On the Movement of a Chemical Reaction Interface

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Communicated by M. KAC

Introduction. This paper is concerned with a free boundary problem in several space variables for parabolic partial differential equations having the form

$$\alpha(u) \frac{\partial u}{\partial t} = \operatorname{div} k(u) \operatorname{grad} u,$$

A precise formulation is given in Section 1. The solution u is required to vanish at the free boundary Γ and, moreover, a condition of the type

$$\nu k_1(u_1) \frac{\partial u_1}{\partial n} + k_2(u_2) \frac{\partial u_2}{\partial n} = 0 \quad \text{on } \Gamma$$

is imposed, where the subscripts 1 and 2 refer to the two sides of Γ .

In the one dimensional case the problem treated here was formulated in the literature as early as 1928 in several different contexts: In the theory of diffusion flames [1] and in the study of the diffusion of oxygen and lactic acid through tissue [10]. In the case of diffusion flames, the free boundary Γ represents the position of a flame created by the combustion of gaseous fuel with oxygen along an interface where the two substances come together. In the case of [10], Γ represents the interface between lactic acid and oxygen along which a chemical reaction occurs. The problem treated here can also be thought of as a mathematical model for the following situation: Two substances in solution diffuse in, say, a beaker. When they come together at Γ they chemically react quickly and completely (and perhaps precipitate out of solution). Hence at Γ the concentrations are zero. A certain amount of substance 1 must flow in from one side of Γ in order to react with a given amount of substance 2 flowing in from the opposite side of Γ .

* Research supported in part by the National Science Foundation under contract G.P. 15724.

† Research supported in part by the Office of Scientific Research of the United States Air Force under contract F44620-69-C-0106.

A different but somewhat related free boundary problem for parabolic equations is the Stefan problem. It has recently received considerable attention in the literature (see [11], [8], [9], [2], [3], [4], [5], as well as chapters in the books [7], [12]). For a discussion of the relation between the Stefan problem and the chemical reaction interface problem, in one dimension, see [13].

Just as was pointed out in [8] for the Stefan problem, the problem considered here presents considerable difficulties in the case of several space variables. These difficulties are inherent in the physical nature of the problem—the topological character of the free boundary Γ may vary considerably in the course of time. Therefore we look for “weak solutions” u of the original problem in which the free boundary Γ is hidden in the set $\{u = 0\}$ where u vanishes. Then the difficulty of giving a precise description of Γ is transferred to a regularity question about the set $\{u = 0\}$. We do not completely resolve this regularity question here.

The concept of a weak solution is formulated in section 2. In section 3, using the techniques of [11], we show that weak solutions are unique. The existence of weak solutions is established in section 4. There we rely on results of Ladyzenskaja and Ural'ceva [12] to solve certain nonlinear approximating problems. Sections 5 and 6 are concerned with the stability and monotone dependence of the solution with respect to its initial data. Analogous results for the Stefan problem were proved in [8]; the methods we use are similar to those of [8]. Using the Holder estimates of [12], the Holder continuity of the weak solution is established in Section 7. Section 8 is concerned with some qualitative aspects of the behavior of the solution. In Section 9 a variational method of Douglas and Dupont [6] is outlined for the construction of numerical approximations to the solution.

1. Classical formulation of the problem. The formulation of the problem given below is intended primarily as heuristic motivation for the generalized formulation described in the next section. The free boundary Γ should not be expected, in general, to actually be as smooth as is alleged in this classical formulation.

Consider the following situation: G is a bounded domain in R^n with a smooth boundary ∂G . Set $G(t) = G \times \{t\}$ and $\partial G(t) = \partial G \times \{t\}$. For any $T, 0 < T \leq \infty$, let $\Omega_T = \bigcup_{0 < t < T} G(t)$ and $S_T = \bigcup_{0 < t < T} \partial G(t)$. When there is no danger of confusion we drop the T and simply write Ω and S . Ω is divided into two parts, Ω_1 and Ω_2 , by the free boundary $\Gamma = \Gamma_T = \bigcup_{0 \leq t \leq T} \Gamma(t)$. Here $\Gamma(t)$ is a hypersurface in $\overline{G(t)}$ given by $\Gamma(t) = \{(x, t) \in G(t) \mid \Phi(x, t) = 0\}$ where $\Phi \in C^1(\overline{\Omega})$ and $\nabla_x \Phi(x, t) \neq 0$ on Γ . We assume $\Phi(x, t) < 0$ in Ω_1 and $\Phi(x, t) > 0$ in Ω_2 . In particular, $\Gamma(0)$ divides the initial region $G(0)$ into two regions $G_1(0)$ and $G_2(0)$. Let $S_i = \overline{\Omega_i} \cap S$ ($i = 1, 2$) and note that some S_i might be void. Figure 1 and 2 show possible geometric configurations.

Given the data $\{\tilde{h}_1, \tilde{h}_2, \Gamma(0)\}$ the problem is to find a solution $\{\tilde{u}_1, \tilde{u}_2, \Phi\}$ of (1.1)–(1.5):

FIGURE 1

FIGURE 2

$$(1.1) \quad \alpha_i(\tilde{u}_i) \frac{\partial \tilde{u}_i}{\partial t} = \sum_{j=1}^n \frac{\partial}{\partial x_j} \left[k_i(\tilde{u}_i) \frac{\partial \tilde{u}_i}{\partial x_j} \right] \quad \text{in } \Omega_i \quad (i = 1, 2),$$

$$(1.2) \quad \tilde{u}_i = \tilde{h}_i, \quad \tilde{h}_i > 0 \quad \text{in } G_i(0) \quad (i = 1, 2),$$

$$(1.3) \quad k_i(\tilde{u}_i) \frac{\partial \tilde{u}_i}{\partial n} = 0 \quad \text{on } S_i \quad (i = 1, 2),$$

$$(1.4) \quad \tilde{u}_i = 0 \quad \text{on } \Gamma \quad (i = 1, 2),$$

$$(1.5) \quad \nu k_1(\tilde{u}_1) \sum_{j=1}^n \frac{\partial \Phi}{\partial x_j} \frac{\partial \tilde{u}_1}{\partial x_j} = -k_2(\tilde{u}_2) \sum_{j=1}^n \frac{\partial \Phi}{\partial x_j} \frac{\partial \tilde{u}_2}{\partial x_j} \quad \text{on } \Gamma.$$

Here ν is a positive constant; $\partial/\partial n$ denotes differentiation in the direction of the outer normal to S . The $\alpha_i(u)$, $k_i(u)$ are sufficiently smooth functions defined for $u \geq 0$ which, for some constants γ_0 , γ_1 , satisfy

$$(1.6) \quad 0 < \gamma_0 < \alpha_i(u) \quad , \quad k_i(u) < \gamma_1 \quad (i = 1, 2).$$

The interpretation in terms of "the movement of a chemical reaction interface" is as follows: $\tilde{u}_i$ represents the concentration of substance i which diffuses according to (1.1) in the region Ω_i ; $\tilde{h}_i$ is the initial concentration. $\Gamma(0)$ is the initial position of the interface separating the two substances. According to (1.3) there is no diffusion across the fixed boundary S . The concentration $\tilde{u}_i$ ought to be strictly positive except on Γ . There it vanishes because of the "fast reaction" which takes place along the free boundary. Equation (1.5) represents the chemical balance governing the reaction.

