A Removable Singularity Theorem for Local Homeomorphisms

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1. Introduction. Let M and N be manifolds and $f: M \rightarrow N$ a continuous mapping. The set B_f of points $p \in M$ at which f is not a local homeomorphism is called the *branch set*. Church and Hemmingsen, [1], have shown that if $m = \dim M \geq 3$, then B_f is a perfect set. This result is false for $m = 2$ because $M \sim \{p\}$ is in general not simply connected.

Interpreted in a different way, the Church–Hemmingsen result becomes a local removable singularity theorem: a possible branch point which is isolated is not a branch point at all. From this point of view it is natural (and in fact desirable for the theory of quasiconformal functions in space) to ask for a removable singularity theorem of this type where the function is not assumed to be continuous or even defined at the isolated point in question.

It is our intention here to provide such a theorem in the case $N = \mathbb{R}^n$: to present a necessary and sufficient condition that a local homeomorphism f in $M \sim \{p\}$ can be so defined at p to become a local homeomorphism of M . Our proof will be independent of the work of Church and Hemmingsen in that we do not first prove that f is continuous at p . Rather, we obtain the full strength of the conclusion at one stroke.

2. Notation and statement of results. For our purposes it suffices to work in n -dimensional Euclidean space $\mathbb{R}^n$, $n \geq 3$, and the corresponding Möbius space $\bar{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$. $\mathcal{B}$ will denote the punctured open unit ball $\{x: 0 < |x| < 1\}$ and $\mathcal{S}$ the $(n-1)$ -sphere $\{x: |x| = 1\}$.

The modulus $M(\Gamma)$ of a family Γ of curves in $\bar{\mathbb{R}}^n$ is defined as follows. We say that a non-negative Borel measurable function ρ is admissible for Γ , and write $\rho \in \mathcal{G}(\Gamma)$ if

$$\int_{\gamma} \rho \, ds \geq 1,$$

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for all $\gamma \in \Gamma$. The integration is with respect to arc length. Then set

$$M(\Gamma) = \inf \left\{ \int_{\mathbb{R}^n} \rho^n d\tau : \rho \in \mathcal{G}(\Gamma) \right\}.$$

Here, τ stands for Lebesgue measure in $\mathbb{R}^n$.

We will prove the following

Theorem. *Suppose $\mathcal{D} \subset \bar{\mathbb{R}}^n$, $n \geq 3$, is a domain containing $\mathbb{B} \cup \mathbb{S}$, that $0 \notin \mathcal{D}$, and that $f : \mathcal{D} \rightarrow \mathbb{R}^n$ is a local homeomorphism. Then $f(0)$ can be so defined that f becomes a homeomorphism of a neighborhood of 0 if and only if f satisfies the modular condition at $x = 0$.*

Modular condition. *For every domain $D \subset \mathcal{D}$ in which f is univalent, and every curve family $\Gamma \subset D$ with the property that $0 \in \text{closure}(\gamma)$ for all $\gamma \in \Gamma$, the modulus $M(f(\Gamma))$ is zero.*

The necessity of the modular condition is clear since the modulus of the family of all curves meeting at any single point is zero, [2].

Mappings which satisfy the condition include the locally quasiconformal mappings (f is a local homeomorphism and for every domain D in which f is univalent, there exists a constant $K = K(D)$ such that for every curve family $\Gamma \subset D$, we have $M(f(\Gamma)) \leq KM(\Gamma)$), and those satisfying the weaker requirement that, under the same quantifications, $M(f(\Gamma)) = 0$ whenever $M(\Gamma) = 0$.

Zoric, [5], answering a long standing question of Lavrent'ev, proved a weaker form of our theorem: *If $f : \bar{\mathbb{R}}^n \sim \{0\} \rightarrow \mathbb{R}^n$ is a locally quasiconformal mapping, then 0 is a removable singularity.* It then follows topologically that $f : \bar{\mathbb{R}}^n \rightarrow \bar{\mathbb{R}}^n$ is a global homeomorphism.

For any set E we will write $\bar{E}$ for the closure of E , and ∂E for the boundary of E . Our point of view is that $f : \mathcal{D} \rightarrow f(\mathcal{D})$ defines $\mathcal{D}$ as a covering space of $f(\mathcal{D})$, so we will use the notation E^* to indicate that the set E^* lies in the domain of f . $\{f^{-1}(E)\}$ will denote the set of all preimages of E , in contrast to $f^{-1}(E)$ which will always refer to the image of E under a specific branch of f^{-1} .

