*Local Analytic Structure in Banach Algebras**

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Let A be a commutative Banach algebra with identity, and X the maximal ideal space of A . In recent years a considerable amount of research has gone into the following question: If $x \in X$, when is it true that A behaves like an algebra of analytic functions “near” x ?

Gleason parts [3] were introduced with the hope that they might be subsets of X possessing analytic structure in some sense. For certain special algebras such is the case—in particular, Dirichlet and logmodular algebras (cf. [8] and [6]). However, several examples—notably the devastating example of Garnett [2]—show that parts do not in general have anything approaching analytic structure.

It seems to us that a more promising approach was given by Gleason in [4], where he shows that if a maximal ideal x of A is algebraically finitely generated then a neighborhood of $x \in X$ is an analytic variety. However, Gleason points out that finitely generated maximal ideals are rare, and that this condition is certainly too strong. Part of the problem is that this is a global condition (a condition on A); whereas, what we need is a local condition on the behavior of A near x .

We propose to approach the question of analytic structure through the use of Aren’s sheaf of germs of locally A -functions [1]. The stalk S_x of this sheaf at $x \in X$ is the algebra of germs at x of functions which are representable in some neighborhood of x as convergent power series in finitely many elements of A . The algebra S_x is always a local algebra. It is not unreasonable to hope that the algebraic structure of S_x might in general determine whether or not there is analytic structure in X near x . For instance, if U is a domain in $\mathbb{C}^n$ and A is the algebra of functions analytic (holomorphic) in U and continuous on $\bar{U}$; then $S_x = \mathcal{O}_x$, the ring of germs of holomorphic functions at x , provided $x \in U$.

A direct analogue of Gleason’s result in [4] would be: If the maximal ideal of S_x is finitely generated, then some neighborhood of x is an analytic variety. We conjecture that this is true if $\{x\}$ is a G_δ (if x is not a G_δ , then there is a simple counter-example which we present in §4). We have not been able to prove

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this as yet. It may be necessary to assume that S_x is Noetherian. Unfortunately, we do not yet have a proof of even this. We shall, however, prove the following: (Theorem 4.1) If S_x is a factor ring of ${}_nO_0$ for some n (i.e. S_x is an "analytic ring" in the sense of [7]) and if $\{x\}$ is a G_δ , then some neighborhood of x is an analytic variety. This is the main (only) theorem of this paper. It shows that the analytic structure in X and the algebraic structure of S_x are closely related and suggests that simple conditions on A which force S_x to be an analytic ring are worth seeking.

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1.1. Henceforth A will be a commutative Banach algebra with identity, X the maximal ideal space of A , $x \in X$ a fixed maximal ideal, and $\hat{a}$ the Gelfand transform of $a \in A$. The stalk at zero of the sheaf of germs of functions holomorphic on $\mathcal{C}^n$ will be denoted by ${}_nO_0$; and the stalk at zero of the sheaf of germs of functions holomorphic on V by ${}_VO_0$, where we assume zero is in V and V is a subvariety of a neighborhood of zero in $\mathcal{C}^n$ for some n .

For information concerning these rings—and holomorphic functions in general—the reader is referred to the text by Gunning and Rossi [5].

Let Y be a topological space and y a point of Y . If f is a continuous complex-valued function defined in a neighborhood of y , then $\gamma_y(f)$ will denote the germ of f at y . This is defined to be the equivalence class of continuous functions identically equal to f in *some* neighborhood of y —the neighborhood depending on the function.

The stalk at x of the sheaf of germs of locally A -functions will be denoted by S_x . We include a construction of S_x which differs from that given by Arens [1], but is similar in spirit and can easily be shown equivalent.

1.2. Let $\mathfrak{A}$ be the set of all finite subsets of A with the property: if $\alpha \in \mathfrak{A}$ and $\alpha = (a_1, \dots, a_i)$ then $\hat{a}_i(x) = 0$ for each i . Define $\hat{\alpha}: X \rightarrow \mathcal{C}^i$ by $\hat{\alpha}(y) = (\hat{a}_1(y), \dots, \hat{a}_i(y))$ for all $y \in X$. Then for any function F holomorphic in a neighborhood of zero in $\mathcal{C}^i$, $F \circ \hat{\alpha}$ is a function continuous on some neighborhood of x . S_x is the algebra of germs of all functions that can be obtained in this manner.

