On the Normalization of Haar Measures in the Representation Theory of Compact Semi-Simple Lie Groups

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1. Introduction. Let $\mathfrak{g}$ be a compact real semi-simple Lie algebra, $\mathfrak{j}$ a Cartan subalgebra of $\mathfrak{g}$; let $\mathfrak{g}_\mathbb{C}$ (respectively $\mathfrak{j}_\mathbb{C}$) denote the complexification of $\mathfrak{g}$ (respectively $\mathfrak{j}$); let $\Phi = \{\alpha\}$ denote the set of roots of the pair $(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$, Φ^+ the positive roots (relative to some ordering of Φ); let W denote the Weyl group of the pair $(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$. Because $\mathfrak{g}$ is compact, the Killing form B of $\mathfrak{g}$ is negative definite; agreeing to use $-B$, we see, therefore, that $\mathfrak{g}$ may be equipped with the structure of a Euclidean space in a natural way. It is well-known that the restriction of B to $\mathfrak{j}$ is non-degenerate; therefore $\mathfrak{j}$ may also be regarded as a Euclidean space in the obvious manner.

Now suppose that G is a connected Lie group with Lie algebra $\mathfrak{g}$; let J denote the analytic subgroup of G with Lie algebra $\mathfrak{j}$ —then J is a maximal torus in G . The group G operates on $\mathfrak{g}$ in the usual way (namely through the adjoint representation); we shall agree to write xX in place of $Ad(x)X$ ($x \in G, X \in \mathfrak{g}$). Let $d_\mathfrak{g}(X)$ (respectively $d_\mathfrak{j}(H)$) denote the canonical element of volume on $\mathfrak{g}$ (respectively $\mathfrak{j}$) as measured by $-B$; let $d_G(x)$ denote normalized Haar measure on G (hence $\int_G d_G(x) = 1$); let π denote the polynomial function on $\mathfrak{j}_\mathbb{C}$ which is defined by the requirement $\pi(H) = \prod_{\alpha > 0} \alpha(H)$ ($H \in \mathfrak{j}_\mathbb{C}$)—then, according to Harish-Chandra [1, p. 105], there exists a positive constant $\text{vol}(G)$ such that, for all continuous functions f on $\mathfrak{g}$ with compact support, we have

$$\int_{\mathfrak{g}} f(X) d_\mathfrak{g}(X) = \text{vol}(G) \int_{\mathfrak{j}} \left| \pi(H) \right|^2 d_\mathfrak{j}(H) \int_G f(xX) d_G(x).$$

The objective of the present note is to compute $\text{vol}(G)$.

Let r denote the cardinality of the set Φ^+ , $[W]$ the order of the Weyl group W ; let $(\cdot, \cdot)$ denote the usual scalar product on the real span (in the dual of $\mathfrak{j}_\mathbb{C}$) of the elements α in Φ ; let ρ be one-half the sum of the positive roots (i.e. let $\rho = 2^{-1} \sum_{\alpha > 0} \alpha$)—then it turns out that

$$\text{vol}(G) = (2\pi)^r / ([W] \prod_{\alpha > 0} (\rho, \alpha)) \quad (\pi = 3.14 \dots).$$

Besides being of interest in its own right, this result has the following significance relative to some work of Kostant (cf. Pukanszky [1]). Thus let us suppose for the moment that G is, in addition, simply connected. As was pointed out by Kostant (and described in Pukanszky's paper), there exists a one-to-one correspondence between the set of all equivalence classes of finite dimensional irreducible representations of G and certain G -orbits in $\mathfrak{g}$ (assumed to be identified with its dual through the Killing form); the G -orbits in question here are those of the form $\Lambda + \rho$, Λ a dominant integral linear function on $\mathfrak{j}_0$; each such orbit admits a G -invariant measure—the K -measure of the orbit—of finite total volume V_Λ (say) (cf. Pukanszky [1, p. 256]); this being so, it can then be shown (cf. Pukanszky [1, p. 262]), for any dominant integral Λ , that

$$V_\Lambda = \text{vol}(G) 2^r(r!) \prod_{\alpha > 0} (\Lambda + \rho, \alpha).$$

