Positivity of Solutions to Integral Equations with Logarithmic Kernels

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1. Introduction. Methods for establishing the positivity of solutions to integral equations of the form

$$(1) \quad \int_{-1}^1 k(x, y)f(y) dy = 1, \quad -1 < x < 1,$$

are lacking in general unless the kernel comes from a differential equation which satisfies a maximum principle. Here, we give sufficient conditions for integral equation (1) to have a strictly positive solution for a class of logarithmic kernels. This result is applied to a problem in linearized viscous hydrodynamics. We show that the local drag distribution is positive in Oseen flow past a flat plate oriented parallel to the oncoming stream.

It is assumed, throughout, that (1) has a solution $f \in L^p[-1, 1]$ for some p , $1 < p < 2$. We require that $k \in K$ where K denotes the class of functions which admit the representation

$$k(x, y) = B(x, y) \operatorname{Log} |x - y| + C(x, y).$$

Here $B(x, y)$ and $C(x, y)$ are required to be analytic functions for complex $x, y \in \bar{D}[-1, 1]$ and $B(x, x) \neq 0$ for $x \in \bar{D}[-1, 1]$, where $D[\alpha, \beta]$ denotes the complex domain $0 \leq |x - \alpha|, |x - \beta| < |\beta - \alpha|$ and $\bar{D}[\alpha, \beta]$ its closure. In Section 2 we consider the integral equation

$$(2) \quad \int_{\alpha}^{\beta} k(x, y)f(y) dy = g(x), \quad -1 \leq \alpha < x < \beta \leq 1,$$

where $k \in K$ and g is analytic for complex $x \in \bar{D}[\alpha, \beta]$. There we extend a theorem due to MacCamy [1], and show that if (2) has a solution $f \in L^p[\alpha, \beta]$, for some p , $1 < p < 2$, then

$$(3) \quad f = \frac{a(x)}{((\beta - x)(x - \alpha))^{1/2}}, \quad \text{a.e.},$$

where $a(x)$ is analytic for complex $x \in D[\alpha, \beta]$. In addition, k is required to satisfy

$$(4) \quad 0 < \operatorname{sgn}(y - x)^n \frac{\partial^n}{\partial x^n} k(x, y) < \infty, \\ -1 \leq x, y \leq 1, \quad x \neq y, \quad n = 0, 1, 2.$$

It is then shown in Section 3 that the solution to (1) is strictly positive almost everywhere. This result is not necessarily limited by the requirement that the inhomogeneous term in (1) be a constant. If an integral equation is encountered where the inhomogeneous term is different from a constant, we need only divide the equation by this term to cast it into the form of (1). If in (2) we let $g = \beta = -\alpha = 1$, then (3) provides a representation for the solution to (1). The proof of positivity depends on the essential properties of (3) which are that the quantity $a(x)/(1 - x^2)^{1/2}$ have a finite number of zeroes on $-1 \leq x \leq 1$ and is twice differentiable on $-1 < x < 1$. If these properties of f are known a priori, then (4) implies that (1) has a strictly positive solution for a much wider class of kernels than that considered here. The methods which establish the positivity of f are also used to derive a uniqueness theorem.

Our results are extended to include integral equations of the form

$$(5) \quad \int_E k(x, y) f(y) dy = 1, \quad x \in E \subseteq \{x : -1 < x < 1\},$$

where $k \in K$ satisfies (4), $f \in L^p[\bar{E}]$ and $E = \bigcup_{i=1}^n \{\alpha_i < x < \beta_i\}$ is a finite number of segments of the real line which satisfy the spacing condition

$$(6) \quad \left(\bigcup_{i=0}^n \{x : \alpha_i > x \geq 2\alpha_i - \beta_i\} \bigcup_{i=0}^n \{x : 2\beta_i - \alpha_i \geq x > \beta_i\} \right) \cap \bar{E} = \emptyset.$$

It is shown that the solution to (5) is strictly positive almost everywhere and a uniqueness theorem is derived. We remark that the spacing condition (6) is needed to obtain a representation of the solution to (5).

2. Representation of the solution. To establish the result on positivity, it is first necessary to derive a representation theorem for the solution f to (1). The desired representation,

$$f = \frac{a(x)}{(1 - x^2)^{1/2}} \quad \text{a.e.},$$

where a is smooth was essentially obtained by MacCamy [1] and we state his result as

