Stability of the Burgers Shock Wave and the Korteweg-de Vries Soliton

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This paper considers the stability of a shock wave solution of the Burgers equation and of a solitary wave solution (soliton) of the Korteweg-de Vries equation with respect to infinitesimal disturbances. It is found that the Burgers shock wave is either asymptotically stable or stable in the Liapunov sense, whereas the Korteweg-de Vries soliton is only stable. The paper is concluded by a brief summary of the results of an application of Liapunov's direct method to the stability of the Burgers shock wave.

1. Introduction. It is well known that in nonlinear wave propagation in a continuum, the steepening of waves due to nonlinear interactions is balanced by the dissipative or dispersive effects, leading respectively to steady shockwaves or solitary waves. Simple but typical examples of such phenomena may be seen in the systems described by the Burgers equation (dissipative system) and the Korteweg-de Vries equation (dispersive system).

In fact, the Burgers equation:

$$(1.1) \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, \quad \nu > 0,$$

has a steady solution of the form

$$(1.2) \quad u_0 = -u_\infty \tanh\left(\frac{u_\infty x}{2\nu}\right), \quad u_\infty > 0,$$

which represents a shock like solution known as the Burgers shock wave [1]. (Throughout the first three sections of this paper, x and t denote, respectively, a one-dimensional space coordinate and time, whereas $u(x, t)$, which is differentiable with respect to x and t as many times as necessary, represents some

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physical quantity whose meaning will depend upon the problem considered.) This solution is not only regarded as a simple model of a gasdynamic shock wave, but it has also received some attention in connection with 'Burbulence,' *i.e.*, a turbulence model based on the Burgers equation [2].

On the other hand, the Korteweg-de Vries equation:

$$(1.3) \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \epsilon \frac{\partial^3 u}{\partial x^3}, \quad \epsilon > 0,$$

has a steady solution of the form

$$(1.4) \quad u_0 = u_\infty \left(1 - 3 \operatorname{sech}^2 \left(\frac{u_\infty}{4\epsilon} x \right) \right), \quad u_\infty > 0,$$

which represents a solitary wave known as the Korteweg-de Vries soliton. (Since ν is the dissipation parameter, physical considerations require that it must always be positive, whereas the dispersion parameter ϵ may be either positive or negative. We may assume $\epsilon > 0$ without loss of generality, since we may obtain an equation of the form $\partial u / \partial t + u \partial u / \partial x = -\epsilon \partial^3 u / \partial x^3$ by making the variable changes $u \rightarrow -u$, $x \rightarrow -x$ and $t \rightarrow t$.) This solution plays an important role in nonlinear dispersive systems [3].

It is of physical importance to decide whether the steady solutions given by (1.2) and (1.4) are ultimately stable or not. As a first step towards this end we shall begin, in this paper, with an analysis based on a framework of linear theory. In the concluding Section of the paper we present the results of an application of Liapunov's direct method to the stability of the Burgers shock wave described by eq. (1.2). Specifically, the disturbance can be shown to tend asymptotically to zero with time with respect to a suitable metric subject only to reasonable limitations on the nature of the initial disturbance.

2. Stability of Burgers shock wave. Following the conventional linear stability theory, we superimpose a small disturbance $v(x, t)$ upon the steady state solution $u_0(x)$, by setting

$$(2.1) \quad u = u_0 + v, \quad |v| \ll |u_0|.$$

Substituting (2.1) into eq. (1.1) and linearizing it with respect to v , we obtain the following equation for v :

$$(2.2) \quad \frac{\partial v}{\partial t} + u_0 \frac{\partial v}{\partial x} + \frac{du_0}{dx} v = \nu \frac{\partial^2 v}{\partial x^2}.$$

Let us now assume that $v(x, t) = f(x)g(t)$, then it may easily be shown that

$$(2.3) \quad g(t) \propto \exp(\sigma t), \quad \sigma = \text{const.},$$

and that $f(x)$ is governed by the equation

$$(2.4) \quad \nu \frac{d^2 f}{dx^2} - u_0 \frac{df}{dx} - \left(\frac{du_0}{dx} + \sigma \right) f = 0.$$

The boundary condition we shall assume for f is

$$(2.5) \quad f \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty.$$

It is obvious that if $\mathcal{R}_s(\sigma) > 0$ the disturbance will grow without bound. This means that the original steady state solution is unstable. On the other hand, if $\mathcal{R}_s(\sigma) < 0$ the disturbance will eventually die away, which means that the steady state solution is asymptotically stable in the Liapunov sense. If, in particular, $\mathcal{R}_s(\sigma) = 0$ then the original steady state solution is merely stable.

