Theorem 6.5 is applicable. {The fact that a polynomially compact seminormal operator is normal is proved in [23].} ■

Theorem 6.7. *Suppose T satisfies (α'') and is reduced by each of its finite-dimensional eigenspaces. Then T is polynomially compact if and only if $\omega(T)$ is finite, and in this case T is normal.*

Proof. Since $(\alpha'') \Rightarrow (\alpha''')$, Weyl's theorem holds for T by Theorem 5.1. Thus the hypotheses of Theorem 6.1 are fulfilled, and we may write $T = A \oplus B$ with A normal and $\sigma(B) \subset \omega(T)$.

If $\omega(T)$ is finite then so is $\sigma(B)$; since B also satisfies (α'') , it follows that B is normal [3]. Then $T = A \oplus B$ is normal, and T is polynomially compact by Theorem 6.4.

Conversely, suppose that T is polynomially compact. Let p be a nonzero polynomial such that $p(T)$ is compact. Let $\delta = \pi_{0f}(T)$ and write $T = T_1 \oplus T_2$

as in Proposition 4.1; thus T_1 is normal and T_2 has no eigenvalues of finite multiplicity. Since $p(T) = p(T_1) \oplus p(T_2)$ is compact, so is $p(T_2)$. But $\pi_{0f}(p(T_2)) \subset p(\pi_{0f}(T_2)) = p(\emptyset) = \emptyset$; since $p(T_2)$ is a compact operator without eigenvalues of finite multiplicity, $\sigma(p(T_2)) = \{0\}$ by the Riesz–Schauder theory. Then $p(\sigma(T_2)) = \sigma(p(T_2)) = \{0\}$, therefore $\sigma(T_2)$ is finite. But T_2 also satisfies (α'') ; as argued earlier, T_2 is normal and therefore so is $T = T_1 \oplus T_2$. Then $\omega(T)$ is finite by Theorem 6.4. ■

The rest of the section deals with operators T such that $\omega(T)$ is a singleton $\{\lambda\}$. If T is any compact operator then $\omega(T) = \{0\}$ (Example 2.12). The converse is false (consider any noncompact quasinilpotent operator). However, if $\omega(T) = \{0\}$ and T is moderately pleasant, then T must be compact and very pleasant:

Theorem 6.8. *Suppose that (i) $\omega(T) = \{0\}$, (ii) Weyl's theorem holds for T , and (iii) T satisfies either (α') or (α'') . Then T is compact and normal.*

Proof. By assumption, $\sigma(T) - \{0\} = \pi_{00}(T)$, thus the nonzero spectrum consists of isolated points. Since, in particular, $\sigma(T)$ has at most one accumulation point, it follows from either (α') or (α'') that T is normal with pure point spectrum. {The proof for (α'') is given in [3, Lemma 4], and the argument is easily adapted for (α') .} The nonzero eigenvalues are of finite multiplicity; since they are either finite in number or form a null sequence, it follows that T is compact. {In particular, if T is seminormal and $\omega(T) = \{0\}$ then T is compact and normal, a result due to Stampfli [21, Corollary 2].} ■

The following corollary is proved in [4, Example 7]:

Corollary 6.9 *If Weyl's theorem holds for T , and T satisfies (α') or (α'') , and if $T = K + N$ with K compact and $\sigma(N) = \{0\}$ then T is compact and normal.*

Proof. $\omega(T) = \omega(N + K) = \omega(N) \subset \sigma(N) = \{0\}$; quote Theorem 6.8. ■

Corollary 6.10. *If (i) $\omega(T) = \{\lambda\}$, (ii) Weyl's theorem holds for T , and (iii) T satisfies either (α') or (α'') , then $T = \lambda I + K$ with K compact and normal.*

Proof. $\omega(T - \lambda I) = \{0\}$, and $T - \lambda I$ also satisfies Weyl's theorem, and (α') or (α'') , as the case may be; quote Theorem 6.8. ■

Another corollary is a characterization of compactness proved by K. Gustafson and J. Weidmann [13]:

Corollary 6.11. *Let K be an operator. In order that*

$$(*) \quad \omega(T + K) = \omega(T) \quad \text{for all operators } T,$$

it is necessary and sufficient that K be compact.