The function Φ is not uniquely determined; it is only one possible parametrization of Γ .

By a *classical solution* of (1.1)–(1.5) we mean a solution $\{\tilde{u}_1, \tilde{u}_2, \Phi\}$ such that $\tilde{u}_i$, $\nabla_x \tilde{u}_i$ are continuous in $\bar{\Omega}_i$ and $D_x^2 \tilde{u}_i$, $D_t \tilde{u}_i$ are continuous in Ω_i .

Unless the contrary is explicitly stated, it is always assumed that T is chosen (arbitrarily) fixed and finite.

2. Generalized formulation of the problem. It is convenient to make even extensions of the functions $\alpha_2(u)$, $k_2(u)$ to negative values of u and define

$$A_i(u) = \int_0^u \alpha_i(\zeta) d\zeta \quad (i = 1, 2),$$

$$K_i(u) = \int_0^u k_i(\zeta) d\zeta \quad (i = 1, 2),$$

so that $A_2(u)$, $K_2(u)$ are odd functions. Let

$$u_1 = \tilde{u}_1, \quad u_2 = -\tilde{u}_2$$

$$h_1 = \tilde{h}_1, \quad h_2 = -\tilde{h}_2.$$

Then (1.1)–(1.5) becomes

$$(1.1') \quad \frac{\partial}{\partial t} A_i(u_i) = \operatorname{div} \operatorname{grad} K_i(u_i) \quad \text{in } \Omega_i \quad (i = 1, 2),$$

$$(1.2') \quad \begin{aligned} u_1 &= h_1 > 0 \quad \text{in } G_1(0) \\ u_2 &= h_2 < 0 \quad \text{in } G_2(0), \end{aligned}$$

$$(1.3') \quad \frac{\partial}{\partial n} K_i(u_i) = 0 \quad \text{on } S_i \quad (i = 1, 2),$$

$$(1.4') \quad u_i = 0 \quad \text{on } \Gamma \quad (i = 1, 2),$$

$$(1.5') \quad \nu \operatorname{grad} \Phi \cdot \operatorname{grad} K_1(u_1) = \operatorname{grad} \Phi \cdot \operatorname{grad} K_2(u_2) \quad \text{on } \Gamma.$$

Here “div” and “grad” refer only to the x -variables.

We consider smooth *test functions* ϕ in R^{n+1} such that

$$(2.1) \quad \begin{aligned} \phi &\equiv 0 \quad \text{on } G(T). \\ \frac{\partial \phi}{\partial n} &\equiv 0 \quad \text{on } S. \end{aligned}$$

For such a ϕ we obtain, after integrating by parts, that

$$-\iint_{\Omega_i} \phi \frac{\partial}{\partial t} A_i(u_i) \, dx \, dt = \iint_{\Omega_i} A_i(u_i) \frac{\partial \phi}{\partial t} \, dx \, dt + \int_{G_i(0)} \phi A_i(h_i) \, dx,$$

since $A_i(u_i) = A_i(0) = 0$ on Γ . Similarly we have

$$\begin{aligned} \iint_{\Omega_i} \phi \operatorname{div} \operatorname{grad} K_i(u_i) \, dx \, dt &= \iint_{\Omega_i} K_i(u_i) \Delta \phi \, dx \, dt \\ &+ \int_0^T \int_{\partial G_i(t)} \phi \frac{\partial}{\partial n} K_i \, dS_x \, dt - \int_0^T \int_{\partial G_i(t)} K_i \frac{\partial \phi}{\partial n} \, dS_x \, dt. \end{aligned}$$

For $i = 1$ we multiply the above equations by ν , add them, and use the facts that $\partial \phi / \partial n = (\partial / \partial n) K_i = 0$ on S_i and $K_i(u_i) = K_i(0) = 0$ on Γ :

$$\begin{aligned} \iint_{\Omega_1} \left[\nu A_1(u_1) \frac{\partial \phi}{\partial t} + \nu K_1(u_1) \Delta \phi \right] \, dx \, dt \\ + \int_{\Gamma} \nu \phi |\operatorname{grad} \Phi|^{-1} \operatorname{grad} \Phi \cdot \operatorname{grad} K_1(u_1) \, dS_x \, dt + \int_{G_1(0)} \phi \nu A_1(h_1) \, dx = 0 \end{aligned}$$

For $i = 2$,

$$\begin{aligned} \iint_{\Omega_2} [A_2(u_2) \frac{\partial \phi}{\partial t} + K_2(u_2) \Delta \phi] \, dx \, dt \\ - \int_{\Gamma} \phi |\operatorname{grad} \Phi|^{-1} \operatorname{grad} \Phi \cdot \operatorname{grad} K_2(u_2) \, dS_x \, dt + \int_{G_2(0)} \phi A_2(h_2) \, dx = 0 \end{aligned}$$

Next let

$$u = \begin{cases} u_1 & \text{in } \Omega_1 \\ 0 & \text{on } \Gamma \\ u_2 & \text{in } \Omega_2, \end{cases}$$

$$h = \begin{cases} h_1 > 0 & \text{in } G_1(0) \\ 0 & \text{on } \Gamma(0) \\ h_2 < 0 & \text{in } G_2(0), \end{cases}$$

and introduce the functions

$$a(u) = \begin{cases} \nu A_1(u), & u > 0 \\ 0, & u = 0 \\ A_2(u), & u < 0, \end{cases}$$

$$b(u) = \begin{cases} \nu K_1(u), & u > 0 \\ 0, & u = 0 \\ K_2(u), & u < 0. \end{cases}$$

Then it follows that

$$(2.2) \quad \iint_{\Omega} [a(u) \frac{\partial \phi}{\partial t} + b(u) \Delta \phi] dx dt + \int_{G(0)} a(h) \phi dx = 0.$$

Definition. By a *weak solution* of (1.1)–(1.5) in Ω we mean a bounded measurable function u in Ω such that (2.2) holds for all test functions ϕ satisfying (2.1).