By setting $u_\infty x/(2\nu) \equiv y$ and $4\nu\sigma/u_\infty^2 \equiv \alpha$ (note that the steady state solution $u_0(x)$ is given by (1.2)), eq. (2.4) and the boundary condition (2.5) reduce, respectively, to the following one-parameter system:

$$(2.6) \quad \frac{d^2 f}{dy^2} + 2 \tanh y \frac{df}{dy} + (2 \operatorname{sech}^2 y - \alpha)f \equiv L(f) = 0,$$

$$(2.7) \quad f \rightarrow 0 \quad \text{as} \quad |y| \rightarrow \infty.$$

Since the differential operator L in eq. (2.6) is even in y , both odd and even functions of y can separately be solutions of eq. (2.6). In fact, the required solutions have the form:

$$(2.8) \quad \begin{cases} f_{\text{odd}} = \operatorname{sech} y (k \sinh ky - \tanh y \cosh ky), \\ f_{\text{even}} = \operatorname{sech} y (k \cosh ky - \tanh y \sinh ky), \end{cases}$$

where $k = (1 + \alpha)^{1/2}$. If, in particular, $k = 0$ then $f_{\text{odd}} = -\operatorname{sech} y \tanh y$ and $f_{\text{even}} \equiv 0$. Therefore we must replace the even solution by

$$f_{\text{even } (k=0)} = \operatorname{sech} y (1 - y \tanh y).$$

Similarly, if $k = 1$, $f_{\text{odd}} \equiv 0$ and $f_{\text{even}} = \operatorname{sech}^2 y$. In this case, $f_{\text{odd } (k=1)} = \tanh y + y \operatorname{sech}^2 y$ should be taken as the odd solution.

Necessary and sufficient conditions under which the solutions (2.8) satisfy the boundary condition (2.7) are given by $\mathcal{R}_s(k) < 1$, from which we obtain the condition $\mathcal{R}_s(\alpha) < -\{\mathcal{I}_m(\alpha)\}^2/4 \leq 0$. Consequently, we have $\mathcal{R}_s(\sigma) < 0$, which shows that the steady shock wave solution (1.2) is asymptotically stable. It should be noted that when $k = 1$ (i.e., $\alpha = 0$) the even solution $f_{\text{even } (k=1)} = \operatorname{sech}^2 y$ represents merely a stable solution, whereas the corresponding odd solution $f_{\text{odd } (k=1)} = \tanh y + y \operatorname{sech}^2 y$ is not an adequate solution because this cannot satisfy the boundary condition (2.7).

Thus we may conclude that the Burgers shock wave is either asymptotically stable or stable with respect to the infinitesimal disturbances.

3. Stability of Korteweg-de Vries Soliton. A similar procedure to that employed in the preceding section leads to a governing equation for the disturbance function f of the form:

$$(3.1) \quad \frac{d^3 f}{dy^3} - 4(1 - 3 \operatorname{sech}^2 y) \frac{df}{dy} - (24 \operatorname{sech}^2 y \tanh y + \alpha)f = 0,$$

where $y \equiv (u_\infty/4\epsilon)^{1/2}x$ and $\alpha \equiv (8\sigma/u_\infty)(\epsilon/u_\infty)^{1/2}$ (note that the steady state solution $u_0(x)$ is given by (1.4) instead of (1.2)). The boundary condition may again be given in the form

$$(3.2) \quad f \rightarrow 0 \quad \text{as} \quad |y| \rightarrow \infty.$$

Three independent solutions of this equation (3.1) can be obtained as follows:

$$(3.3) \quad \begin{aligned} f_k &= \lambda_k(\lambda_k - 2)^2 e^{\lambda_k y} + 4 \frac{d^2}{dy^2} \{e^{(\lambda_k-1)y} \operatorname{sech} y\} \\ &= e^{\lambda_k y} [\lambda_k(\lambda_k - 2)^2 + 4e^{-y} \operatorname{sech} y \\ &\quad \times \{\lambda_k(\lambda_k - 2) - 2(\lambda_k - 1) \tanh y + 2 \tanh^2 y\}], \quad (k = 1, 2, 3), \end{aligned}$$

where the λ_k are the distinct roots of the cubic equation:

$$(3.4) \quad \lambda^3 - 4\lambda - \alpha = 0.$$

If, in particular, $\lambda_k = 0, 2$ or -2 , that is, if $\alpha = 0$ thereby representing a stable case, then all the f_k are proportional to $\operatorname{sech}^2 y \tanh y$ for $k = 1, 2, 3$. Therefore, in this case, we must take for our three independent solutions the following functions:

$$(3.5) \quad \begin{cases} f_1 = \operatorname{sech}^2 y \tanh y \equiv f_0, & \text{say,} \\ f_2 = \tanh^2 y - 2 \operatorname{sech}^2 y + 3yf_0, \\ f_3 = 2 \sinh^2 y + 7 \tanh^2 y - 8 \operatorname{sech}^2 y + 15yf_0. \end{cases}$$