Theorem 1 (MacCamy). *Let $\tilde{f}$ be a function which is continuous and satisfies Hölder condition on $\alpha < x < \beta$ and such that $[(x - \alpha)(\beta - x)]^\gamma \tilde{f}(x)$ is continuous at $x = \alpha, \beta$ for some $\gamma, 1 > \gamma > 0$. Furthermore let $\tilde{f}$ be a solution to*

$$* \int_\alpha^\beta \left\{ \frac{\tilde{A}(x, y)}{x - y} + \tilde{B}(x, y) \operatorname{Log} |x - y| + \tilde{C}(x, y) \right\} \tilde{f}(y) dy = \tilde{g}(x), \quad \alpha < x < \beta,$$

where $\tilde{A}(x, y)$, $\tilde{B}(x, y)$, $\tilde{C}(x, y)$ are analytic for complex $x, y \in \tilde{D}[\alpha, \beta]$, $\tilde{g}(x)$ is analytic for complex $x \in \tilde{D}[\alpha, \beta]$ and if $\tilde{A}(x, x) \neq 0$, then $\tilde{A}(x, x) \neq 0$ for complex $x \in \tilde{D}[\alpha, \beta]$ or if $\tilde{A}(x, x) \equiv 0$, then $\tilde{B}(x, x) \neq 0$ for complex $x \in \tilde{D}[\alpha, \beta]$. Then

$$\tilde{f} = \frac{\tilde{a}(x)}{((\beta - x)(x - \alpha))^{1/2}},$$

where $\tilde{a}(x)$ is analytic for complex $x \in D[\alpha, \beta]$.

For generality as well as reasons which will be evident in the proof of Theorem 5 we consider in this section the integral equation,

$$(7) \quad \int_{\alpha}^{\beta} k(x, y) f(y) dy = g(x), \quad -1 \leq \alpha < x < \beta \leq 1$$

where $k \in K$ and $g(x)$ is analytic for complex $x \in \tilde{D}[\alpha, \beta]$. Theorem 1 cannot be applied directly to (7) if we assume only that $f \in L^p[\alpha, \beta]$, for some p , $1 < p < 2$. To help surmount this difficulty we write (7) as

$$(8) \quad \int_{\alpha}^{\beta} [B(x, x) \operatorname{Log} |x - y| + (B(x, y) - B(x, x)) \operatorname{Log} |x - y| + C(x, y)] h(y) dy = g(x), \quad \alpha < x < \beta,$$

where

$$k(x, y) = B(x, y) \operatorname{Log} |x - y| + C(x, y),$$

$$h(x) = f(x) - \frac{c}{\pi} \frac{1}{((\beta - x)(x - \alpha))^{1/2}},$$

$$g(x) = g(x) - \frac{c}{\pi} \int_{\alpha}^{\beta} k(x, y) \frac{dy}{((\beta - y)(y - \alpha))^{1/2}} \quad \text{and} \quad c = \int_{\alpha}^{\beta} f(x) dx.$$

We assert that

$$\int_{\alpha}^{\beta} k(x, y) \frac{dy}{((\beta - y)(y - \alpha))^{1/2}}$$

is analytic for complex $x \in \tilde{D}[\alpha, \beta]$ and hence $g(x)$ is analytic for complex $x \in \tilde{D}[\alpha, \beta]$. To preserve the continuity of the argument, the proof of this assertion is delayed until Lemma 1. Furthermore we observe that since $\int_{\alpha}^{\beta} h(x) dx = 0$,

$$\int_{\alpha}^{\beta} \operatorname{Log} |x - y| h(y) dy = \int_{\alpha}^{\beta} \frac{\int_{\alpha}^y h(t) dt}{x - y} dy.$$

Then integration by parts in (8) yields the integral equation

$$(9) \quad \int_{\alpha}^{\beta} \left\{ \frac{B(x, y)}{x - y} - B_v(x, y) \operatorname{Log} |x - y| - \frac{B(x, x) - B(x, y)}{x - y} - C_v(x, y) \right\} \cdot \int_{\alpha}^y h(t) dt dy = g(x), \quad \alpha < x < \beta$$

for the unknown function $\int_{\alpha}^x h(y) dy$. We are assuming that (7) has a solution $f \in L^p[\alpha, \beta]$, for some p , $1 < p < 2$. Therefore, (8) has a solution $h \in L^p[\alpha, \beta]$. Moreover, since (8) has solution h , (9) has a solution $\int_{\alpha}^x h(y) dy$ which satisfies the inequality

$$\left| \int_{\alpha}^x h(t) dt - \int_{\alpha}^y h(t) dt \right| = \left| \int_y^x h(t) dt \right| \leq |x - y|^{p/p-1} \|h\|_p,$$

and is thereby Hölder continuous. Application of Theorem 1 to (9) gives

$$(10) \quad \int_{\alpha}^x h(y) dy = \frac{\hat{d}(x)}{((\beta - x)(\alpha - x))^{1/2}},$$

where $\hat{d}(x)$ is analytic for complex $x \in D[\alpha, \beta]$ and $\hat{d}(\alpha) = \hat{d}(\beta) = 0$ since the right hand side of (10) vanishes at these points. In (10) insert the definition of h and differentiate with respect to x to obtain

$$f(x) = \frac{a(x)}{((\beta - x)(x - \alpha))^{1/2}} \quad \text{a.e.},$$

where $a(x) = c + \hat{d}_x(x) + \frac{1}{2}\hat{d}(x)[(\beta - x)^{-1} - (x - \alpha)^{-1}]$ is analytic for complex $x \in D[\alpha, \beta]$. Thus we have just proved