The preceding discussion is summarized by the following theorem.

Theorem 1. A classical solution of (1.1)–(1.5) in Ω is also a weak solution.

At this point it would seem natural to define weak versions of Ω_1 , Ω_2 and Γ as the sets where $\{u > 0\}$, $\{u < 0\}$ and $\{u = 0\}$, respectively, corresponding to any weak solution u . However a weak solution, by definition, is determined uniquely only up to a set of measure zero. Thus the weak free boundary corresponding to the set $\{u = 0\}$ would be determined only up to a set of measure zero. In Section 7 it is shown that weak solutions can be found which are Holder continuous, so that the sets $\{u > 0\}$, $\{u < 0\}$ are open and $\{u = 0\}$ is closed. But we have not achieved a complete description of the set $\{u = 0\}$ —it may or may not contain interior points.

An alternate more intrinsic recuperation of Γ can be had as follows: Let $B(P; r)$ be the open ball of radius r centered at the point $P \in R^{n+1}$ and denote Lebesgue measure in R^{n+1} by μ . Corresponding to any weak solution u we define

$$E = \{(x, t) \in \Omega \mid u(x, t) \geq 0\}$$

and

$$\zeta(P; r) = \frac{\mu\{E \cap B(P; r)\}}{\mu\{\Omega \cap B(P; r)\}}.$$

Then Γ should correspond to the essential boundary $\mathcal{F}_*E$ of E . A point $P \in \mathcal{F}_*E$ if and only if

$$0 < \liminf_{r \downarrow 0} \zeta(P; r) \leq \limsup_{r \downarrow 0} \zeta(P; r) < 1.$$

Note that $\mathcal{F}_*E$ does not change if u is changed on a set of measure zero. The question of regularity for $\{u = 0\}$ or $\mathcal{F}_*E$ remains unsolved.

Theorem 2. *A sufficiently regular weak solution of (1.1)–(1.5) in Ω is also a classical solution.*

Proof. To be more precise, let u be a weak solution of (1.1)–(1.5) in Ω . Assume that a function $\Phi \in C^1(\bar{\Omega})$ exists such that $\{(x, t) \in \bar{\Omega} \mid \Phi(x, t) = 0\} = \{(x, t) \in \bar{\Omega} \mid u(x, t) = 0\} \equiv \Gamma(t)$ with $\nabla_x \Phi \neq 0$ on $\Gamma(t)$, $\Phi < 0$ in $\Omega_1 = \bigcup_{0 < t < T} G_1(t)$, and $\Phi > 0$ in $\Omega_2 = \bigcup_{0 < t < T} G_2(t)$. Here $\Gamma(t)$ divides $G(t)$ into the two regions $G_1(t)$ and $G_2(t)$. Setting $u_i = u$ in Ω_i ($i = 1, 2$), we assume that u_i , $\nabla_x u_i$ are continuous in $\bar{\Omega}_i$ and $D_x^2 u_i$, $D_t u_i$ are continuous in Ω_i . Then we have a classical solution of (1.1)–(1.5) in Ω .

In (2.2) we may take test functions of ϕ having compact support in Ω_i . Hence u_i is a weak solution of (1.1') in Ω_i and, since u_i is smooth, it follows that u_i is a classical solution. Next take a test function ϕ with support in a neighborhood of a point $(x_0, t_0) \in \Gamma$ with $t_0 > 0$, $x_0 \notin S$. Then from (2.2) we have

$$\iint_{\Omega_1} [\nu A_1(u_1) \frac{\partial \phi}{\partial t} + \nu K_1(u_1) \Delta \phi] dx dt + \iint_{\Omega_2} [A_2(u_2) \frac{\partial \phi}{\partial t} + K_2(u_2) \Delta \phi] dx dt = 0.$$

Integrating by parts we obtain

$$\int_{\Gamma} \phi |\text{grad } \Phi|^{-1} \{ \nu \text{grad } \Phi \cdot \text{grad } K_1(u_1) - \text{grad } \Phi \cdot \text{grad } K_2(u_2) \} dS_x dt = 0,$$

from which it follows that (1.5') is satisfied at (x_0, t_0) . By continuity it follows that (1.5') holds everywhere on Γ . In a similar fashion the other boundary conditions in (1.1')–(1.5') can be demonstrated to hold. Since (1.1')–(1.5') is equivalent to (1.1)–(1.5), we see that $\{u_1, u_2, \Phi\}$ is indeed a classical solution.

It is useful to observe that if $T_1 < T_2$ and if u is a weak solution of (1.1)–(1.5) in Ω_{T_2} , then u is also a weak solution of (1.1)–(1.5) in Ω_{T_1} .

3. Uniqueness. These assumptions will be needed for uniqueness:

(i) G is bounded and $\partial G \in C^{2+\alpha}$ for some $\alpha > 0$.

(ii) For $i = 1, 2$ the functions $\alpha_i(u)$, $k_i(u)$ are continuous even functions of u .

For all u we require that (1.6) should hold.

Remark. We make no explicit assumptions concerning the initial interface position $\Gamma(0)$ corresponding to the set $\{h = 0\}$. In order to have a reasonable

physical problem, of course, h should be taken such that $\Gamma(0)$ is “ $(n - 1)$ -dimensional”; e.g. a smooth hypersurface.

Under the assumptions listed above we have the following theorem.

Theorem 3. *Weak solutions of (1.1)–(1.5) in Ω are unique.*

The proof of theorem 3 requires several lemmas.

Lemma 1. *For any real u and v , $u \neq v$,*

$$a(u) - a(v) = \bar{a}(u, v)(u - v),$$

$$b(u) - b(v) = \bar{b}(u, v)(u - v),$$

where $\bar{a}(u, v)$ and $\bar{b}(u, v)$ are positive numbers such that

$$0 < \bar{\gamma}_0 < \bar{a}(u, v), \quad \bar{b}(u, v) < \bar{\gamma}_1,$$

with $\bar{\gamma}_0 = \min(\nu\gamma_0, \gamma_0)$, $\bar{\gamma}_1 = \max(\nu\gamma_1, \gamma_1)$.