The problem of explicitly computing $\text{vol}(G)$ was, however, left open there. Using the above stated value for $\text{vol}(G)$, we see that actually

$$V_\Lambda = (4\pi)^r r! [W]^{-1} \left(\prod_{\alpha > 0} (\Lambda + \rho, \alpha) / \prod_{\alpha > 0} (\rho, \alpha) \right) = (4\pi)^r r! [W]^{-1} d_\Lambda,$$

d_Λ the dimension of the irreducible representation determined by Λ (we have, of course, recalled Weyl's well-known formula for d_Λ). It is instructive to compare the above formula for V_Λ/d_Λ with the observations in Pukanszky [1, p. 264] on the nilpotent case.

The present method of calculating $\text{vol}(G)$ depends upon some results of de Rham [1] and Schwartz [1] which deal with the fundamental solutions to certain partial differential equations; this technique has also been utilized by Harish-Chandra [2] in his study of the invariant integral on a non-compact semi-simple Lie algebra.

2. Preliminary Results.

(A) *Results of de Rahm.* Let E be an n -dimensional real vector space, Q a positive definite bilinear form on E ; fix an orthonormal basis $\{X_i\}$ in E relative to the Euclidean norm on E determined by Q ; let $x_1, \dots, x_n$ denote the corresponding set of coordinates; put $dX = dx_1 \cdots dx_n$ —thus dX is the usual canonical element of volume on E . Let n_1, n_2 be two non-negative integers such that $n_1 + n_2 = n \geq 2$. Put

$$u = \sum_1^{n_1} x_i^2 - \sum_{n_1+1}^n x_i^2, \quad \square = \sum_1^{n_1} (\partial/\partial x_i)^2 - \sum_{n_1+1}^n (\partial/\partial x_i)^2.$$

De Rham [1] has shown that there exists a function ξ on $\mathbb{R}$ (the real numbers) with the following properties. Let $\Xi(X) = \xi(u(X))(X \in E; X = \sum_{i=1}^n x_i X_i)$ —then Ξ is a locally summable function on E and

$$\square^{[n/2]}. \Xi = \delta$$

in the sense of distributions. [Here δ denotes the Dirac measure concentrated at the origin while $[n/2]$ is the greatest integer in $n/2$.] It is actually possible to give explicit expressions for ξ in terms of the Heaviside function $t \mapsto Y(t)$ ($Y(t)$ being 0 or 1 according as $t \leq 0$ or $t > 0$, $t \in \mathbb{R}$). There are four cases:

- (1) $\xi(t) = (-1)^{(n_1-1)/2} a_n^{-1} Y(t) |t|^{-1/2}$ if $n \equiv n_1 \equiv 1 \pmod{2}$;
- (2) $\xi(t) = (-1)^{n_1/2} a_n^{-1} Y(-t) |t|^{-1/2}$ if $n \equiv n_2 \equiv 1 \pmod{2}$;
- (3) $\xi(t) = -(-1)^{n_1/2} b_n^{-1} \log |t|$ if $n \equiv n_1 \equiv 0 \pmod{2}$;
- (4) $\xi(t) = (-1)^{(n_1-1)/2} c_n^{-1} Y(t)$ if $n_1 \equiv n_2 \equiv 1 \pmod{2}$.

Here

$$a_n = 2^{n-1} \pi^{(n-1)/2} \Gamma((n-1)/2), \quad b_n = \pi c_n = 2^n \pi^{n/2} \Gamma(n/2),$$

Γ the classical gamma function.

(B) *Results of Schwartz.* Retain the notations introduced in (A). Given $X \in E$, write $X = \sum_{i=1}^n x_i X_i$ and set $d(X) = (x_1^2 + \cdots + x_n^2)^{1/2}$; put $\Delta = \sum_{i=1}^n (\partial/\partial x_i)^2$ —then Schwartz [1, p. 47] has established the following points.