Theorem 2. Let $k \in K$ and $g(x)$ be analytic for complex $x \in \bar{D}[\alpha, \beta]$. If integral equation (7) has a solution $f \in L^p[\alpha, \beta]$, for some p , $1 < p < 2$, then

$$f(x) = \frac{a(x)}{((\beta - x)(x - \alpha))^{1/2}}$$

a.e. where $a(x)$ is analytic for complex $x \in D[\alpha, \beta]$.

We now prove the earlier assertion.

Lemma 1. Let $k \in K$ and $-1 \leq \alpha < \beta \leq 1$, then

$$\bar{g}(x) = \int_{\alpha}^{\beta} k(x, y) \frac{dy}{((\beta - y)(y - \alpha))^{1/2}}$$

is analytic for complex $x \in \bar{D}[\alpha, \beta]$.

Proof. For $k \in K$, $\bar{g}$ may be expressed as

$$(11) \quad \bar{g}(x) = \int_{\alpha}^{\beta} [B(x, y) \operatorname{Log} |x - y| + C(x, y) \frac{dy}{((\beta - y)(y - \alpha))^{1/2}}].$$

Let $\begin{pmatrix} x \\ y \end{pmatrix} = (\beta - \alpha) \begin{pmatrix} s \\ t \end{pmatrix} + \alpha$, $\bar{g}(x) = \tilde{g}(s)$, $B(x, y) = \tilde{B}(s, t)$ and $C(x, y) = \tilde{C}(s, t) - \operatorname{Log} |\beta - \alpha| \tilde{B}(s, t)$. Then (11) becomes

$$(12) \quad \tilde{g}(s) = \int_0^1 [\tilde{B}(s, t) \operatorname{Log} |t - s| + \tilde{C}(s, t)] \frac{dt}{(t(1 - t))^{1/2}},$$

where $\tilde{B}(s, t)$ and $\tilde{C}(s, t)$ are analytic for complex $t, s \in \bar{D}[0, 1]$. We could have

equally well let $\begin{pmatrix} x \\ y \end{pmatrix} = (\alpha - \beta) \begin{pmatrix} \hat{s} \\ \hat{t} \end{pmatrix} + \beta$, $\bar{g}(x) = \bar{g}(\hat{s})$, $B(x, y) = \hat{B}(\hat{s}, \hat{t})$, and $C(x, y) = \hat{C}(\hat{s}, \hat{t}) - \text{Log } |\beta - \alpha| \hat{B}(\hat{s}, \hat{t})$. Then (11) would become

$$\bar{g}(\hat{s}) = \int_0^1 [\hat{B}(\hat{s}, \hat{t}) \text{Log } |\hat{t} - \hat{s}| + \hat{C}(\hat{s}, \hat{t})] \frac{d\hat{t}}{(\hat{t}(1 - \hat{t}))^{1/2}},$$

where $\hat{B}(\hat{s}, \hat{t})$ and $\hat{C}(\hat{s}, \hat{t})$ are analytic for complex $\hat{s}$, $\hat{t} \in \bar{D}[0, 1]$. If $\bar{g}(s)$ is analytic for complex s on $\text{Re}(s) \leq \frac{1}{2}$ and $0 \leq |s| \leq 1$, then $\bar{g}(\hat{s})$ must be analytic for complex $\hat{s}$ on $\text{Re}(\hat{s}) \leq \frac{1}{2}$ and $0 \leq |\hat{s}| \leq 1$. However, this statement implies that $\bar{g}(x)$ is analytic for complex $x \in \bar{D}[\alpha, \beta]$. Therefore to prove the lemma it is sufficient to show that $\bar{g}(s)$ given by (12) is analytic for complex s on $\text{Re}(s) \leq \frac{1}{2}$ and $0 \leq |s| \leq 1$.