Hence, if f is continuous on some neighborhood of x , then $\gamma_x(f) \in S_x$ if and only if there is an $\alpha \in \mathfrak{A}$ and an F holomorphic on a neighborhood of zero in $\mathcal{C}^i$ (j depending on α) such that $\gamma_x(f) = \gamma_x(F \circ \hat{\alpha})$.

Similarly, ${}_nO_0 = \{\gamma_0(F) : F \text{ is holomorphic on some neighborhood of } 0 \in \mathcal{C}^n\}$, and ${}_VO_0 = \{\gamma_0(F) : F \text{ is holomorphic on } V \text{ in some neighborhood of } 0 \in V\}$.

Each of these is a local algebra where, for example, the maximal ideal m of S_x is $\{\gamma_x(f) : f(x) = 0\}$ and the maximal ideal M of ${}_nO_0$ is $\{\gamma_0(F) : F(0) = 0\}$.

2.1. From here on we assume that there exists a fixed algebraic homomorphism η from ${}_nO_0$ onto S_x .

2.2. *Definition.* If $\gamma_x(f_i) \in M$ (the maximal ideal of S_x) for $1 \leq i \leq n$ and if $f = (f_1, \dots, f_n)$, then we define the homomorphism $f^* : {}_nO_0 \rightarrow S_x$ by $f^*\gamma_0(F) = \gamma_x(F \circ f)$ where $f(y) = (f_1(y), \dots, f_n(y))$.

The objective is to show that there is a tuple $f = (f_1, \dots, f_n)$ of functions defined on a neighborhood of x , such that $f^* = \eta$.

2.3. *Lemma.* $\bigcap_{k=1}^{\infty} M^k = \{0\}$.

Proof. If $I = \text{kernel of } \eta$, then $I = \text{radical of } I$. To prove this suppose $\gamma_0(F)^k \notin I$. By our assumption 2.1 we can find an f such that $\eta\gamma_0(F) = \gamma_x(f)$. Then

$$\gamma_x(f^k) = \gamma_x(f)^k = [\eta\gamma_0(F)]^k = \eta[\gamma_0(F)^k] = 0.$$

Hence $f^k = 0$ in a neighborhood of x ; therefore, $f = 0$ in a neighborhood of x , or equivalently $\gamma_x(f) = 0$. Consequently, $\eta\gamma_0(F) = 0$ and $\gamma_0(F) \in I$.

By the Nullstellensatz [5, 90], there is a subvariety V of some neighborhood of zero in $\mathcal{C}^m$ such that $I = \{\gamma_0(F) \in {}_nO_0 : F \equiv 0 \text{ on } U \cap V \text{ for some neighborhood } U \text{ of zero}\}$.

In particular S_x is isomorphic to ${}_vO_0$. Since ${}_vO_0$ is a Noetherian local ring, the lemma follows from Krull's Theorem [9, 216].

2.4. *Proposition.* $\eta = f^*$ for some f .

Proof. Let $(z_1, \dots, z_n)$ be a coordinate system for $\mathcal{C}^m$. Choose $f_1, \dots, f_n$ continuous on some common neighborhood of x , such that $\eta\gamma_0(z_i) = \gamma_x(f_i)$ for $1 \leq i \leq n$. Let $f = (f_1, \dots, f_n)$ —"f" will always denote this fixed function.

i) We must first show that f satisfies the conditions of Definition 2.2. Since S_x is a factor space of ${}_nO_0$, and ${}_nO_0$ is a local algebra, it follows that $\eta M = m$. And in particular $\gamma_x(f_i) = \eta\gamma_0(z_i) \in m$ for each i . Hence we can define $f^*: {}_nO_0 \rightarrow S_x$ according to Definition 2.2.

ii) We now show that $(\eta - f^*)$ annihilates the polynomials. Let P be a polynomial in n -variables, then $\eta\gamma_0 P(z_1, \dots, z_n) = P(\eta\gamma_0 z_1, \dots, \eta\gamma_0 z_n) = P(\gamma_x f_1, \dots, \gamma_x f_n) = \gamma_x[P(z_1, \dots, z_n) \circ f] = f^*\gamma_0 P(z_1, \dots, z_n)$. Thus the assertion follows.

iii) $\eta = f^*$.

Let $\gamma_0(F) \in {}_nO_0$. If we consider the power series expansion of F at zero, it follows that we may write $F = \sum P_i$ where the P_i 's are homogeneous polynomials of degree i . Let $Q_i = \sum_{j=i}^{\infty} P_j$, then $F = P_0 + P_1 + \dots + P_{i-1} + Q_i$.