Proof. If u, v have the same sign the lemma follows from the definitions of $a(u)$, $b(u)$ and (1.6) by the Mean Value Theorem. Otherwise suppose, say, that $u > 0$ and $v < 0$. Then $\nu\gamma_0 u < a(u)$, $b(u) < \nu\gamma_1 u$ and $\gamma_1 v < a(v)$, $b(v) < \gamma_0 v$. Let α, β be such that $\nu\gamma_0 u < \alpha < \nu\gamma_1 u$, $\gamma_1 v < \beta < \gamma_0 v$ and consider the function

$$s(\alpha, \beta) = \frac{\alpha - \beta}{u - v}.$$

It is easy to see that

$$\frac{\nu\gamma_0 u - \gamma_0 v}{u - v} < s(\alpha, \beta) < \frac{\nu\gamma_1 u - \gamma_1 v}{u - v},$$

and an examination of the maximum and the minimum of the right and left-hand sides of this relation leads to the desired inequalities. The case $u < 0$ and $v > 0$ is handled similarly.

Corresponding to any two functions $u = u(x, t)$, $v = v(x, t)$, we define the function

$$(3.1) \quad e(x, t) \equiv \begin{cases} \frac{b(u) - b(v)}{a(u) - a(v)}, & u \neq v \\ 0, & u = v. \end{cases}$$

Note that $a(u) \neq a(v)$ for $u \neq v$. The next lemma is a trivial consequence of Lemma 1.

Lemma 2. *There is a positive constant $C_1 = C_1(\bar{\gamma}_0, \bar{\gamma}_1)$ such that*

$$0 \leq e(x, t) \leq C_1$$

for all (x, t) .

We need to establish bounds for the solutions ϕ_m ($m = 1, 2, 3, \dots$) of the following problem:

$$(3.2) \quad \frac{\partial \phi_m}{\partial t} + e_m \Delta \phi_m = f \quad \text{in } \Omega,$$

$$(3.3) \quad \phi_m(x, T) = 0 \quad \text{on } G(T),$$

$$(3.4) \quad \frac{\partial \phi_m}{\partial n} = 0 \quad \text{on } S.$$

Here and in what follows T , $0 < T < \infty$, is fixed, So is $f \in C_0^\infty(\Omega)$, and $e_m = e_m(x, t)$ is assumed to be a function in $C^\infty(\bar{\Omega})$ such that

$$(3.5) \quad \frac{1}{m} \leq e_m(x, t) \leq C_2, \quad (x, t) \in \bar{\Omega}$$

for some constant C_2 independent of m . The existence of smooth solutions ϕ_m to (3.2)–(3.4) follows from the hypothesis we have made on Ω , as is well-known [7], [12].

Lemma 3. *There is a positive constant $C_3 = C_3(T, f)$, independent of m , such that*

$$\max_{\bar{\Omega}} |\phi_m| \leq C_3(T, f).$$

Proof. The hypotheses made on ∂G insure the existence of a positive C^2 function $\psi = \psi(x)$ in $\bar{G}$ having a positive exterior normal derivative at every point of ∂G . Choose

$$B > C_2 \max_{\partial} |\Delta \psi| + \max_{\bar{\Omega}} |f|$$

and set

$$w(x, t) = \psi(x) + B(T - t).$$

Consider $z^\pm = w \pm \phi_m$. In Ω we have

$$(3.6) \quad \frac{\partial z^\pm}{\partial t} + e_m \Delta z^\pm = e_m \Delta \psi - B \pm f < 0.$$

On $G(T)$, $z^\pm = w > 0$ and

$$\frac{\partial z^\pm}{\partial n} = \frac{\partial w}{\partial n} = \frac{\partial \psi}{\partial n} > 0$$

on S . It follows that $z^\pm \geq 0$ on $\bar{\Omega}$. For if $z^\pm$ were negative anywhere, $z^\pm$ would have to assume its minimum at a point of $\Omega \cup G(0)$. At such a minimum we would have $\partial z^\pm / \partial t \geq 0$, $\Delta z^\pm \geq 0$ which contradicts (3.6). Thus $w \geq \pm \phi_m$ and therefore

$$\max_{\bar{\Omega}} |\phi_m| \leq \max_{\partial} \psi + BT = C_3.$$

This completes the proof of the lemma.

Still under the assumptions stated for Lemma 3, we derive the following result:

Lemma 4. *There is a positive constant $C_4 = C_4(T, f)$, independent of m , such that*

$$\|e_m^{1/2} \Delta \phi_m\|_{L^2(\Omega)} \leq C_4(T, f).$$

Proof. Multiplying (3.2) by $\Delta \phi_m$ and integrating over Ω , we obtain

$$\int_0^T \int_{G(t)} \frac{\partial \phi_m}{\partial t} \Delta \phi_m \, dx \, dt + \int \int_{\Omega} e_m |\Delta \phi_m|^2 \, dx \, dt = \int \int_{\Omega} f \Delta \phi_m \, dx \, dt.$$

But

$$\begin{aligned} \int_0^T \int_{G(t)} \frac{\partial \phi_m}{\partial t} \Delta \phi_m \, dx \, dt &= \int_0^T \int_{\partial G(t)} \frac{\partial \phi_m}{\partial t} \frac{\partial \phi_m}{\partial n} \, dS_x \, dt - \int_0^T \int_{G(t)} \operatorname{grad} \frac{\partial \phi_m}{\partial t} \cdot \operatorname{grad} \phi_m \, dx \, dt \\ &= -\frac{1}{2} \int_0^T \int_{G(t)} \frac{\partial}{\partial t} |\operatorname{grad} \phi_m|^2 \, dx \, dt \\ &= \frac{1}{2} \int_{G(0)} |\operatorname{grad} \phi_m(x, 0)|^2 \, dx \end{aligned}$$

because of (3.4) and (3.3). Also, since $f \in C_0^\infty(\Omega)$,

$$\iint_{\Omega} f \Delta \phi_m \, dx \, dt = \iint_{\Omega} \phi_m \Delta f \, dx \, dt.$$

Hence

$$(3.7) \quad \frac{1}{2} \int_{G(0)} |\operatorname{grad} \phi_m|^2 \, dx + \iint_{\Omega} e_m |\Delta \phi_m|^2 \, dx \, dt = \iint_{\Omega} \phi_m \Delta f \, dx \, dt.$$

It follows from (3.7) and Lemma 3 that

$$\left\{ \iint_{\Omega} e_m |\Delta \phi_m|^2 \, dx \, dt \right\}^{1/2} \leq C_4(T, f),$$

which is the content of Lemma 4.