Let R denote the domain $0 < |z| < 1$ of a Riemann surface and let z^+ be the image of a point z after a counterclockwise circuit of the origin. For $0 < |z| < 1$ and $0 \leq \arg z \leq 2\pi$, define

$$v(z) = \int_0^1 [\tilde{B}(z, t) \text{Log}(t - z) + \tilde{C}(z, t)] \frac{dt}{(t(1 - t))^{1/2}},$$

and

$$(13) \quad I(z) = 2\pi i \int_0^+ \tilde{B}(z, t) \frac{dt}{(t(1 - t))^{1/2}},$$

where the logarithm is intended as that branch which is real for real positive t and real negative z . The functions $v(z)$ and $I(z)$ are analytic on the sector $0 < \arg z < 2\pi$ of R and satisfy for $\arg(z) = 0$,

$$(14) \quad v(z^+) - v(z) = I(z)$$

$$(15) \quad v(z^+) + v(z) = 2 \int_0^1 [\tilde{B}(z, t) \text{Log } |t - z| + \tilde{C}(z, t)] \frac{dt}{(t(1 - t))^{1/2}} = 2\tilde{g}(z).$$

More details of this development can be found in MacCamy [1]. In (13) let $t = sz$ to conclude that for $\arg(z) = 0$,

$$(16) \quad I(z^+) + I(z) = 0.$$

Then (14) can be written as

$$(v(z^+) + \frac{1}{2}I(z^+)) - (v(z) + \frac{1}{2}I(z)) = 0,$$

which is just a statement of the single-valueness of $(v(z) + \frac{1}{2}I(z))$. Therefore,

$$(17) \quad v(z) = -\frac{1}{2}I(z) + h(z)$$

where, since $v(z)$ has no poles, $h(z)$ is analytic on $0 \leq |z| < 1$. Equations (15), (16) and (17) give for $\arg(z) = 0$,

$$(18) \quad \tilde{g}(z) = h(z).$$

Hence $\tilde{g}(z)$ is analytic for $0 \leq |z| < 1$. To complete the proof we note that $v(z)$ and $I(z)$ can be analytically continued to the domain $0 < \arg(z) < 2\pi$ and $0 < |z| \leq 1$, and therefore by (17) and (18) $\tilde{g}(z)$ can be continued to this same domain. Consequently, $\tilde{g}(s)$ is analytic for complex s on $\operatorname{Re}(s) \leq \frac{1}{2}$ and $0 \leq |s| \leq 1$. *q.e.d.*

3. Main Results. Our attention is now directed at the integral equation

$$(19) \quad \int_{-1}^1 k(x, y)f(y) dy = 1, \quad -1 < x < 1$$

whose kernel $k \in K$ satisfies (4) and whose solution $f \in L^p[-1, 1]$, $1 < p < 2$, exists. Theorem 2 immediately yields

$$f = \frac{a(x)}{(1-x^2)^{1/2}}, \quad \text{a.e.},$$

where $a(x)$ is analytic for complex $x \in D[-1, 1]$. To show that $f > 0$ a.e., it suffices to take

$$(20) \quad f = \frac{a(x)}{(1-x^2)^{1/2}},$$

and then show that $f > 0$.

First we shall prove by contradiction that f is non-negative. Therefore assume that there exists a point x , $-1 \leq x \leq 1$, such that $f(x) < 0$, and define,

$$E_p = \overline{\{x : f(x) > 0\}}$$

$$E_n = \overline{\{x : f(x) < 0\}}$$

$$u^p(x) = \int_{E_p} k(x, y)f(y) dy$$

$$u^n(x) = \int_{E_n} k(x, y)f(y) dy.$$

From the analyticity properties of $a(x)$ in the representation of f given by (20) we conclude that f has only a finite number of zeroes and that if f is different from zero at a point, it must be different from zero on an interval. Furthermore we observe that f must be strictly positive on some interval in order that (19) have a solution, because by (4) the kernel k is strictly positive. Consequently the sets E_p and E_n are nonempty and in particular consist of a finite number of closed line segments.

With the definitions above we express integral equation (19) as

$$(21) \quad u^p(x) + u^n(x) = 1, \quad -1 \leq x \leq 1,$$

where the domain of validity of (19) has been extended by continuity to include the endpoints $x = \pm 1$. We now show that u^p which is a continuous function on the interval $-1 \leq x \leq 1$ has no maximum on this interval. From the definition

of E_p and (4) we note that

$$(22) \quad u_{xx}^p(x) = \int_{E_p} k_{xx}(x, y) f(y) dy > 0, \quad x \notin E_p,$$

and likewise that $u_{xx}^n(x) < 0$, $x \notin E_n$ which imply that $\max_{-1 \leq x \leq 1} u^p(x)$ occurs for $x \in E_p \cup \{x: x = \pm 1\}$ and that $\min_{-1 \leq x \leq 1} u^n(x)$ occurs for $x \in E_n \cup \{x: x = \pm 1\}$. From (21) we observe that

$$(23) \quad \begin{aligned} \max_{-1 \leq x \leq 1} u^p(x) &= \max_{-1 \leq x \leq 1} (1 - u^n(x)) \\ &= 1 - \min_{-1 \leq x \leq 1} u^n(x). \end{aligned}$$