From ii) we have $(\eta - f^*)\gamma_0(F) = (\eta - f^*)\gamma_0(Q_i)$ for all j .

Since η and f^* are homomorphisms $(\eta - f^*)M^j \subseteq m^j$. Hence, $(\eta - f^*)\gamma_0(Q_i)$ is an element of m^j for every j since $\gamma_0(Q_i)$ is in M^j for every j . By lemma 2.3: $(\eta - f^*)\gamma_0(F) = 0$.

3.1. *Proposition.* If $\{x\}$ is a G_δ and φ is a mapping from a neighborhood of x into a topological space Y such that:

For each $a \in A$ there exists a function F defined in a neighborhood of $\varphi(x)$ such that $\gamma_x(\hat{a}) = \gamma_x(F \circ \varphi)$.

Then φ is a monomorphism on some neighborhood of x .

Proof. Since X is a compact Hausdorff space it follows that $\{x\}$ is a G_δ if and only if $\{x\}$ has a countable descending neighborhood base $\{U_i\}$. We may assume that each U_i is contained in the domain of φ . We shall prove the proposition by contradiction; hence, suppose there is a sequence of pairs $\{(x_i, y_i)\}$ of points of X such that for all j :

- 1) x_i and y_i are elements of U_i
- 2) $x_i \neq y_i$ but $\varphi(x_i) = \varphi(y_i)$.

By hypothesis we can find, for each $a \in A$, an F such that $\gamma_x(\hat{a}) = \gamma_x(F \circ \varphi)$. Hence there is a j such that $\hat{a} = F \circ \varphi$ on U_j . This implies:

- (*): For each $a \in A$ there is a j such that $\hat{a}(x_k) = \hat{a}(y_k)$ for all $k > j$.
 We shall construct an element of A that does not satisfy (*) for any j .
 By using (*) we can find a sequence $\{a_i\}$ of elements of A such that:

- 3) $\|a_i\| = 1$
- 4) $\hat{a}_i(x_i) \neq \hat{a}_i(y_i)$
- 5) $\hat{a}_i(x_k) = \hat{a}_i(y_k)$ for all $k > j$.

(Note: we can certainly do this for some subsequence of $\{(x_i, y_i)\}$ and that is clearly sufficient.)

Let $c_1 = a_1$ and $c_{i+1} = (a_{i+1}/6)[\hat{c}_i(x_i) - \hat{c}_i(y_i)]$ for $j \geq 1$.

It is trivial to prove by induction that $\hat{c}_i(x_i) \neq \hat{c}_i(y_i)$.

Let $c = \sum c_i$; we assert that c does not satisfy (*). To prove this we use the following estimate:

$$6) \quad \|c_{i+k}\| \leq \frac{1}{2 \cdot 3^k} |\hat{c}_i(x_i) - \hat{c}_i(y_i)| \quad (j, k = 1, 2, \dots).$$

We prove 6) by induction on k . For $k = 1$ we have the obvious estimate from the equation defining c_{i+1} . Assuming 6) is true for $m = k$ we have:

$$\|c_{i+m+1}\| \leq \frac{1}{6} |\hat{c}_{i+m}(x_{i+m}) - \hat{c}_{i+m}(y_{i+m})| \leq \frac{1}{3} \|c_{i+m}\| \leq \frac{1}{2 \cdot 3^{m+1}} |\hat{c}_i(x_i) - \hat{c}_i(y_i)|.$$

From 6) we have that

$$\|c_{1+k}\| \leq \frac{1}{2 \cdot 3^k} |\hat{c}_1(x_1) - \hat{c}_1(y_1)| \leq \frac{\|c_1\|}{3^k} = \frac{1}{3^k}.$$

Therefore $\sum c_i$ is absolutely summable, and c is a well-defined element of A . Finally, we notice from the defining equation for c_{i+1} and 5) that

$$\hat{c}(x_i) - \hat{c}(y_i) = \sum_{k=1}^{\infty} (\hat{c}_k(x_i) - \hat{c}_k(y_i)) = \sum_{k=i}^{\infty} (\hat{c}_k(x_i) - \hat{c}_k(y_i)).$$

We obtain the contradiction from the following computation showing that c does not satisfy (*).