We will be working with spheres $S_r(a) = \{y : |y - a| = r\}$ and balls $B_r(a) = \{y : |y - a| < r\}$. Let L be a directed ray from $y = a$ and for each y let $\phi(y)$ be the angle between the vector y and the ray L . The spherical cap $C_{*,r}$ on $S_r(a)$ with respect to L is defined as

$$C_{*,r} = \{y : y \in S_r(a), \phi(y) < s\}$$

At the expense of some abuse of notation we will write

$$\partial C_{*,r} = \{y : y \in S_r(a), \phi(y) = s\}.$$

In this connection we comment that if Γ is the family of all curves in $C_{*,r}$ joining the axis L to a single point $y_0 \in \partial C_{*,r}$, and if ρ is admissible for Γ , then

$$(1) \quad \int_{S_r(a)} \rho^n d\omega \geq \frac{\mathcal{K}_n}{r}.$$

Here, $\mathcal{K}_n$ is a dimensional constant, and ω is surface area. For the proof, see [4, Theorem 3.6] for a similar result regarding the case $n = 3$. Our case offers only slight additional difficulty.

3. Preliminary remarks. We will frequently use two familiar topological principles. The first is the Jordan Separation Theorem, [3], which asserts that a homeomorphic image S^* in $\mathbb{R}^n$ of the $(n - 1)$ -sphere separates $\bar{\mathbb{R}}^n$ into two parts, each of which has S^* as its boundary. The part not containing ∞ is called $\text{Int}(S^*)$ and the part containing ∞ is called $\text{Ext}(S^*)$.

The other is the Monodromy Theorem. We will use it in the following form. Suppose R is a *simply connected* region (every loop in R is contractible to a point) or a sphere with the relative topology, and $R^* \subset \mathfrak{D}$ is such that $f: R^* \rightarrow R$ defines R^* as an *unlimited* covering space of R (given $p \in R$ and any $p^* \in R^*$ over p , then any arc from p in R can be lifted from p^*). Then f is in fact a homeomorphism of R^* onto R so that there is a uniquely determined branch of f^{-1} defined on R which is a homeomorphism of R onto R^* . If $x \in R^*$, and $y = f(x)$, this branch of f^{-1} is uniquely determined by the requirement $f^{-1}(y) = x$.

We shall also find the following three simple properties of f useful.

- (A) If f is univalent on a compact connected set $E^* \subset \mathfrak{D}$, then f is a homeomorphism of some open set containing E^* .
- (B) If R is an open ball, punctured ball, punctured $\bar{\mathbb{R}}^n$, or spherical cap, and if f^{-1} has a well defined branch from R into $\mathfrak{B}$, then for each $y_0 \in \partial R$,

$$\lim_{\substack{y \rightarrow y_0 \\ y \in R}} f^{-1}(y)$$

exists.

- (C) For any $y \in \bar{\mathbb{R}}^n$, the set of numbers $r > 0$ for which $y \in f(S_r(0))$ is discrete.

The proof of Property C is trivial. Property A is easy, and a proof may be found in [5]. We will sketch the proof of Property B. If the limit does not exist *and equal zero*, then there is a sequence $y_n \in R$, $y_n \rightarrow y_0$, with $f^{-1}(y_n) \rightarrow x_0 \in \mathfrak{B} \cup \mathfrak{S}$. But f is a local homeomorphism at x_0 and hence is the inverse of f^{-1} near y_0 . Therefore x_0 is the only limit point.

4. Proof. This will be accomplished in five steps. We show that of the three alternatives resulting from Step 2, one leads to the desired result (Step 3), while the other two are impossible (Step 5).

Step 1. Consider the family of concentric spheres $S_r = S_r(a)$, $r_1 < r < r_2$, and a ray emanating from $y = a$ with I denoting the interval $r_1 < r < r_2$ on L . Assume that

- (i) $S_r \cap f(\mathfrak{S}) = \emptyset$, $r_1 < r < r_2$, and
- (ii) a single valued branch of f^{-1} is specified in a neighborhood U of I .