Therefore $\max_{-1 \leq x \leq 1} u^p(x)$ occurs for $x \in E_p \cap E_n \cup \{x: x = \pm 1\}$. To establish that $u_x^p(x)$ exists in a neighborhood of $x_1 \in E_p \cap E_n$ we introduce

Lemma 2. *Let $u(x) = \int_{E_0} k(x, y) f(y) dy$ where $k \in K$, f is given by (20) and $E_0 \subseteq \{x: -1 \leq x \leq 1\}$ consists of a finite number of line segments. If $f(x_1) = 0$, where $x_1 \neq \pm 1$ is a limit point of E_0 , then $u_x(x)$ exists on $|x - x_1| < \delta$ for some $\delta > 0$. Moreover, if $f_x(x_1) = 0$ as well, then $u_{xx}(x)$ exists on $|x - x_1| < \delta$.*

Proof. For x_1 , a limit point of E_0 , we choose $\delta > 0$ so small such that $\{x: 0 < x - x_1 \leq \delta\}$ or $\{x: 0 < x_1 - x \leq \delta\} \subset E_0$ and such that $E_\delta = \{x: |x - x_1| \leq \delta\} \subset \{x: -1 \leq x \leq 1\}$. For x in E_δ but not in E_0 we define $f(x) = 0$, and note that with this definition f is continuous for $x \in E_\delta$ and f_x may have a discontinuity at $x = x_1$. We write $u(x)$ as

$$(24) \quad \begin{aligned} u(x) &= \int_{E_0 - E_\delta} k(x, y) f(y) dy \\ &\quad + \int_{E_\delta} [(B(x, y) - B(x, x_1)) \text{Log } |x - y| + C(x, y)] f(y) dy \\ &\quad + B(x, x_1) \int_{E_\delta} \text{Log } |x - y| f(y) dy, \end{aligned}$$

where $k(x, y) = B(x, y) \text{Log } |x - y| + C(x, y)$. We observe that for $|x - x_1| < \delta$, $k_x(x, y)$ is continuous for $y \in E_0 - E_\delta$ and that

$$(25) \quad \begin{aligned} \frac{d}{dx} \int_{E_\delta} \text{Log } |x - y| f(y) dy \\ &= \frac{d}{dx} \left\{ (y - x) [\text{Log } |x - y| - 1] f(y) \right\} \Big|_{y=x_1-\delta}^{y=x_1+\delta} \\ &\quad + \int_{E_\delta} (x - y) [\text{Log } |x - y| - 1] f_y(y) dy \\ &= \text{Log } |x - y| f(y) \Big|_{y=x_1+\delta}^{y=x_1-\delta} + \int_{E_\delta} \text{Log } |x - y| f_y(y) dy \Big\}. \end{aligned}$$

Therefore $u_x(x)$ exists on $|x - x_1| < \delta$ and

$$(26) \quad u_x(x) = \int_{E_0 - E_\delta} k_x(x, y) f(y) dy \\ + \int_{E_\delta} \left[B_x(x, y) \operatorname{Log} |x - y| + \frac{B(x, y) - B(x, x)}{x - y} + C_x(x, y) \right] f(y) dy \\ + B(x, x) \left\{ \int_{E_\delta} \operatorname{Log} |x - y| f_y(y) dy + \operatorname{Log} |x - y| f(y) \right\}_{y=x_1+\delta}^{y=x_1-\delta}.$$

For the second part of the lemma, we note that for $|x - x_1| < \delta$ and $f_x(x_1) = 0$, each of the three integrals in (26) is a differentiable function of x ; the first because the kernel is continuously differentiable, the second because the kernel belongs to K while the conclusion of the first part of the lemma applies, and the third because $f_x(x)$ is continuous for $x \in E_\delta$ and an analysis similar to (25) applies. Clearly the remaining term is differentiable on this interval. Hence $u_{xx}(x)$ exists on $|x - x_1| < \delta$. *q.e.d.*

For $x_1 \in E_p \cap E_n$, $f(x_1) = 0$ and Lemma 2 yields that $u_x^p(x)$ exists in a neighborhood of x_1 . This fact together with (22) is sufficient to insure that $\max_{-1 \leq x \leq 1} u^p(x)$ does not occur for $x \in E_p \cap E_n$. Then it must occur for $x = \pm 1$. However, suppose $u^p(1)$ is the maximum of $u^p(x)$ on $-1 \leq x \leq 1$, then $u_x^p(1) \geq 0$. From earlier remarks about the sets E_p and E_n , the point $x = 1 \notin E_p \cap E_n$. If $x = 1 \notin E_p$, then from (4) $u_x^p(1) = \int_{E_p} k_x(1, y) f(y) dy < 0$, which implies that $x = 1 \in E_p$ or equivalently $x = 1 \notin E_n$. But for $x = 1 \notin E_n$, $u_x^n(1) > 0$ which by differentiation of (21) gives $u_x^p(1) < 0$. Therefore $u^p(1)$ is not a maximum. Thus $u^p(-1)$ must be the $\max_{-1 \leq x \leq 1} u^p(x)$, but a similar argument rules out this possibility. Hence we have shown that $u^p(x)$ a continuous function on $-1 \leq x \leq 1$ has no maximum on this interval, thereby effecting a contradiction. Consequently the assumption that there exists an x , $-1 \leq x \leq 1$, such that $f(x) < 0$ has led to a contradiction and must therefore be false. We have now shown that