Let $e = e(x, t)$ be (as in Lemma 2) a bounded non-negative measurable function in Ω . By convolving e with appropriate mollification kernels, one can find a sequence of functions $\bar{e}_m \in C^\infty(\bar{\Omega})$ with

$$0 \leq \bar{e}_m(x, t) \leq \sup_{\Omega} e$$

in $\bar{\Omega}$ and such that

$$(3.8) \quad \|e - \bar{e}_m\|_{L^2(\Omega)} < \frac{1}{m}$$

for $m = 1, 2, \dots$. Set

$$(3.9) \quad e_m = \bar{e}_m + \frac{1}{m}.$$

Lemma 5. *There is a positive constant C_5 , independent of m , such that*

$$\left\| \frac{e}{e_m} \right\|_{L^2(\Omega)} \leq C_5.$$

Proof. In what follows $\|\cdot\|$ stands for the L^2 norm in Ω and μ denotes $(n+1)$ -dimensional Lebesgue measure. We have

$$\begin{aligned} \|e - e_m\| &\leq \|e - \bar{e}_m\| + \|\bar{e}_m - e_m\| \\ &\leq \frac{1}{m} (1 + \mu(\Omega)^{1/2}). \end{aligned}$$

Hence

$$\begin{aligned} \left\| \frac{e - e_m}{e_m} \right\| &\leq (\min e_m)^{-1} \|e - e_m\| \\ &\leq 1 + \mu(\Omega)^{1/2}. \end{aligned}$$

It follows that

$$\begin{aligned} \left\| \frac{e}{e_m} \right\| &= \left\| \frac{e - e_m}{e_m} + 1 \right\| \\ &\leq 1 + 2\mu(\Omega)^{1/2}, \end{aligned}$$

and the proof of Lemma 5 is complete.

Proof of Theorem 3. Let u and v be two weak solutions of (1.1)–(1.5) in Ω corresponding to the same initial data h . Then $|u|, |v| \leq M$, for some M . For any function $f \in C_0^\infty(\Omega)$ we shall establish that

$$(3.10) \quad \iint_{\Omega} [a(u) - a(v)] f \, dx \, dt = 0,$$

from which it follows that $a(u) = a(v)$, and hence that $u = v$, almost everywhere.

For any test function ϕ satisfying (2.1) we have

$$\iint_{\Omega} \left\{ [a(u) - a(v)] \frac{\partial \phi}{\partial t} + [b(u) - b(v)] \Delta \phi \right\} dx \, dt = 0.$$

Since $a(u) = a(v)$ and $b(u) = b(v)$ on the set where $u = v$, the above expression implies that

$$\iint_{\Omega} [a(u) - a(v)] \left\{ \frac{\partial \phi}{\partial t} + e \Delta \phi \right\} dx \, dt = 0,$$

with e defined by (3.1). Introduce a sequence of smooth functions $\bar{e}_m$ and e_m as

was done in (3.8) and (3.9). Then the corresponding solutions ϕ_m of (3.2)–(3.4) are allowed test functions. Hence

$$\iint_{\Omega} [a(u) - a(v)] \left\{ \frac{\partial \phi_m}{\partial t} + e_m \Delta \phi_m + (e - e_m) \Delta \phi_m \right\} dx dt = 0,$$

and it follows that

$$\begin{aligned} \left| \iint_{\Omega} [a(u) - a(v)] f dx dt \right| &= \left| \iint_{\Omega} [a(u) - a(v)] (e_m - e) \Delta \phi_m dx dt \right| \\ &\leq M_1 \iint_{\Omega} |e_m - e| \cdot |\Delta \phi_m| dx dt \\ &\equiv M_1 I_m, \end{aligned}$$

for some constant M_1 , independent of m . Thus it will suffice to show that

$$(3.11) \quad \lim_{m \rightarrow \infty} I_m = 0.$$

Consider

$$\begin{aligned} I_m &= \iint_{\Omega} (e_m^{1/2} + e^{1/2}) |e_m^{1/2} - e^{1/2}| \cdot |\Delta \phi_m| dx dt \\ &= \iint_{\Omega} e_m^{1/2} |e_m^{1/2} - e^{1/2}| \cdot |\Delta \phi_m| dx dt + \iint_{\Omega} e^{1/2} |e_m^{1/2} - e^{1/2}| \cdot |\Delta \phi_m| dx dt \\ &\equiv I_m^1 + I_m^2. \end{aligned}$$

For I_m^1 we obtain ($\|\cdot\|$ denotes the L^2 norm in Ω)

$$\begin{aligned} |I_m^1| &\leq \|e_m^{1/2} \Delta \phi_m\| \cdot \|e_m^{1/2} - e^{1/2}\| \\ &\leq C_4 \|e_m^{1/2} - e^{1/2}\| \end{aligned}$$

by the Schwarz inequality and Lemma 4. But

$$\begin{aligned} |e_m^{1/2} - e^{1/2}|^2 &= |e_m^{1/2} - e^{1/2}| \cdot |e_m^{1/2} - e^{1/2}| \\ &\leq |e_m^{1/2} - e^{1/2}| (e_m^{1/2} + e^{1/2}) \\ &= |e_m - e|. \end{aligned}$$

Hence (μ denotes $(n+1)$ -dimensional Lebesgue measure)

$$\|e_m^{1/2} - e^{1/2}\|^2 \leq \int_{\Omega} |e_m - e| dx dt \leq \mu(\Omega)^{1/2} \|e_m - e\|,$$

again by Schwarz. Since $e_m \rightarrow e$ in $L^2(\Omega)$ it follows that

$$\lim_{m \rightarrow \infty} I_m^1 = 0.$$

For I_m^2 we need a more subtle inequality. First consider.

$$E_m = \{(x, t) \in \Omega \mid |e_m^{1/2} - e^{1/2}| \geq \Theta > 0\},$$

and observe that

$$\begin{aligned}\Theta\mu(E_m) &\leq \iint_{E_m} |e_m^{1/2} - e^{1/2}| \, dx \, dt \\ &\leq \mu(\Omega)^{1/2} \|e_m^{1/2} - e^{1/2}\| \\ &\leq \mu(\Omega)^{3/4} \|e_m - e\|^{1/2}.\end{aligned}$$