If n is odd

$$\Delta^{k+(n-1)/2} \cdot d^{2k-1} = c_{k,n} \delta \quad (k \geq 1)$$

and if n is even

$$\Delta^{k+n/2} \cdot (d^{2k} \log d) = c'_{k,n} \delta \quad (k \geq 0).$$

Here $c_{k,n}$ and $c'_{k,n}$ are non-zero numbers and the above relations are taken in the sense of the theory of distributions. The constants $c_{k,n}$, $c'_{k,n}$ are:

$$\begin{aligned} c_{k,n} &= (2(k + (n-1)/2) - n)(2(k + (n-1)/2) - 2 - n) \cdots (4 - n)(2 - n) \\ &\quad \cdot 2^{k+(n-1)/2-1} (k-1 + (n-1)/2)! \frac{2(\sqrt{\pi})^n}{\Gamma(n/2)}; \\ c'_{k,n} &= [(2(k + n/2) - n)(2(k + n/2) - 2 - n) \cdots (4 - n)(2 - n)] \\ &\quad \cdot 2^{k-1+n/2} (k-1 + n/2)! \frac{2(\sqrt{\pi})^n}{\Gamma(n/2)} \end{aligned}$$

where in the formula $[[\cdots]]$ the factor zero is omitted.

(C) *Results of Harish-Chandra.* (1) Let $S(\mathfrak{g}_e)$ denote the symmetric algebra over $\mathfrak{g}_e$, $I(\mathfrak{g}_e)$ the corresponding set of G -invariants in $S(\mathfrak{g}_e)$; we shall identify $S(\mathfrak{g}_e)$ with the algebra of polynomial functions on $\mathfrak{g}_e$ (via the identification of $\mathfrak{g}_e$ with its dual through the Killing form). Let us also bear in mind that the elements of $S(\mathfrak{g}_e)$ can be regarded, whenever it is convenient to do so, as differential operators on $\mathfrak{g}$; if $p \in S(\mathfrak{g}_e)$, then the corresponding differential operator on $\mathfrak{g}$ will be denoted by $\partial(p)$ —the subalgebra $\mathfrak{P}(\mathfrak{g}_e)$ of the algebra of all differential operators on $\mathfrak{g}$ which is generated by $S(\mathfrak{g}_e) \cup \partial(S(\mathfrak{g}_e))$ is called the algebra

of polynomial differential operators (on $\mathfrak{g}$). If f is a C^∞ function on $\mathfrak{g}$, D a differential operator on $\mathfrak{g}$, then, in what follows, we shall usually write $f(X; D)$ instead of $Df(X)(X \in \mathfrak{g})$. Similar definitions and notations apply to $\mathfrak{j}$.

(2) Given $f \in C_c^\infty(G)$, set

$$\phi_f(H) = \pi(H) \int_{\mathfrak{g}} f(xH) d_{\mathfrak{g}}(x) \quad (H \in \mathfrak{j}).$$

It is easy to prove that $\phi_f \in C_c^\infty(\mathfrak{j})$; moreover it can be shown (cf. Harish-Chandra [1, p. 103]) that

$$\phi_{\partial(p)f} = \partial(\tilde{p})\phi_f \quad (\text{all } p \in I(\mathfrak{g}_\alpha)).$$

Here $\tilde{p}$ denotes the restriction of the polynomial function p to $\mathfrak{j}_\alpha$. Harish-Chandra [1, p. 104] has also established that

$$\phi_f(0; \partial(\pi)) = [W] \prod_{\alpha > 0} (\rho, \alpha) f(0) \quad (\rho = 2^{-1} \sum_{\alpha > 0} \alpha).$$

(3) Let ω denote the Casimir polynomial of $\mathfrak{g}_\alpha$, $\tilde{\omega}$ its restriction to $\mathfrak{j}_\alpha$ (thus $\omega(X) = \text{tr}(ad(X))^2$, $X \in \mathfrak{g}_\alpha$); let R and L , respectively, denote the mappings $D \mapsto D \circ \partial(\tilde{\omega})$, $D \mapsto \partial(\tilde{\omega}) \circ D$ ($D \in \mathfrak{P}(\mathfrak{j}_\alpha)$) of $\mathfrak{P}(\mathfrak{j}_\alpha)$ into itself (the circle denoting composition of differential operators); put $\nabla = L - R$. Let r denote the number of positive roots of the pair $(\mathfrak{g}_\alpha, \mathfrak{j}_\alpha)$ —then clearly the greatest integer in $n/2$, $[n/2]$, is $[l/2] + r$ (n the dimension of $\mathfrak{g}_\alpha$, l the dimension of $\mathfrak{j}_\alpha$). Harish-Chandra [2, p. 572] has shown that