$$\begin{aligned}
 |\hat{c}(x_i) - \hat{c}(y_i)| &\geq |\hat{c}_i(x_i) - \hat{c}_i(y_i)| - \sum_{k=j+1}^{\infty} |\hat{c}_k(x_i) - \hat{c}_k(y_i)| \\
 &\geq |\hat{c}_i(x_i) - \hat{c}_i(y_i)| - 2 \sum_{k=j+1}^{\infty} \|c_k\| \\
 &\geq |\hat{c}_i(x_i) - \hat{c}_i(y_i)| - \sum_{k=1}^{\infty} \frac{1}{3^k} |\hat{c}_i(x_i) - \hat{c}_i(y_i)| \\
 &= |\hat{c}_i(x_i) - \hat{c}_i(y_i)| \left(1 - \sum_{k=1}^{\infty} \frac{1}{3^k}\right) \\
 &= \frac{1}{2} |\hat{c}_i(x_i) - \hat{c}_i(y_i)| > 0.
 \end{aligned}$$

3.2. Corollary. If $\{x\}$ is a G_δ , then f is a monomorphism on some neighborhood of x .

Proof. Take $\varphi = f$ and $Y = \mathcal{C}^n$. Then since $\eta = f^*$ and η is an epimorphism, it follows that for any $a \in A$ there is an F such that $\gamma_x(\hat{a}) = \eta\gamma_0(F) = f^*\gamma_0(F) = \gamma_x(F \circ f)$.

4.1. Theorem. Let $\eta: {}_nO_0 \rightarrow S_x$ be an algebraic epimorphism, and suppose that $\{x\}$ is a G_δ . Then there is a homeomorphism f from an open neighborhood U of x onto a subvariety V of $\mathcal{C}^n$ such that $\eta = f^*$ and $\hat{a} \circ f^{-1}$ is holomorphic on V for every $a \in A$.

Proof. We have already constructed f . The only thing left to show is that f is a homeomorphism onto some subvariety of $\mathcal{C}^n$.

Choose a compact neighborhood W of x such that f is 1-1 on W (Corollary 3.2). Then f is a homeomorphism of W onto $W_f = \{f(y) : y \in W\}$. For each $a \in A$, $\hat{a} \circ f^{-1}$ is a continuous function on W_f .

Let $D_i = \{z \in \mathcal{C}^n : |z_i| < 1/j, z = (z_1, \dots, z_n), 1 \leq i \leq n\}$. Then in terms of functions on W_f the statement that $f^*(= \eta) : {}_nO_0 \rightarrow S_x$ is onto, implies that given an $a \in A$ there is an F holomorphic on D_i (some j) such that $F = \hat{a} \circ f^{-1}$ on $D_i \cap W_f$ — i.e., F extends $\hat{a} \circ f^{-1}$ to D_i .

Let A_i^k be the set of all $a \in A$ with the property that there is an F holomorphic on D_i extending $\hat{a} \circ f^{-1}$ to D_i , and such that $\sup_{z \in D_i} |F(z)| \leq k\|a\|$.

Obviously $A = \cup A_i^k$ where $(j, k = 1, 2, \dots)$. Furthermore each A_i^k is closed in A . For suppose $\{a_i\}$ is a sequence from A_i^k converging in norm to a . Let $\{F_i\}$ be a corresponding sequence of extensions. Since the family $\{F_i\}$ is a bounded family of holomorphic functions on D_i it is a normal family [5, 11]. Without loss of generality we can assume that $\{F_i\}$ converges pointwise on D_i to a function F holomorphic on D_i . Clearly, $\sup_{z \in D_i} |F(z)| \leq k\|a\|$. Therefore we need to show that F extends $\hat{a} \circ f^{-1}$. Since $\{a_i\}$ converges in norm to a , then $\{a_i\}$ converges uniformly to $\hat{a}$ on X . Hence $\{a_i \circ f^{-1}\}$ converges uniformly to $\hat{a} \circ f^{-1}$ on

W_f , forcing $\{F_i\}$ to converge uniformly on $W_f \cap D_i$ to $\hat{a} \circ f^{-1}$. Therefore F must indeed extend $\hat{a} \circ f^{-1}$ to D_i .

By the Baire category theorem (and a very mundane argument), we conclude that $A = A_{j_0}^M$ for some M and j_0 .