It then follows that $e_m^{1/2} \rightarrow e^{1/2}$ in measure. Given arbitrary δ , $\Theta > 0$, choose m so large that $\mu(E_m) < \delta$. Then

$$\begin{aligned}|I_m^2| &= \iint_{E_m} e^{1/2} |e_m^{1/2} - e^{1/2}| \cdot |\Delta\phi_m| \, dx \, dt + \iint_{\Omega - E_m} e^{1/2} |e_m^{1/2} - e^{1/2}| \cdot |\Delta\phi_m| \, dx \, dt \\ &\leq \sup_{\Omega} (e_m^{1/2} + e^{1/2}) \iint_{E_m} \frac{e^{1/2}}{e_m^{1/2}} e_m^{1/2} |\Delta\phi_m| \, dx \, dt \\ &\quad + \theta \iint_{\Omega - E_m} \frac{e^{1/2}}{e_m^{1/2}} e_m^{1/2} |\Delta\phi_m| \, dx \, dt.\end{aligned}$$

Finally, letting C stand for a generic constant not depending on m , δ or Θ , we apply the previous lemmas and Schwarz's inequality to obtain:

$$\begin{aligned}|I_m^2| &\leq C \|e_m^{1/2} \Delta\phi_m\|_{L^2(E_m)} \cdot \left\| \frac{e^{1/2}}{e_m^{1/2}} \right\|_{L^2(E_m)} \\ &\quad + \Theta \|e_m^{1/2} \Delta\phi_m\|_{L^2(\Omega - E_m)} \cdot \left\| \frac{e^{1/2}}{e_m^{1/2}} \right\|_{L^2(\Omega - E_m)} \\ &\leq C \left\| \frac{e^{1/2}}{e_m^{1/2}} \right\|_{L^2(E_m)} + \Theta C \left\| \frac{e^{1/2}}{e_m^{1/2}} \right\|_{L^2(\Omega - E_m)} \\ &\leq C \left\| \frac{e}{e_m} \right\|_{L^2(E_m)}^{1/2} \cdot \mu(E_m)^{1/4} \\ &\quad + \Theta C \left\| \frac{e}{e_m} \right\|_{L^2(\Omega - E_m)}^{1/2} \cdot \mu(\Omega - E_m)^{1/4} \\ &\leq C \delta^{1/4} + C\Theta.\end{aligned}$$

Thus

$$\lim_{m \rightarrow \infty} I_m^2 = 0.$$

This establishes (3.11) and the proof of Theorem 3 is complete.

4. Existence. Let $H^1(G)$ be the Hilbert space of functions having first (strong) derivatives in $L^2(G)$. $H^1(G)$ can be obtained as the completion of functions in $C^\infty(G)$ with respect to the norm

$$\|u\|_{H^1(G)}^2 = \int_G \{|u|^2 + |\nabla_x u|^2\} \, dx.$$

Similarly $H^1(\Omega)$ is the analogous space for the region Ω with norm

$$||u||_{H^1(\Omega)}^2 = \iint_{\Omega} \{|u|^2 + |\nabla_x u|^2 + |u_t|^2\} dx dt.$$

For the following existence theorem we need conditions (i)–(ii) of Section 3 and, for simplicity, the additional assumptions:

(iii) The initial function h , defined in $G(0)$, is bounded and measurable.

(iv) The initial function $h \in H^1(G)$.

Condition (iv), however, can be relaxed (see the remarks at the end of the proof of Theorem 5).

Theorem 4. *There exists a weak solution u of (1.1)–(1.5) in Ω .*

With condition (iv), the solution u constructed in the proof of Theorem 4 will be in $H^1(\Omega)$. In Section 7 we show that u is Hölder continuous. Additional properties of the solution are developed in other sections.

As a preliminary to the proof of Theorem 4, first observe that the continuous functions $a(u)$, $b(u)$ can be approximated, uniformly on compact subsets of R^1 , by a sequence of functions $a_m(u)$, $b_m(u) \in C^\infty(R^1)$ with $a_m(0) = b_m(0) = 0$. Moreover, this can be done so that for all u ,

$$(4.1) \quad 0 < \bar{\gamma}_0 \leq a'_m(u) \quad , \quad b'_m(u) \leq \bar{\gamma}_1 .$$

Also the function h can be approximated by a sequence of functions $h_m \in C^\infty(\bar{G})$ in $H^1(G)$ such that

$$\lim_{m \rightarrow \infty} ||h - h_m||_{H^1(G)} = 0,$$

and

$$\max_{G(0)} |h_m| \leq \sup_{G(0)} |h|.$$

Motivated by (2.1) and (2.2), we consider the following sequence of approximating problems:

$$(4.2) \quad \frac{\partial}{\partial t} a_m(u_m) = \operatorname{div} \operatorname{grad} b_m(u_m) \quad \text{in } \Omega,$$

$$(4.3) \quad u_m = h_m \quad \text{in } G(0),$$

$$(4.4) \quad \frac{\partial}{\partial n} b_m(u_m) = 0 \quad \text{on } S.$$

The existence of a classical solution u_m to (4.2)–(4.4) follows from the work of Ladyzhenskaja and Uraltseva [12; pp. 475–492]. To be more precise, let

$$\begin{aligned} v_m &= b_m(u_m) \\ &= \int_0^{u_m} b'_m(\xi) d\xi. \end{aligned}$$

Since $b'_m > 0$ there is a C^∞ inverse function ζ_m such that $u_m = \zeta_m(v_m)$. Setting

$$c_m(v_m) = \{a'_m[\zeta_m(v_m)]\zeta'_m(v_m)\}^{-1},$$

we see that (4.2)–(4.4) can be reformulated as

$$(4.5) \quad \frac{\partial v_m}{\partial t} = c_m(v_m) \Delta v_m \quad \text{in } \Omega,$$

$$(4.6) \quad v_m = b_m(h_m) \quad \text{in } G(0),$$

$$(4.7) \quad \frac{\partial}{\partial n} v_m = 0 \quad \text{on } S.$$

Note that (4.1) implies that c_m is bounded and strictly positive. Therefore Theorem 7.4 of [12; p. 491] is applicable. This means that solutions v_m exist in a Holder class, and it follows by standard Schauder estimates that the $v_m \in C^\infty$. According to Theorem 7.3 of the same reference (p. 487) there is a constant $C_6 = C_6(T)$ (and hence also constants $C_7 - C_9$), independent of m , such that

$$\begin{aligned} \max_{\bar{\Omega}} |v_m| &\leq C_6(T) \max_{G(0)} |v_m| \\ &\leq C_7(T) \max_{G(0)} |h_m| \end{aligned}$$

Thus we arrive at the useful estimate

$$(4.8) \quad \max_{\bar{\Omega}} |u_m| \leq C_8(T) \sup_{G(0)} |h| \leq C_9(T, h),$$

for $m = 1, 2, \dots$.