It should also be noted that the cubic equation (3.4) has double roots, $\lambda = \pm 2/3^{1/2}$, when $\alpha = \mp 16/3^{3/2}$. In this case two of the solutions given by (3.3) become identically equal to each other, say $f_1 \equiv f_2$. Therefore we must then choose three new independent solutions as follows:

$$(3.6) \quad \begin{cases} f_1 = \lambda_0(\lambda_0 - 2)^2 e^{\lambda_0 y} + 4 \frac{d^2}{dy^2} \{e^{(\lambda_0-1)y} \operatorname{sech} y\}, \\ f_2 = yf_1 + (3\lambda_0^2 - 8\lambda_0 + 4)e^{\lambda_0 y} + 8 \frac{d}{dy} \{e^{(\lambda_0-1)y} \operatorname{sech} y\}, \\ f_3 = \lambda_3(\lambda_3 - 2)^2 e^{\lambda_3 y} + 4 \frac{d^2}{dy^2} \{e^{(\lambda_3-1)y} \operatorname{sech} y\}, \end{cases}$$

where λ_0 denotes either $2/3^{1/2}$ or $-2/3^{1/2}$, and λ_3 is either $-4/3^{1/2}$ or $4/3^{1/2}$, according as λ_0 is $2/3^{1/2}$ or $-2/3^{1/2}$. It is obvious that the cubic equation (3.4) has no triple roots.

Now it can easily be seen that any linear combination of the solutions given by (3.3), (3.5) or (3.6) cannot satisfy the required boundary condition (3.2) except for the f_0 given in (3.5) which represents a stable solution.

Thus we must conclude that the Korteweg-de Vries soliton is stable in the Liapunov sense with respect to the infinitesimal disturbances.

4. Stability of Burgers Shock Wave by Liapunov's Direct Method. An approach to the question of the stability of the Burgers shock wave and the Korteweg-de Vries soliton may also be made using Liapunov's direct method. It is the object of this Section to summarise the results of such an analysis, to be published in a subsequent paper, concerning the stability of systems of non-linear parabolic equations of which Burgers equation is a special example.

We consider a system of parabolic equations of the form:

$$(4.1) \quad \frac{\partial u}{\partial t} + a(x, t; u) \frac{\partial u}{\partial x} + b(x, t; u) + c\left(x, t; u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) \frac{\partial^2 u}{\partial x^2} = 0,$$

where u is an $(n \times 1)$ dependent variable vector involving the dependent variables $u_1, u_2, \dots, u_n$; a and c are $(n \times n)$ matrices with elements depending on x, t and u , whilst b is an $(n \times 1)$ column vector with elements depending on x, t and u . It is assumed that a, b and c are differentiable functions with respect to the arguments indicated. The system of equations (4.1) degenerates to the Burgers equation when u, a, b and c are scalars, with $a = u, b = 0$ and $c = -\nu$ where $\nu > 0$.

In the general case let $\tilde{u}$ represent a disturbance superimposed on the steady state solution u_0 , so that

$$(4.2) \quad u = u_0 + \tilde{u}.$$

Then $\tilde{u}$ must be the solution of

$$(4.3) \quad \frac{\partial \tilde{u}}{\partial t} + a_0 \frac{\partial \tilde{u}}{\partial x} + \tilde{a} \frac{\partial u_0}{\partial x} + \tilde{a} \frac{\partial \tilde{u}}{\partial x} + \tilde{b} + c_0 \frac{\partial^2 \tilde{u}}{\partial x^2} + \tilde{c} \frac{\partial^2 u_0}{\partial x^2} + \tilde{c} \frac{\partial^2 \tilde{u}}{\partial x^2} = 0,$$

where $a(x, t; u_0 + \tilde{u}) \equiv a_0 + \tilde{a}$, $a_0 \equiv a(x, t; u_0)$ and similar expressions are defined for $\tilde{b}, \tilde{c}$ and c_0 .

Now provided $\tilde{u}$ is such that its support on the initial line is the finite x -interval I , it can be shown that when c is negative definite the trivial solution $\tilde{u} \equiv 0$ is asymptotically stable in the Liapunov sense with respect to the metric $\rho\{\tilde{u}\} = \langle ||\tilde{u}||^2 + ||\partial \tilde{u} / \partial x||^2 \rangle$, and that $||\tilde{u}|| \rightarrow 0$ as $t \rightarrow \infty$ provided that $\tilde{u}$ and $\partial \tilde{u} / \partial x$ are bounded and continuous.

Here we have used the notations

$$(4.4) \quad \begin{cases} |f| = \max |f^n|, \\ ||f|| = \langle \sum_n |f^n|^2 \rangle^{1/2}, \end{cases}$$

for the modulus and norm, respectively, of a column vector f with components f^n , and $\langle g(x, t) \rangle$ to denote the integral $\langle g(x, t) \rangle = \int_I g(x, t) dx$, where $g(x, t)$ is an integrable scalar function. When this general result is applied to the special case of the Burgers equation, it is found that depending on the value of k , in addition to asymptotic stability it is also possible to have stability in the sense of Liapunov.

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