Then for almost every r , $r_1 < r < r_2$, f^{-1} can be extended from $S_r \cap U$ to become

a homeomorphism of S_r onto $S_r^* = f^{-1}(S_r)$ with $S_r^* \subset \mathfrak{B}$. Furthermore, for each r such that $B_r^* = \text{Int}(S_r^*)$ lies in $\mathfrak{B}$, (i.e. $0 \notin \text{Int}(S_r^*)$), f^{-1} can be extended from S_r to be a homeomorphism of either $\text{Int}(S_r)$ or $\text{Ext}(S_r)$ onto B_r^* .

Proof. For each r , $r_1 < r < r_2$, consider the spherical caps $C_{t,r}$ on S_r which are constructed with respect to the ray L and set $t = t(r)$ equal to the supremum of numbers s with the property that f^{-1} has a homeomorphic extension from $U \cap C_{s,r}$ to $C_{s,r}$. By Property B and hypothesis (i), either f^{-1} has a homeomorphic extension to $\bar{C}_{t,r}$ and into $\mathfrak{B}$, or there exists a point $y(r) \in \partial C_{t,r}$ with

$$(2) \quad \lim_{\substack{y \rightarrow y(r) \\ y \in C_{t,r}}} f^{-1}(y) = 0.$$

In the former alternative, we observe that by Property A, f will have a homeomorphic extension to an open set containing $f^{-1}(\bar{C}_{t,r})$, which contradicts the maximality of t , unless $t = \pi$. Thus in fact either f^{-1} has a homeomorphic extension to S_r , or there exists a point $y(r)$ on $\partial C_{t,r}$ satisfying (2). Let E be the set of values of r for which this latter alternative holds true. We are to show that E has measure zero.

Again by Property A it is clear that $t(r)$ is lower semicontinuous and positive, and therefore that

$$D = \bigcup_{r \notin E} S_r \cup \bigcup_{r \in E} C_{t,r}, \quad (r_1 < r < r_2),$$

is a domain, and that f^{-1} is a homeomorphism of D . For each $r \in E$, let Γ_r be the family of curves in $C_{t,r}$ joining I to $y(r)$, and let $\Gamma = \bigcup \Gamma_r$. If ρ is admissible for Γ , then by (1),

$$\int_{S_r} \rho^n d\omega \geq \frac{\mathcal{K}_n}{r}.$$

Integrating over all r , we find

$$\int_{\mathbf{R}^n} \rho^n d\tau \geq \int_E dr \int_{S_r} \rho^n d\omega \geq \mathcal{K}_n \int_E \frac{1}{r} dr,$$

therefore

$$M(\Gamma) \geq \mathcal{K}_n \int_E \frac{1}{r} dr.$$

Because of (2), each curve in $\Gamma^* = f^{-1}(\Gamma)$ terminates at the point 0, and according to the modular condition, it must be true that $M(\Gamma) = 0$. It follows that E has measure zero.

To prove the final statement of Step 1, assume that $r \notin E$ and that $B_r^* = \text{Int}(S_r^*)$ lies in $\mathfrak{B}$. Consider the totality of lifts $\{f^{-1}(S_r)\}$ of points on S_r , and let R^* be the component of $B_r^* \sim \{f^{-1}(S_r)\}$ which is adjacent to S_r^* . We see that $\partial f(R^*) \subset S_r$, that $f(R^*) \cap S_r = \emptyset$, and hence that $R = f(R^*)$ is either $\text{Int}(S_r)$ or $\text{Ext}(S_r)$. In either case f defines R^* as an unlimited covering of R .

By the Monodromy Theorem, f is actually a homeomorphism of R^* onto R , implying that $R^* = B^*$.

Step 2. By composing f with a Möbius transformation if necessary we may assume that there is a sequence $x_n \rightarrow 0$ for which $y_n = f(x_n) \rightarrow 0$. Using Property C and replacing $\mathcal{S}$ by a smaller concentric sphere if necessary we can assume in addition that $0 \notin f(\mathcal{S})$ and $|y_n| < \text{dist}(0, f(\mathcal{S}))$ for all n .