Lemma 6. *There is a positive constant $C_{10} = C_{10}(T)$, independent of m , such that*

$$\iint_{\Omega} |u_m|^2 dx dt \leq C_{10}(T).$$

Proof. This follows from (4.8), but we give an independent proof. Observe that

$$\text{grad } a_m(u_m) \cdot \text{grad } b_m(u_m) = a'_m b'_m |\text{grad } u_m|^2 \geq 0.$$

It follows, upon multiplying equation (4.2) by $a_m(u_m)$ and integrating by parts over $G(t)$, that

$$\int_{G(t)} \frac{1}{2} \frac{\partial}{\partial t} |a_m(u_m)|^2 dx = - \int_{G(t)} \text{grad } a_m(u_m) \cdot \text{grad } b_m(u_m) dx \leq 0.$$

Therefore

$$\int_{G(t)} |a_m(u_m)|^2 dx \leq \int_{G(0)} |a_m(h_m)|^2 dx.$$

Since $a_m(0) = 0$ and (4.1) holds, the above inequality implies

$$(4.9) \quad (\bar{\gamma}_0)^2 \int_{G(t)} |u_m|^2 dx \leq (\bar{\gamma}_1)^2 \int_{G(0)} |h_m|^2 dx.$$

Hence

$$\iint_{\Omega} |u_m|^2 dx dt \leq (\bar{\gamma}_1/\bar{\gamma}_0)^2 T \int_{G(0)} |h_m|^2 dx \leq \text{const.},$$

where the existence of the constant is a consequence of the convergence of h_m to h in $H^1(G)$.

Lemma 7. *There is a positive constant C_{11} , independent of m , such that*

$$\iint_{\Omega} \left| \frac{\partial u_m}{\partial t} \right|^2 dx dt \leq C_{11}$$

and for each t , $0 \leq t \leq T$,

$$\int_{G(t)} |\nabla_x u_m|^2 dx \leq C_{11}.$$

Proof. Note that

$$\left[\frac{\partial}{\partial t} b_m(u_m) \right] \left[\frac{\partial}{\partial t} a_m(u_m) \right] = b'_m(u_m) a'_m(u_m) \left| \frac{\partial u_m}{\partial t} \right|^2.$$

This time we multiply equation (4.2) by $(\partial/\partial t)b_m(u_m)$ and integrate by parts to obtain

$$\int_{G(t)} b'_m a'_m \left| \frac{\partial u_m}{\partial t} \right|^2 dx = - \int_{G(t)} \frac{1}{2} \frac{\partial}{\partial t} |\text{grad } b_m(u_m)|^2 dx.$$

Hence for $0 \leq t \leq T$ we have

$$\iint_{\Omega_t} b'_m a'_m \left| \frac{\partial u_m}{\partial t} \right|^2 dx = -\frac{1}{2} \int_{G(t)} (b'_m)^2 |\nabla_x u_m|^2 dx + \frac{1}{2} \int_{G(0)} (b'_m)^2 |\nabla_x h_m|^2 dx.$$

Using (4.1) we conclude that

$$\begin{aligned} (\bar{\gamma}_0)^2 \iint_{\Omega_t} \left| \frac{\partial u_m}{\partial t} \right|^2 dx dt + \frac{1}{2} (\bar{\gamma}_0)^2 \int_{G(t)} |\nabla_x u_m|^2 dx \\ \leq \frac{1}{2} (\bar{\gamma}_1)^2 \int_{G(0)} |\nabla_x h_m|^2 dx \leq \text{const.}, \end{aligned}$$

since $h_m \rightarrow h$ in $H^1(G)$. The assertions of the lemma now follow from the above inequality.

The next lemma is a trivial consequence of Lemmas 6 and 7

Lemma 8. *The sequence of $\{u_m\}$ of approximating functions is bounded,*

$$\|u_m\|_{H^1(\Omega)} < C_{12}(T),$$

in $H^1(\Omega)$.

Proof of Theorem 4. In what follows we shall select various subsequences from $\{u_m\}$ and, to avoid inundation by subscripts, always denote the subsequence again by $\{u_m\}$. Lemma 8 implies, by Rellich's Lemma, that there is a subsequence u_m and a function $u \in L^2(\Omega)$ such that

$$(4.10) \quad \|u_m - u\|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

It also follows from Lemma 8, as is well known, that $u \in H^1(\Omega)$ and that the first (strong) derivatives of u are the weak limits,

$$(4.11) \quad \begin{aligned} \nabla_x u_m &\rightarrow \nabla_x u \quad \text{weakly in } L^2(\Omega), \\ \frac{\partial u_m}{\partial t} &\rightarrow \frac{\partial u}{\partial t} \quad \text{weakly in } L^2(\Omega), \end{aligned}$$

of the corresponding derivatives of u . In particular, $u_m \rightarrow u$ in measure and, by choosing another subsequence, we can arrange that $u_m \rightarrow u$ almost everywhere in Ω . Therefore u can be taken to be bounded in Ω , by (4.8). Also, using (4.9) and Lemma 7, it is not hard to prove that

$$(4.12) \quad \operatorname{ess\,sup}_{0 \leq t \leq T} \int_{G(t)} |\nabla_x u|^2 dx < \infty.$$

In order to complete the proof of Theorem 4 it remains to show that u is a weak solution. Let ϕ be a test function satisfying (2.1) and consider a u_m from the subsequence. Since u_m is a classical solution of (4.2)–(4.4), it is also a weak solution; that is,

$$\iint_{\Omega} \left[a_m(u_m) \frac{\partial \phi}{\partial t} + b_m(u_m) \Delta \phi \right] dx dt + \int_{G(0)} a_m(h_m) \phi dx = 0.$$

Therefore

$$\begin{aligned} \iint_{\Omega} \left[a(u) \frac{\partial \phi}{\partial t} + b(u) \Delta \phi \right] dx dt + \int_{G(0)} a(h) \phi dx \\ = \iint_{\Omega} [a(u) - a_m(u_m)] \frac{\partial \phi}{\partial t} dx dt + \iint_{\Omega} [b(u) - b_m(u_m)] \Delta \phi dx dt \\ + \int_{G(0)} [a(h) - a_m(h_m)] \phi dx \equiv J_m^1 + J_m^2 + J_m^3. \end{aligned}$$

Thus it will suffice to prove that

$$\lim_{m \rightarrow \infty} J_m^k = 0 \quad (k = 1, 2, 3).$$

Consider first

$$J_m^1 = \iint_{\Omega} [a(u) - a_m(u)] \frac{\partial \phi}{\partial t} dx dt + \iint_{\Omega} [a_m(u) - a_m(u_m)] \frac{\partial \phi}{\partial t} dx dt \equiv J'_m + J''_m.$$

Clearly $J'_m \rightarrow 0$ since $a_m \rightarrow a$ uniformly on compact sets and u is bounded. From

(4.1) we obtain

$$[a_m(u) - a_m(u_m)]^2 \leq (\tilde{\gamma}_1)^2 |u - u_m|^2;$$

hence

$$|J_m''| \leq \text{const } \|u - u_m\|_{L^1(\Omega)} \rightarrow 0$$

by Schwarz's Inequality. Thus $J_m^1 \rightarrow 0$. Similar arguments show that $J_m^2 \rightarrow 0$ and $J_m^3 \rightarrow 0$. This completes the proof of Theorem 4.