Proof. Sufficiency is Corollary 2.7. Conversely, suppose $(*)$ holds. For $T = 0$ this yields $\omega(K) = \{0\}$. Then $\omega(K^* + K) = \omega(K^*) = (\omega(K))^* = \{0\}$, therefore

$K^* + K$ is compact by Theorem 6.8. Similarly $K - K^*$ is compact, therefore $(K^* + K) + (K - K^*) = 2K$ is also compact. ■

7. Weyl spectrum and commutators. A *commutator* is an operator of the form $AB - BA$. A. Brown and C. Pearcy [6] have characterized the noncommutators (assuming $\dim \mathfrak{H} = \aleph_0$) as the set of operators of the form $\lambda I + K$, where λ is a nonzero scalar and K is compact; for such an operator, $\omega(\lambda I + K) = \omega(\lambda I) = \{\lambda\}$. Thus, assuming $\dim \mathfrak{H} = \aleph_0$,

$$(*) \quad T \text{ a noncommutator} \Rightarrow \omega(T) = \{\lambda\}, \quad \lambda \neq 0,$$

and in particular,

$$(**) \quad 0 \in \omega(T) \Rightarrow T \text{ is a commutator.}$$

(The special case of (**) for T normal was proved in [5, Theorem 5].)

The implication reverse to (*) generally fails (e.g., let $T = I + N$, where N is quasinilpotent and noncompact [6, Corollary 5.4]). However, if T is pleasant the implication can be reversed (without any separability hypothesis):

Theorem 7.1. *If (i) $\omega(T) = \{\lambda\}$, $\lambda \neq 0$, (ii) Weyl's theorem holds for T , and (iii) T satisfies either (α') or (α'') , then T is a normal noncommutator.*

Proof. By Corollary 6.10, $T = \lambda I + K$, where K is compact (and normal), therefore T is a noncommutator [cf. 14, Problem 182, Corollary]. ■

Lemma 7.2. *If A and B are operators, the following conditions are equivalent: (a) $A \otimes B$ is Fredholm; (b) $A \otimes B$ is invertible; (c) A and B are invertible.*

Proof. Since $\sigma(A \otimes B) = \sigma(A)\sigma(B)$ [7], (b) and (c) are equivalent. Of course (b) implies (a).

(a) $\Rightarrow$ (b): Suppose $A \otimes B$ is Fredholm. We assert that A and B are injective. If, on the contrary, say $Ax = 0$, $x \neq 0$, then for all y one has $(A \otimes B)(x \otimes y) = Ax \otimes By = 0$, thus $N(A \otimes B)$ is infinite-dimensional, a contradiction. It follows that A^* and B^* have dense range, consequently $(A \otimes B)^* = A^* \otimes B^*$ has dense range, and this implies that $A \otimes B$ is injective. But $A^* \otimes B^* = (A \otimes B)^*$ is also Fredholm, so $A^* \otimes B^*$ is injective. Then $N(A \otimes B) = N((A \otimes B)^*) = \{0\}$, therefore $A \otimes B$ is invertible by Proposition 1.1. ■

Theorem 7.3. *(Assuming $\dim \mathfrak{H} = \aleph_0$) If either A or B is singular, then $A \otimes B$ is a commutator.*

Proof. By Lemma 7.2, $A \otimes B$ is not Fredholm; all the more, it is not Fredholm of index 0, that is, $0 \in \omega(A \otimes B)$, and so $A \otimes B$ is a commutator by (**). ■