Let $V_0 = \{z \in D_{j_0} : F(z) = 0 \text{ for all } F \text{ holomorphic on } D_{j_0} \text{ and zero on } W_f \cap D_{j_0}\}$. By construction: for any a in A there is a *unique* holomorphic extension F_a of $\hat{a} \circ f^{-1}$ from $W_f \cap D_{j_0}$ to V_0 . Moreover, the mapping $a \rightarrow F_a$ is obviously a continuous homomorphism (with norm at most M) of A into the bounded holomorphic functions on V_0 . Hence there is a continuous mapping $\varphi: V_0 \rightarrow X$ such that $F_a = \hat{a} \circ \varphi$ on V_0 .

Choose $G_1, \dots, G_k$ holomorphic in a neighborhood of zero such that $\{\gamma_0(G_i)\}$ generates the kernel of f^* (see Section 2.3). Choose $j_1 \geq j_0$ such that: each G_i is holomorphic on D_{j_1} , $V_0 \cap D_{j_1} \subseteq \varphi^{-1}(W)$, and $f^{-1}(D_{j_1}) \subseteq W$.

Since the composition of holomorphic functions is holomorphic; $f_i \circ \varphi$ is holomorphic on $V_0 \cap D_{j_1}$ for each i , where $(f_1, \dots, f_n) = f$. On W_f we also have that $\hat{a} \circ f^{-1} = F_a$. This implies that $f^{-1} = \varphi$ on W_f . In particular, $f \circ \varphi$ is the identity function on $V_0 \cap W_f$.

Since $f^*\gamma_0(G_i) = 0$, there is a neighborhood W_1 of x such that $G_i \circ f = 0$ on W_1 for each i . Choose $j_2 \geq j_1$ such that $V_0 \cap D_{j_2} \subseteq \varphi^{-1}(W_1)$.

Let $V = \{z \in D_{j_2} : F(z) = 0 \text{ for all } F \text{ holomorphic on } D_{j_2} \text{ and zero on } D_{j_2} \cap W_f\}$. Then:

i) $V \subseteq V_0 \cap D_{j_2}$. This is clear from the construction of V and V_0 . Hence φ restricts to V , and $\hat{a} \circ \varphi$ is holomorphic on V for every a in A .

ii) $W_f \cap V = W_f \cap D_{j_2}$. Also by construction. Hence $U = f^{-1}(D_{j_2}) = f^{-1}(V)$ is an open neighborhood of x .

Since $f_i \circ \varphi$ is holomorphic on V and $f_i \circ \varphi(z) = z_i$ on $W_f \cap D_{j_2}$ for each i , then $f_i \circ \varphi = z_i$ on V . This implies that $f \circ \varphi(z) = z$ for all z in V .

Finally, since $f \circ \varphi = \text{identity}$ on V and f is a monomorphism on $f^{-1}(V) = U$ it follows that f is a homeomorphism of U onto V and $\varphi = f^{-1}$. That $f^* = \eta$ is the statement of Proposition 2.4.

4.2. Corollary. Suppose $\{x\}$ is a G_δ . If there exist elements $a_1, \dots, a_n$ of A such that for each $a \in A$ there is a function F_a holomorphic on a neighborhood of $(\hat{a}_1(x), \dots, \hat{a}_n(x)) \in \mathcal{C}^n$ and $\hat{a} = F_a \circ (\hat{a}_1, \dots, \hat{a}_n)$ in some neighborhood of x , then the conclusions of Theorem 4.1 hold where $f = (\hat{a}_1, \dots, \hat{a}_n)$ and $f^* = \eta$.

Proof. This is obvious from the construction of f in section 2.4, and the definition of S_x ; namely, the homomorphism η in Theorem 4.1 is the homomorphism f^* translated to ${}_nO_0$.

4.3. That $\{x\}$ be a G_δ is a necessary condition can easily be seen in the following example. Let A be the algebra of bounded continuous functions with the supremum norm on the space Ω consisting of the set of all countable ordinals union the first uncountable ordinal (with the order topology). If ω is the first

uncountable ordinal then S_ω is isomorphic to the complex field ${}_0O_0$. Hence, we have the hypotheses of Theorem 4.1 satisfied with $n = 0$, except that $\{\omega\}$ is not a G_δ . The conclusion of Theorem 4.1 fails here since $\{\omega\}$ is not an open set in Ω . (Note that by Theorem 3.2 if $\{x\}$ is a G_δ and S_x is isomorphic to ${}_0O_0$ then x must be an isolated point of X .) On the other hand if we restrict our attention to separable algebras A , then this condition is always satisfied. The reason being that the maximal ideal space is now second countable, consequently every singleton point is a G_δ set.

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