$$\begin{aligned} \pi \circ \partial(\tilde{\omega}^{[n/2]}) &= (L - \nabla)^{[n/2]} \pi \\ &= \partial(\tilde{\omega}^{[l/2]}) \circ \eta \end{aligned}$$

where

$$\eta = \sum_{i=0}^r \binom{[l/2]+r}{i} (-1)^i L^{r-i} \nabla^i \pi,$$

the first term in the expression for η being the usual binomial coefficient. The local expression for η at the origin can be computed (cf. Harish-Chandra [2, p. 572])—the result is

$$\eta_0 = (-1)^r [l/2]([l/2] + 1) \cdots ([l/2] + r - 1) 2^r \partial(\pi).$$

[If D is any differential operator on $\mathfrak{j}$ (say), then the local expression of D at a point $H \in \mathfrak{j}$ is that unique element $p \in S(\mathfrak{j}_\alpha)$ such that $f(H; D) = f(H; \partial(p))$ for all $f \in C^\infty(\mathfrak{j})$.]

3. Computation of $\text{vol}(G)$. We shall agree to retain the notations introduced in n° s 1 and 2.

The computation of $\text{vol}(G)$ will be divided into two separate cases, according to whether $l \geq 2$ or $l = 1$.

The Case $l \geq 2$. We shall employ the results of de Rham outlined in 2(A). Let ω denote the Casimir polynomial of $\mathfrak{g}_0$, $\tilde{\omega}$ the restriction of ω to $\mathfrak{j}_0$. Fix an orthonormal basis for $(\mathfrak{g}, \mathfrak{j})$ relative to $(-B)$ and let $x_1, \dots, x_n$ denote the corresponding set of coordinates—then, in the notations of 2(A),

$$u = -\sum_1^n x_i^2 = \omega, \quad \square = -\sum_1^n (\partial/\partial x_i)^2 = \partial(\omega), \quad \Xi(X) = \xi(\omega(X))(X \mathfrak{z} \mathfrak{g}).$$

[Hence we are taking $n_1 = 0$, $n_2 = n$ —plainly $n \geq 2$, $\mathfrak{g}$ being semi-simple.] Since $n \equiv l \pmod{2}$, cases (2) and (3) of 2(A) are applicable simultaneously to both $\partial(\omega)$ and $\partial(\tilde{\omega})$ according to whether l is odd or even, respectively. Let $\tilde{\Xi}$ denote the function corresponding to the pair $(\mathfrak{j}, \partial(\tilde{\omega}))$ —then we have

$$g(0) = \int_{\mathfrak{j}} \tilde{\Xi}(H) g(H; \partial(\tilde{\omega}^{l/2})) d_{\mathfrak{j}}(H)$$

for all $g \in C_c^\infty(\mathfrak{j})$. On the other hand, for $g \in C_c^\infty(\mathfrak{j})$,

$$\int_{\mathfrak{j}} \Xi(H) g(H; \partial(\tilde{\omega}^{l/2})) d(H) = \begin{cases} 2^{l-n} \pi^{(l-n)/2} \frac{\Gamma((l-1)/2)}{\Gamma(n-1)/2} g(0) & \text{if } l \text{ is odd;} \\ 2^{l-n} \pi^{(l-n)/2} \frac{\Gamma(l/2)}{\Gamma(n/2)} g(0) & \text{if } l \text{ is even.} \end{cases}$$