For each n the requirement $f_n^{-1}(y_n) = x_n$ uniquely determines a single valued branch f_n^{-1} of f^{-1} in a spherical neighborhood U_n of y_n . We assume that $U_n \subset \{y : |y| < \text{dist}(0, f(\mathcal{S}))\}$ and that $U_n^* = f_n^{-1}(U_n) \rightarrow 0$. Let L_n denote the ray from 0 through y_n and let I_n be $L_n \cap U_n$. Applying Step 1, there exists a sphere $S_n = S_{r_n}(0)$ with $S_n \cap U_n \neq \emptyset$ such that f_n^{-1} can be extended from $S_n \cap U_n$ to a homeomorphism, again denoted by f_n^{-1} , of S_n into $\mathcal{B}$. For each n fix one such sphere S_n . We may assume that the radii r_n are strictly decreasing to 0.

By taking a subsequence if necessary, we can assume one of the following three possibilities occurs:

(a) $0 \in \text{Int}(f_n^{-1}(S_n))$ for all n , or $B_n^* = \text{Int}(f_n^{-1}(S_n))$ lies in $\mathcal{B}$ and f_n^{-1} extends to a homeomorphism of either

(b) $\text{Int}(S_n) = B_n$ onto B_n^* for all n , or

(c) $\text{Ext}(S_n) = B'_n$ onto B_n^* for all n .

Step 3. If possibility (a) of Step 2 occurs, then the theorem is true.

Proof. We have constructed the S_n so that they form a nested sequence of spheres shrinking to $y = 0$. Let R_n denote the spherical annulus bounded by S_1 and S_n , and $\hat{R}_n$ the relatively compact region in $\mathcal{B}$ bounded by S_1^* and S_n^* . Denote the component of $\hat{R}_n \sim \{f^{-1}(S_1 \cup S_n)\}$ which is adjacent to S_1^* by R_n^* . Then $\partial f(R_n^*) \subset S_1 \cup S_n$ and $f(R_n^*) \cap (S_1 \cup S_n) = \emptyset$, and therefore $f(R_n^*)$ is either $\text{Ext}(S_1)$, R_n , or $\text{Int}(S_n)$. Because $n \geq 3$, any of this trio is simply connected so that by the Monodromy Theorem, whichever one is $f(R_n^*)$ is homeomorphic to R_n^* . But R_n^* is not a topological ball so that in fact $f(R_n^*) = R_n$ and $f : R_n^* \rightarrow R_n$ is a homeomorphism, whose inverse is an extension of f_1^{-1} .

Letting $n \rightarrow \infty$, we see that there exists an extension of f_1^{-1} on $R = B_1 \sim \{0\}$ which maps R homeomorphically onto $f_1^{-1}(R) = R^* \subset \mathcal{B}$. By Property B, $\lim_{y \rightarrow 0} f_1^{-1}(y) = x_0$ exists, and therefore f_1^{-1} maps $R \cup \{0\}$ onto $R^* \cup \{x_0\}$. Topologically, this is only possible if $x_0 = 0$, and $R^* = \text{Int}(S_1^*) \sim \{0\}$. The theorem is true in this event.

Step 4. Assume possibility (b) of Step 2 holds. For each n we are going to apply the following expansion process to f_n^{-1} and $B_n = B_{r_n}(0)$. Let $t = t_n$ be the supremum of numbers $s \geq r_n$ with the property that f_n^{-1} has an extension to $B_s = B_s(0)$, mapping into $\mathcal{B}$. Then $t < \infty$, and f_n^{-1} has an extension to $B_t \cup S_t = \bar{B}_t$, mapping into $\mathcal{B} \cup \mathcal{S}$ such that $f_n^{-1}(S_t)$ intersects $\mathcal{S}$.

Proof. If $t = \infty$, then $\lim_{y \rightarrow \infty} f_n^{-1}(y)$ exists (Property B) so that f_n^{-1} becomes

a homeomorphism of $\mathbb{R}^n$ into $\mathfrak{B}$. This is topologically impossible, hence $t < \infty$.

By Property B, for each $y_0 \in S_t$,

$$\lim_{\substack{y \rightarrow y_0 \\ y \in B_t}} f_n^{-1}(y)$$

exists and will be denoted by $f_n^{-1}(y_0)$. If none of the limit values $f_n^{-1}(y_0)$ are zero, and if none lie on $\mathfrak{s}$, then t is not maximal (Property A). We complete the proof by showing that none are zero.