An instructive application of Theorem 7.3 to von Neumann algebras is as follows. Let $\mathfrak{M}$ be a von Neumann algebra of finite class, on a separable infinite-dimensional Hilbert space. If $T \mapsto T^{\mathfrak{h}}$ is the center-valued trace of $\mathfrak{M}$, then the center-valued trace for the finite von Neumann algebra $\mathfrak{M} \otimes \mathfrak{M}$ satisfies $(S \otimes$

$T)^{\natural} = S^{\natural} \otimes T^{\natural}$ [10, p. 249, Théorème 3, and p. 56, Proposition 2]. Suppose $A \in \mathfrak{M}$, $A \geq 0$, $A \neq 0$, A singular (such an A exists, unless $\mathfrak{M}$ is one-dimensional); by Theorem 7.3, $A \otimes A$ is a commutator. Since $A^{\natural} \neq 0$ [10, p. 249], $(A \otimes A)^{\natural} = A^{\natural} \otimes A^{\natural} \neq 0$. Thus: *$A \otimes A$ is a commutator, but its trace is not zero.* Therefore, although a representation $A \otimes A = XY - YX$ exists, one must reach outside $\mathfrak{M} \otimes \mathfrak{M}$ for X and Y . (The underlying reason for this is the role of unilateral shifts in the construction of commutators; such operators are foreign to von Neumann algebras of finite class.)

A *self-commutator* is an operator (necessarily self-adjoint) of the form $T^*T - TT^*$. The self-commutators in separable, infinite-dimensional Hilbert space have been characterized by H. Radjavi [17, Theorem 8] as follows: (Assuming $\dim \mathfrak{H} = \aleph_0$) *A self-adjoint operator S is a self-commutator if and only if $0 \notin \text{conv } \omega(S)$.* If S is semidefinite (i.e., $S \geq 0$ or $S \leq 0$), then $\omega(S) \geq 0$ or $\omega(S) \leq 0$, and the condition $0 \notin \text{conv } \omega(S)$ is equivalent to $0 \notin \omega(S)$. On the other hand, for any normal operator S , the condition $0 \notin \omega(S)$ is equivalent to “ $N(S)$ is infinite-dimensional or $R(S)$ is not closed”; indeed, since $N(S) = N(S^*) = R(S)^{\perp}$, the definition of “ S is Fredholm of index 0” reduces to “ $N(S)$ is finite-dimensional and $R(S)$ is closed”. Thus, Radjavi’s theorem has the following corollary: (Assuming $\dim \mathfrak{H} = \aleph_0$) *If S is semidefinite, the following conditions are equivalent: (a) S is a self-commutator; (b) $0 \notin \omega(S)$; (c) either $N(S)$ is infinite-dimensional or $R(S)$ is not closed.*

Two special cases are worthy of note: (1) If S is a compact self-adjoint operator, then $\omega(S) = \{0\}$ and therefore S is a self-commutator by Radjavi’s theorem [cf. 14, Problem 186]. (2) If T is seminormal then, by the above corollary, the operator $S = T^*T - TT^*$ either has infinite-dimensional null space or has nonclosed range, and therefore is not invertible [cf. 14, Problem 188].

8. Von Neumann’s converse of Weyl’s theorem. *Throughout this section, the underlying Hilbert space $\mathfrak{H}$ is assumed to be separable.*

Weyl’s classical result, noted at the end of Section 5, can be put as follows: If A and B are self-adjoint operators such that $A - B$ is compact, then $\tau(A) = \tau(B)$ (i.e., $\omega(A) = \omega(B)$). J. von Neumann proved the following striking converse [15, Satz II; 18, p. 365]:

Theorem 8.1. (von Neumann) *If A and B are self-adjoint operators such that $\tau(A) = \tau(B)$, then there exists a unitary operator U such that $UAU^* - B$ is compact.*

Von Neumann’s theorem has remarkable consequences for spectral theory in the Calkin algebra L/L_c . By Corollary 5.6, the condition in Theorem 8.1 may be written $\omega(A) = \omega(B)$, or, in view of (7), $\sigma(\hat{A}) = \sigma(\hat{B})$. The theorem grows more startling with reformulation; to this end, we introduce some general definitions.