$$(27) \quad f(x) \geq 0, \quad -1 \leq x \leq 1.$$

At this point we remark that the arguments which led to the contradiction above are insensitive to the value of the constant which appears as the inhomogeneous term in (19), except for the initial assumption. Indeed if instead of (19) we considered the integral equation,

$$(28) \quad \int_{-1}^1 k(x, y) f(y) dy = 0, \quad -1 < x < 1,$$

and assumed that it had a solution different from zero we would have arrived at the same contradiction. Consequently the only solution of (28) is $f \equiv 0$.

To show that $f > 0$, we first assume that there exists an x_1 , $-1 < x_1 < 1$, such that $f(x_1) = 0$, which implies by (27) that $f_x(x_1) = 0$. Define:

$$u^1(x) = \int_{-1}^{x_1} k(x, y) f(y) dy,$$

$$u^2(x) = \int_{x_1}^1 k(x, y)f(y) dy.$$

Then integral equation (19) can be expressed as

$$(29) \quad u^1(x) + u^2(x) = 1, \quad -1 \leq x \leq 1.$$

From (27) and (4) we observe that

$$(30) \quad u_{xx}^1(x) = \int_{-1}^{x_1} k_{xx}(x, y)f(y) dy > 0, \quad x_1 < x \leq 1$$

and likewise that $u_{xx}^2(x) > 0$, $-1 \leq x < x_1$. Then differentiation of (29) yields

$$(31) \quad u_{xx}^1(x) < 0, \quad -1 \leq x < x_1.$$

From Lemma 2 we conclude that $u_{xx}^1(x)$ exists in a neighborhood of x_1 and therefore, by (30) and (31) we have that $u_{xx}^1(x_1) = 0$. However from (30) we note that

$$u_{xx}^1(x) \geq c^2 \int_{-1}^{x_1} f(y) dy \geq 0, \quad x_1 < x \leq 1,$$

where $c^2 = \min_{-1 \leq x, y \leq 1} k_{xx}(x, y)$ is a non zero, positive constant for $k \in K$ satisfying (4). In order that $u_{xx}^1(x_1) = 0$, this inequality implies that $\int_{-1}^{x_1} f(y) dy = 0$. However f is non negative and we have that $f(x) \equiv 0$, $-1 \leq x \leq x_1$ which is clearly impossible by the representation of f given in (20). Consequently, this contradiction gives

$$f(x) > 0, \quad -1 < x < 1.$$

Finally we show that $f(\pm 1) > 0$. Assume that $f(1) = 0$. This implies by representation (20) for f that as x approaches 1, $f(x)$ approaches zero at least as fast as $(1 - x)^{1/2}$ and that $f_x(x)$ has at worst a square root singularity at $x = 1$. For $k \in K$, let $k = B(x, y) \text{Log } |x - y| + C(x, y)$. Then integral equation (19) can be expressed as

$$\begin{aligned} & \int_{-1}^{1-\epsilon} k(x, y)f(y) dy \\ &= 1 - \int_{1-\epsilon}^1 \{ [B(x, y) - B(x, x)] \text{Log } |x - y| + C(x, y) \} f(y) dy \\ & \quad - B(x, x) \int_{1-\epsilon}^1 \text{Log } |x - y| f(y) dy, \quad -1 < x < 1, \end{aligned}$$

where $0 < \epsilon < 2$. Restrict x to the interval $1 - \epsilon < x < 1$; then differentiate this equation with respect to x ; using (25) to evaluate the derivative of the last integral; and take the limit as x approaches 1 to achieve

$$\begin{aligned} & \int_{-1}^{1-\epsilon} k_x(1, y)f(y) dy = - \int_{1-\epsilon}^1 \left\{ \frac{B(1, y) - B(1, 1)}{1 - y} + B_x(1, y) \text{Log } |1 - y| \right. \\ & \quad \left. + C_x(1, y) \right\} f(y) dy - B(1, 1) \left\{ \int_{1-\epsilon}^1 \text{Log } |1 - y| f_y(y) dy + \text{Log } \epsilon f(1 - \epsilon) \right\}. \end{aligned}$$