Thus, for example, consider the case when l is odd—then

$$\begin{aligned} g(0) &= \int_{\mathfrak{j}} a_i^{-1} Y(-\tilde{\omega}(H)) |\tilde{\omega}(H)|^{-1/2} g(H; \partial(\tilde{\omega}^{l/2})) d_{\mathfrak{j}}(H) \\ &= a_i^{-1} / a_n^{-1} \int_{\mathfrak{j}} a_n^{-1} Y(-\tilde{\omega}(H)) |\tilde{\omega}(H)|^{-1/2} g(H; \partial(\tilde{\omega}^{l/2})) d_{\mathfrak{j}}(H) \\ &= a_i^{-1} / a_n^{-1} \int_{\mathfrak{j}} \Xi(H) g(H; \partial(\tilde{\omega}^{l/2})) d(H). \end{aligned}$$

Consequently

$$\int_{\mathfrak{j}} \Xi(H) g(H; \partial(\tilde{\omega}^{l/2})) d_{\mathfrak{j}}(H) = \frac{a_l}{a_n} g(0) = 2^{l-n} \pi^{(l-n)/2} \frac{\Gamma((l-1)/2)}{\Gamma((n-1)/2)} g(0).$$

The case of even l is treated in a similar way. Now choose an $f \in C_c^\infty(\mathfrak{g})$ such that $f(0) = 1$ —then, as has been noted above, $\phi_f \in C_c^\infty(\mathfrak{j})$. Therefore

$$\begin{aligned} 1 = f(0) &= \int_{\mathfrak{g}} \Xi(X) \partial(\omega^{ln/2}) f(X) d_{\mathfrak{g}}(X) \\ &= \text{vol}(G) (-1)^r \int_{\mathfrak{j}} \Xi(H) \pi(H) \phi_{\partial(\omega^{ln/2}), f}(H) d_{\mathfrak{j}}(H) \\ &= \text{vol}(G) (-1)^r \int_{\mathfrak{j}} \Xi(H) \pi(H) \phi_f(H; \partial(\tilde{\omega}^{ln/2})) d_{\mathfrak{j}}(H) \quad (\text{cf. } 2(C) - (2)) \end{aligned}$$

$$\begin{aligned}
&= \text{vol } (G) (-1)^r \int_{\mathfrak{I}} \Xi(H) \phi_r(H; \pi \circ \partial(\tilde{\omega}^{(n/2)})) d_{\mathfrak{I}}(H) \\
&= \text{vol } (G) (-1)^r \int_{\mathfrak{I}} \Xi(H) \phi_r(H; \partial(\tilde{\omega}^{(l/2)}) \circ \eta) d_{\mathfrak{I}}(H)
\end{aligned}$$

where the η in the last expression is as in 2(C)–(3). We recall that the local expression for η at zero, η_0 , is:

$$\eta_0 = (-1)^r \prod_{i=0}^{r-1} ([l/2] + i) 2^r \partial(\pi).$$

Hence

$$\begin{aligned}
1 &= \text{vol } (G) (-1)^r \int_{\mathfrak{I}} \Xi(H) \phi_r(H; \partial(\tilde{\omega}^{(l/2)}) \circ \eta) d_{\mathfrak{I}}(H) \\
&= \text{vol } (G) \begin{cases} (-1)^r 2^{l-n} \pi^{(l-n)/2} \frac{\Gamma((l-1)/2)}{\Gamma((n-1)/2)} \phi_r(0; \eta) & \text{if } l \text{ is odd;} \\ (-1)^r 2^{l-n} \pi^{(l-n)/2} \frac{\Gamma(l/2)}{\Gamma(n/2)} \phi_r(0; \eta) & \text{if } l \text{ is even,} \end{cases}
\end{aligned}$$

and so, keeping in mind that $\phi_r(0; \partial(\pi)) = [W] \prod_{\alpha > 0} (\rho, \alpha)$ (cf. 2(C)–(2)), we find that

$$1 = \begin{cases} \left\{ [W] \prod_{\alpha > 0} (\rho, \alpha) \prod_{i=0}^{r-1} ([l/2] + i) 2^{r+l-n} \pi^{(l-n)/2} \frac{\Gamma((l-1)/2)}{\Gamma((n-1)/2)} \right\} \text{vol } (G) & \text{if } l \text{ is odd;} \\ \left\{ [W] \prod_{\alpha > 0} (\rho, \alpha) \prod_{i=0}^{r-1} ([l/2] + i) 2^{r+l-n} \pi^{(l-n)/2} \frac{\Gamma(l/2)}{\Gamma(n/2)} \right\} \text{vol } (G) & \text{if } l \text{ is even.} \end{cases}$$