We say that $A, B \in L(\mathfrak{H})$ are *essentially similar* if there exists an invertible operator S such that $SAS^{-1} - B$ is compact (in other words, A is similar to a

perturbation of B by a compact operator). Call A, B *essentially equivalent* if there exists a unitary operator U such that $UAU^* - B$ is compact. If A and B are essentially similar, it follows from Corollary 2.7 that $\omega(A) = \omega(B)$. Von Neumann's theorem can be rephrased as a partial converse: If A and B are self-adjoint operators such that $\omega(A) = \omega(B)$, then A and B are essentially equivalent.

In the Calkin algebra L/L_c , the terms *similar* and (unitarily) *equivalent* are defined in the usual way. The relation between similarity in L/L_c and essential similarity in L is clouded by the fact that an invertible element of L/L_c need not have an invertible lifting (see the remarks following Corollary 2.8). The same remark pertains to equivalence in L/L_c and essential equivalence in L , since a unitary element in L/L_c need not have a unitary (or even normal) lifting:

Example 8.2. If $u = \hat{U}$, where U is the unilateral shift, then u is a unitary element of L/L_c which has no normal lifting. {Proof: For any compact K , $U + K$ is Fredholm with $i(U + K) = i(U) = -1$ [20, Theorem 2.8], therefore $U + K$ cannot be normal [cf. 14, Solution 145].}

However, if $a \in L/L_c$ is self-adjoint, then it has a self-adjoint lifting; indeed, if A is any lifting then $(1/2)(A + A^*)$ is a self-adjoint lifting.

The foregoing considerations, and von Neumann's theorem, motivate the following definitions. Let us say that $a, b \in L/L_c$ are *strongly similar* if there exists an invertible operator $S \in L(\mathcal{H})$ such that $\hat{S}a\hat{S}^{-1} = b$ (in other words, a and b have similar liftings). Call a, b *strongly equivalent* if there exists a unitary operator $U \in L(\mathcal{H})$ such that $\hat{U}a\hat{U}^* = b$ (i.e., a and b have equivalent liftings). These definitions simply shift the perspective to L/L_c : clearly A and B are essentially similar [essentially equivalent] if and only if $\hat{A}$ and $\hat{B}$ are strongly similar [strongly equivalent]. Von Neumann's theorem can now be given a striking reformulation in the Calkin algebra:

Corollary 8.3. Let a and b be self-adjoint elements of L/L_c . The following conditions imply one another: (i) a and b are strongly equivalent; (ii) a and b are similar; (iii) $\sigma(a) = \sigma(b)$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) trivially (for any a and b).

(iii) $\Rightarrow$ (i): Write $a = \hat{A}$, $b = \hat{B}$, with A and B self-adjoint. Citing (7) we have $\omega(A) = \sigma(a) = \sigma(b) = \omega(B)$, therefore A and B are essentially equivalent by von Neumann's theorem. ■

Corollary 8.3 does not remain true if "self-adjoint" is replaced by "unitary":

Example 8.4. There exist unitaries u, w in L/L_c such that $\sigma(u) = \sigma(w) = C$ (the unit circle) but u and w are not similar.

Proof. If $u = \hat{U}$ and $w = \hat{W}$, where U is a unilateral shift and W is a bilateral shift (each with respect to a suitably indexed orthonormal basis), then u and w are unitary and $\sigma(u) = \sigma(w) = C$. Assume to the contrary that there exists

an invertible $s \in L/L_c$ such that $su = ws$. Choose $S \in L(\mathcal{H})$ with $\hat{S} = s$; S is a Fredholm operator, by Atkinson's theorem [14, Problem 142]. Then $SU = WS + K$, K compact, therefore $i(S) + i(U) = i(SU) = i(WS + K) = i(WS) = i(S)$ by various properties of index [20, Theorems 2.5, 2.8], which contradicts $i(U) = -1$. ■