For ϵ sufficiently small the right hand side of this equation is $O(\epsilon^{1/2} \text{Log } \epsilon)$ while the left hand side can be bounded away from zero by (4) and the fact that $f(x) > 0$, $-1 < x < 1$. Thus

$$-c^2 \int_{-1}^1 f(x) dx \geq \int_{-1}^{1-\epsilon} k_x(1, y)f(y) dy = O(\epsilon^{1/2} \text{Log } \epsilon),$$

where $c^2 = -\text{Max}_{-1 \leq x, y \leq 1} k_x(x, y)$ is a non zero, positive constant by (4). The left hand side of this equation is in contradiction with the right hand side for all ϵ sufficiently small. Therefore, the assumption that $f(1) = 0$ was false, and thus $f(1) > 0$. Similarly we can show that $f(-1) > 0$. Hence, $f(x) > 0$, $-1 \leq x \leq 1$ and we have proved

Theorem 3. *Let $k \in K$ satisfy (4) and let $f \in L^p[-1, 1]$, for some p , $1 < p < 2$, be a solution to (19), then $f > 0$ a.e.*

Theorem 4. *For $k \in K$ satisfying (4) the only solution $f \in L^p[-1, 1]$, $1 < p < 2$, to (28) is $f = 0$ a.e.*

Our results can readily be extended to include integral equations of the form

$$(32) \quad \int_E k(x, y)f(y) dy = 1, \quad x \in E \subseteq \{x: -1 \leq x \leq 1\}$$

where $k \in K$ satisfies (4), $E = \bigcup_{i=1}^n E_i$, $E_i \equiv \{x: \alpha_i < x < \beta_i\}$, and $n < \infty$. We assume (32) has a solution $f \in L^p[\bar{E}]$ for some p , $1 < p < 2$. If we let $g_i(x) = 1 - \int_{E-E_i} k(x, y)f(y) dy$, then the integral equation

$$\int_{E_i} k(x, y)f(y) dy = g_i(x), \quad x \in E_i$$

has a solution $f \in L^p[\alpha_i, \beta_i]$. Furthermore if we require

$$(33) \quad \bigcup_{i=1}^n \{x: \alpha_i > x \geq 2\alpha_i - \beta_i\} \bigcup_{i=1}^n \{x: 2\beta_i - \alpha_i \geq x > \beta_i\} \cap \bar{E} = 0,$$

then $g_i(x)$ is analytic for complex $x \in \bar{D}[\alpha_i, \beta_i]$ and Theorem 2 yields

$$f = \frac{a_i(x)}{((\beta_i - x)(x - \alpha_i))^{1/2}} \quad \text{a.e.,} \quad \alpha_i < x < \beta_i, \quad i = 1, 2, \dots, n$$

where $a_i(x)$ is analytic for complex $x \in D[\alpha_i, \beta_i]$. The proofs of Theorems 3 and 4 can now be followed making minor changes where necessary (e.g. in (21) replacing $-1 \leq x \leq 1$ by $x \in \bar{E}$) to derive their analogies which we state as

Theorem 5. *Let $k \in K$ and E satisfy (4) and (33) respectively, and let $f \in L^p[\bar{E}]$ for some p , $1 < p < 2$, be a solution to (32), then $f > 0$ a.e.*

Theorem 6. *For $k \in K$ and E satisfying (4) and (33) respectively, the only solution $f \in L^p[\bar{E}]$, $1 < p < 2$ to*

$$\int_E k(x, y)f(y) dy = 0, \quad x \in E$$

is $f = 0$, a.e.

3. Application of Results. A classic problem in viscous hydrodynamics is flow past a flat plate, of length l , oriented parallel to the oncoming stream. This flow has been studied within the framework of the linearized, time-independent hydrodynamical equations, suggested by Oseen [2],

$$(34) \quad \nu \Delta \mathbf{w} - U \frac{\partial}{\partial x_1} \mathbf{w} - \frac{1}{\rho} \nabla p = 0, \quad \nabla \cdot \mathbf{w} = 0,$$

together with the boundary conditions,

$$(35) \quad \begin{aligned} \mathbf{w}(x) &= 0, \quad -\frac{l}{2} < x_1 < \frac{l}{2}, \quad x_2 = 0, \\ \mathbf{w}(x) &\rightarrow U \mathbf{i}_1, \quad |x| \rightarrow \infty, \end{aligned}$$

where U is the magnitude of the freestream velocity, ν the kinematic viscosity and ρ , the density. To insure uniqueness, this flow is required to be symmetric about the plane $x_2 = 0$. It has been shown [3] that this problem is equivalent to solving the integral equation,

$$(36) \quad \int_{-1}^1 \left[\frac{1}{2\pi} \int_0^{\text{Re}/4} e^{\beta(r-s)} K_0(\beta|r-s|) d\beta \right] \sigma(s) ds = 1, \quad -1 < r < 1,$$

where $\rho U^2 \sigma(2x_1/l) = \rho U l / 2 \partial w_1 / \partial x_2(x_1, 0)$ can be interpreted as the pointwise force distribution acting on the plate in the direction $\mathbf{i}_1$ (local drag), $\text{Re} = Ul/\nu$ is the Reynolds number, and K_0 is the zeroeth order modified Bessel function of the second kind. Once the solution σ is obtained, the velocity field $\mathbf{w}$ and pressure field p are known within a quadrature (see [2] or [4] for example).