Let $z \in \mathbb{C}$; if z is neither zero nor a negative integer, then, as is well known,

$$\prod_{i=0}^{r-1} (z + i) = \frac{\Gamma(z + r)}{\Gamma(z)} \quad (r \geq 1).$$

Using this relation in the last expression for 1 above, we find that

$$\text{vol } (G) = (2\pi)^r / ([W] \prod_{\alpha > 0} (\rho, \alpha))$$

whether l be odd or even. Therefore the claimed value for $\text{vol } (G)$ has been established in the case $l \geq 2$.

(ii) *The Case $l = 1$.* In this situation $\mathfrak{g}$ is isomorphic to the compact real form for $sl(2, \mathbb{C})$ —hence $n = 3$. Let us use 2(B) with $n = 3$, $k = 1$ —thus

$$c_{1,3}f(0) = \int_{\mathfrak{I}} d(X) \Delta^2 f(X) d_{\mathfrak{I}}(X) \quad (f \in C_c^\infty(\mathfrak{g})).$$

Here $\Delta = -\partial(\omega)$, $d(X) = (-\omega(X))^{1/2}(X \in \mathfrak{g})$. Let $\tilde{d}$ denote the distance function on $\mathfrak{h}$. Choose $f \in C_c^\infty(\mathfrak{g})$ such that $f(0) = 1$ —then

$$\begin{aligned} c_{1,3} &= \int_{\mathfrak{g}} d(X) \Delta^2 f(X) d_{\mathfrak{g}}(X) \\ &= \text{vol}(G)(-1) \int_{\mathfrak{h}} \tilde{d}(H) \pi(H) \phi_{\partial(\omega^2)_r}(H) d_{\mathfrak{h}}(H) \\ &= \text{vol}(G)(-1) \int_{\mathfrak{h}} \tilde{d}(H) \phi_r(H; \pi \circ \partial(\tilde{\omega}^2)) d_{\mathfrak{h}}(H). \end{aligned}$$

Employing the notations introduced in 2(C)–(3), note that $\pi \circ \partial(\tilde{\omega}^2) = R^2\pi = (L - \nabla)^2\pi = (L^2 - 2L\nabla + \nabla^2)\pi = (L^2 - 2L\nabla)\pi = L(L - 2\nabla)\pi = \partial(\tilde{\omega}) \circ \mu$ where $\mu = (L - 2\nabla)\pi$. Now $L = R + \nabla$ —thus $\mu = (R - \nabla)\pi$ and so $\mu_0 = -(\nabla\pi)_0 = -2\partial(\pi)$. Therefore

$$\begin{aligned} c_{1,3} &= \text{vol}(G)(-1) \int_{\mathfrak{h}} \tilde{d}(H) \phi_r(H; \partial(\tilde{\omega}) \circ \mu) d_{\mathfrak{h}}(H) \\ &= \text{vol}(G)(-1)^2 c_{1,1} \phi_r(0; \mu) \quad (\text{cf. } 2(\text{B})) \\ &= \text{vol}(G)(-1)^2 c_{1,1}(-2) \phi_r(0; \partial(\pi)) \\ &= (-2)[W](\rho, \alpha) c_{1,1} \text{vol}(G). \end{aligned}$$

Consequently

$$\text{vol}(G) = c_{1,3}/(-2)[W](\rho, \alpha) c_{1,1}.$$

The values of $c_{1,3}$ and $c_{1,1}$ are known—see 2(B) above. Moreover in the case at hand, $[W] = 2$, $(\rho, \alpha) = 2^{-1}(\alpha, \alpha) = 1/4$. Bearing in mind that $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(3/2) = \sqrt{\pi}/2$, we get

$$\text{vol}(G) = \frac{(-4)2(\sqrt{\pi})^3\Gamma(1/2)}{(-2)2(1/4)4\sqrt{\pi}\Gamma(3/2)} = 4\pi,$$

as desired.

The computation of $\text{vol}(G)$ is now complete.

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