Suppose then that for some $y_0 \in S_t$, $f_n^{-1}(y_0) = 0$. Replacing $\mathfrak{s}$ by a smaller concentric sphere if necessary (Property C), we may assume $y_0 \in f(\mathfrak{s})$. Using Step 1, we can find a sequence of spheres $S_\eta(y_0)$ shrinking to y_0 such that f_n^{-1} has an extension from $S_\eta(y_0) \cap B_t$ to $S_\eta(y_0)$ which is a homeomorphism of $S_\eta(y_0)$ onto a compact set S_η^* in $\mathfrak{B}$. Our assumption $\lim_{x \rightarrow 0} f(x) = y_0$ of Step 2 implies that it is not the case that $\lim_{x \rightarrow 0} f(x) = y_0$. Therefore, in view of Step 3, we may assume that $0 \notin \text{Int}(S_\eta^*)$. Therefore, by the last part of Step 1, f_n^{-1} has an extension which maps either $\text{Int}(S_\eta(y_0))$ or $\text{Ext}(S_\eta(y_0))$ onto $\text{Int}(S_\eta^*)$. However, the former possibility can be excluded at once since $f_n^{-1}(y_0) = 0 \notin \text{Int}(S_\eta^*)$.

Consequently allowing $S_\eta(y_0)$ to shrink to y_0 we find that f_n^{-1} can be extended to a homeomorphism of $\mathbb{R}^n \sim \{y_0\}$ and even (Property B) of $\mathbb{R}^n$ into $\mathfrak{B}$. We have already observed that this is topologically impossible.

Step 4'. Assume that possibility (c) of Step 2 holds and set $t = t_n$ equal to the infimum of numbers $s \leq r_n$ such that f_n^{-1} has an extension from $\text{Ext}(S_n)$ to $\text{Ext}(S_n(0))$. Then $t > 0$, and f_n^{-1} has a homeomorphic extension from $\bar{B}_{t_n} = \text{Ext}(S_t) \cup S_t$ into $\mathfrak{B} \cup \mathfrak{s}$ such that $f_n^{-1}(S_t)$ intersects $\mathfrak{s}$.

The proof is the same as for Step 4.

Step 5. The conclusions of Steps 4 and 4' lead to a contradiction.

Proof. Because they are concentric, any two extremal spheres S_{t_n} are either disjoint or identical and the same is true of the lifts $S_{t_n}^* = f_n^{-1}(S_{t_n})$. Furthermore each $S_{t_n}^*$ meets $\mathfrak{s}$ yet no $B_{t_n}^* = \text{Int}(S_{t_n}^*)$ contains points of $\mathfrak{s}$. Hence two topological balls $B_{t_n}^*$ are either identical or disjoint.

Now recall that f_n^{-1} was initially determined in Step 2 by the requirement that $f_n^{-1}(y_n) = x_n$ and that each $S_n^* = f_n^{-1}(S_n)$ passes through a neighborhood $f_n^{-1}(U_n) = U_n^*$ of x_n which tends to zero as $n \rightarrow \infty$. Because $S_n^* \subset B_{t_n}^*$, we conclude that $B_{t_m}^* \cap B_{t_n}^* \neq \emptyset$ for $m \neq n$, with at most a finite number of exceptions, and therefore we may assume this to be true for all $m \neq n$.

By taking another subsequence if necessary, we may assume that there are points $x'_n \in S_{t_n}^* \cap \mathfrak{s}$ such that $\lim x'_n = x'_0 \in \mathfrak{s}$. Therefore $\lim f(x'_n) = f(x'_0) = y'_0$. The requirement $f^{-1}(y'_0) = x'_0$ uniquely determines a single valued branch f_0^{-1} of f^{-1} in a neighborhood U of y'_0 . But $U \cap S_{t_n} \neq \emptyset$, and $f_0^{-1}(U) \cap S_{t_n}^* \neq \emptyset$ for all $n > N$, some N . It follows that for all $n > N$, f_n^{-1} is an extension of f_0^{-1} .

Because the B_{t_n} are concentric, for any pair $n, m > N$, we must have $B_{t_n} \subset B_{t_m}$ or $B_{t_m} \subset B_{t_n}$. Assuming the former, we find $B_{t_n}^* = f_0^{-1}(B_{t_n}) \subset f_0^{-1}(B_{t_m})$

$= B_m^*$, which is in contradiction to the second paragraph of the present Step 5. This completes the proof.

Added in proof. It has just been brought to our attention that Zoric [Mat. Sb., **81** (123) (1970), 634–636] has published an independent proof of the main theorem of this paper, at least for the case of the locally quasiconformal mappings.

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