The pathology in Example 8.4 cannot occur when the spectrum is a proper subset of the unit circle:

Corollary 8.5. *Let u and v be unitary elements of L/L_c whose spectra are proper subsets of the unit circle. The following conditions imply one another: (i) u and v are strongly equivalent; (ii) u and v are similar; (iii) $\sigma(u) = \sigma(v)$.*

Proof. Rotating by a scalar factor of absolute value one, we can suppose $1 \notin \sigma(u) = \sigma(v)$. The corollary now follows routinely from Corollary 8.3, the Cayley transform, and the spectral mapping theorem for the rational functional calculus [9, p. 278]. ■

In view of (7), Corollary 8.5 implies a variant of von Neumann's theorem for unitary operators whose Weyl spectrum is a proper subset of the unit circle:

Corollary 8.6. *If U and V are unitary operators such that $\omega(U) = \omega(V)$ is a proper subset of the unit circle, then U and V are essentially equivalent.*

Note that for a unitary operator U , $\omega(U) \subset C$ properly if and only if $\sigma(U) \subset C$ properly; this is immediate from the formula $\omega(U) = \sigma(U) - \pi_{00}(U)$. It is implicit in the proof of Corollary 8.5 that if $u \in L/L_c$ is unitary and $\sigma(u) \subset C$ properly, then u has a unitary lifting.

Corollary 8.7. *If A and B are seminormal operators such that $\sigma(A) = \sigma(B)$ is connected, then $\operatorname{Re} A$ is essentially equivalent to $\operatorname{Re} B$, and $\operatorname{Im} A$ is essentially equivalent to $\operatorname{Im} B$.*

Proof. It is sufficient to consider real parts. If $\sigma(A) = \sigma(B)$ is a singleton $\{\lambda\}$, then $A = B = \lambda I$ (seminormal operators are convexoid [16, p. 47, Corollary 2]). Otherwise, the connected set $\sigma(A) = \sigma(B)$ can have no isolated points, and in particular $\pi_{00}(A) = \pi_{00}(B) = \emptyset$; since Weyl's theorem holds for A and B (see Section 5) we conclude that

$$(*) \quad \omega(A) = \sigma(A) = \sigma(B) = \omega(B),$$

and (*) holds even if $\sigma(A) = \sigma(B)$ is a singleton. Citing a projection theorem of Putnam [16, p. 46, Theorem 3.4.1] we have

$$(**) \quad \sigma(\operatorname{Re} A) = \operatorname{Re} \sigma(A) = \operatorname{Re} \sigma(B) = \sigma(\operatorname{Re} B);$$

moreover, the set described in (**) is the projection of a connected set, so it must be an interval (possibly degenerate). We may therefore replace A, B in (*) by $\operatorname{Re} A, \operatorname{Re} B$, obtaining $\omega(\operatorname{Re} A) = \omega(\operatorname{Re} B)$, and so $\operatorname{Re} A$ and $\operatorname{Re} B$ are essentially equivalent by von Neumann's theorem. ■

Corollary 8.8. *If $a, b \in L/L_c$ are seminormal and $\sigma(a) = \sigma(b)$, then $\operatorname{Re} a$ and $\operatorname{Re} b$ are strongly equivalent, and $\operatorname{Im} a$ and $\operatorname{Im} b$ are strongly equivalent.*

Proof. Again citing Putnam's projection theorem, we have $\sigma(\operatorname{Re} a) = \operatorname{Re} \sigma(a) = \operatorname{Re} \sigma(b) = \sigma(\operatorname{Re} b)$, therefore $\operatorname{Re} a$ and $\operatorname{Re} b$ are strongly equivalent by Corollary 8.3. ■

The foregoing results suggest the following question: Does von Neumann's theorem hold with "self-adjoint" replaced by "normal"?

Added in Proof (11-10-70): Yes [W. Sikonja (preprint), and G. Edgar, J. Ernest and S. G. Lee (preprint)].

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