Within the framework of the Oseen equation (34) Olmstead and Gautesen [5] have recently shown that those solutions which correspond to flows that adhere to the exterior of an obstacle and tend to a uniform flow at infinity, have the unexpected property that the magnitude of the drag acting on the obstacle is left unchanged when the direction of uniform flow is reversed. We emphasize that this result is independent of any obstacle symmetry. As a consequence of the force and aft symmetry of the flat plate, the flow we consider does not suffer from this paradox. Therefore conclusions drawn from this mathematical model may be in good agreement with physical experience. However, a better linearization constant than U may be required (Lewis and Carrier [6] have suggested the value $\frac{1}{3}U$). Physical experience would suggest that the flow does not separate from the plate. Hence the local drag $\rho U^2 \sigma$ should be positive in order to insure that no separation will occur. Numerical investigations [7-10], indicate this for various values of the Reynolds number. We rigorously verify the positivity of the local drag for all values of the Reynolds number. In addition, our results are applied to Oseen flow past a series of coplanar flat plates.

To show that our results are applicable to Oseen flow past a flat plate, we must verify that the difference kernel $h(r) = 1/2\pi \int_0^{\text{Re}/4} e^{\beta r} K_0(\beta|r|) d\beta$ of (36) belongs to K and satisfies (4). For the first part we observe that $h(r)$ can be

expressed as

$$h(r) = B_1(r) \operatorname{Log} |r| + C_1(r),$$

where

$$B_1(r) = -\frac{1}{2\pi} \int_0^{\operatorname{Re}/4} e^{\beta r} I_0(\beta r) d\beta, \quad C_1(r) = -\frac{1}{2\pi} \int_0^{\operatorname{Re}/4} e^{\beta r} \cdot \left[\left(\operatorname{Log} \left| \frac{\beta}{2} \right| + \gamma \right) I_0(\beta r) - \sum_{n=0}^{\infty} \frac{\psi(n+1) + \gamma}{(n!)^2} \left(\frac{\beta r}{2} \right)^{2n} \right] d\beta,$$

I_0 is the zeroeth order modified Bessel function of the first kind, $\gamma = 0.577\dots$ is Euler's constant and $\psi(n)$ is the Psi function. Clearly $h(r - s) \in K$ for all finite values of the Reynolds number Re . For the second part we write h as

$$h(r) = \frac{1}{2\pi} \int_0^{\operatorname{Re}/4} d\beta \int_0^{\infty} dt e^{\beta(1 - \operatorname{sgn}(r) \cosh t)r}$$

and conclude that $h(r - s)$ satisfies (4) for all finite values of Re . Then Theorem 3 applies and we have that the local drag $\rho U^2 \sigma > 0$ a.e.

If we consider instead, the problem of Oseen flow past a series of coplanar flat plates the governing equations (34) and the boundary condition at infinity remain the same. However the adherence boundary condition in (35) becomes

$$w(x) = 0, \quad x_1 \in \sum, \quad x_2 = 0$$

where $\sum = \bigcup_{i=1}^n \{x_1 : 1/2\alpha_i l < x_1 < 1/2\beta_i l\}$ and $\beta_n = -\alpha_1 = 1$. This problem is equivalent to solving the integral equation,

$$\int_E h(r - s) \delta(s) ds = 1, \quad r \in E,$$

where $E = \bigcup_{i=1}^n \{r : \alpha_i < r < \beta_i\}$ and $\rho U^2 \delta(2x_1/l)$ can be interpreted as the local drag acting on $\sum$. If we require that the plates be spaced sufficiently far apart such that E satisfies (33), then Theorem 5 yields that the local drag, $\rho U^2 \delta > 0$ a.e.

We remark that the solutions of these problems which have physical meaning are

$$\sigma(r) = \frac{a(r)}{(1 - r^2)^{1/2}} > 0, \quad -1 \leq r \leq 1,$$

$$\delta(r) = \frac{a_i(r)}{((\beta_i - r)(r - \alpha_i))^{1/2}} > 0, \quad \alpha_i \leq r \leq \beta_i, \quad i = 1, 2, \dots, n,$$

and that there exists a square root singularity at each of the end points of each plate.

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