Remark 1. From the above proof it follows that every subsequence of the original sequence $\{u_m\}$ has itself a subsequence which converges to the (unique) weak solution u . Hence the entire original sequence u_m converges to u .

Remark 2. Throughout this paper we have been assuming that T is finite. But Theorem 3 also implies the uniqueness of a weak solution in Ω_∞ and Theorem 4 implies the existence of a weak solution u_n in Ω_n for $n = 1, 2, \dots$. Obviously $u_m = u_n$ a.e. in Ω_m if $m \leq n$ by Theorem 3. Therefore our existence and uniqueness results are also valid in Ω_∞ for the case $T = \infty$.

5. Stability. Consider two weak solutions $u, \hat{u}$ of (1.1) – (1.5) in Ω corresponding, respectively, to the initial functions $h, \hat{h}$. Hence we assume that h and $\hat{h}$ satisfy the hypotheses of theorem 4. Set

$$\lambda = \sup_{G(0)} \max \{h, \hat{h}\}.$$

Theorem 5. *There is a constant $C_{13}(T, \lambda)$ (which may increase with λ) such that*

$$\|u - \hat{u}\|_{L^1(\Omega)} \leq C_{13}(T, \lambda) \|h - \hat{h}\|_{L^1(G)}$$

Proof. For consistency with our previous notation we set $v = \hat{u}$. Let $f \in C_0^\infty(\Omega)$ and consider the solutions ϕ_m of (3.2)–(3.4). These are allowed test functions. Hence, using (3.1) we have

$$\begin{aligned} \iint_{\Omega} [a(u) - a(v)] \left\{ \frac{\partial \phi_m}{\partial t} + e \Delta \phi_m \right\} dx dt \\ = \iint_{\Omega} \left\{ [a(u) - a(v)] \frac{\partial \phi_m}{\partial t} + [b(u) - b(v)] \Delta \phi_m \right\} dx dt \\ = \int_{G(0)} [a(\hat{h}) - a(h)] \phi_m dx. \end{aligned}$$

Therefore

$$\begin{aligned} (5.1) \quad \iint_{\Omega} [a(u) - a(v)] f dx dt \\ = \int_{G(0)} [a(\hat{h}) - a(h)] \phi_m dx + \iint_{\Omega} [a(u) - a(v)] (e_m - e) \Delta \phi_m dx dt. \end{aligned}$$

As $m \rightarrow \infty$ the second term on the right in (5.1) tends to zero by precisely the same argument used in section 3 to demonstrate (3.11). From the proof of Lemma 3 we see that

$$\max_{\bar{\Omega}} |\phi_m| \leq C_3(T, f)$$

where the constant C_3 may depend on

$$\max_{\bar{\Omega}} |f|,$$

but is independent of m . Hence

$$(5.2) \quad \left| \iint_{\Omega} [a(u) - a(v)] f \, dx \, dt \right| \leq C_3(T, f) \int_{G(0)} |a(\hat{h}) - a(h)| \, dx.$$

Now $[a(u) - a(v)]$ is a bounded measurable function in Ω . It can be approximated in $L^2(\Omega)$ by a sequence $f_i \in C_0^\infty(\Omega)$ such that all the f_i are bounded in Ω by the bound on $[a(u) - a(v)]$ which, in turn, can be estimated in terms of λ by Lemma 1 and (4.8). Thus (5.2) holds for each f_i . Letting $j \rightarrow \infty$ we obtain

$$\iint_{\Omega} [a(u) - a(v)]^2 \, dx \, dt \leq C_3 \int_{G(0)} |a(\hat{h}) - a(h)| \, dx \, dt.$$

An application of Lemma 1 and Schwarz's Inequality then yields

$$\iint_{\Omega} |u - v|^2 \, dx \, dt \leq C_{13}^2 \int_{G(0)} |h - \hat{h}|^2 \, dx$$

for some constant $C_{13} = C_{13}(T, \lambda)$. This completes the proof of Theorem 5.

Remark 1. Using the stability theorem we can easily remove condition (iv) from the hypotheses of the existence Theorem 4. Indeed, let h be an initial function which is only bounded and measurable. Then there is a sequence of functions $h_m \in C^\infty(\bar{G})$ which converges to h in $L^2(G)$ and such that

$$\max_{\bar{G}} |h_m| \leq \sup_{\bar{G}} |h|.$$

According to Theorem 4 there is a weak solution u_m of (1.1)–(1.5) in Ω corresponding to each h_m . The h_m form a Cauchy sequence in $L^2(G)$; so do the u_m (in $L^2(\Omega)$) by Theorem 5. Hence the u_m converges in $L^2(\Omega)$ to a bounded function $u \in L^2(\Omega)$. It is clear that the functions $a(u_m)$, $b(u_m)$ and $a(h_m)$ converge to $a(u)$, $b(u)$ and $a(h)$, respectively, in L^2 . Taking weak limits in

$$\iint_{\Omega} \left[a(u_m) \frac{\partial \phi}{\partial t} + b(u_m) \Delta \phi \right] \, dx \, dt + \int_{G(0)} a(h_m) \phi \, dx = 0,$$

one obtains (2.2). Hence u is a weak solution of (1.1)–(1.5) in Ω .

Remark 2. Similar arguments show that Theorem 5 remains valid without the requirements (iv) on h and $\hat{h}$.

6. Monotone dependence. Let h and $\hat{h}$ be two initial functions satisfying condition (iii) and consider the corresponding weak solutions u and $\hat{u}$ of (1.1)–(1.5) in Ω .

Theorem 6. *If $h \geq \hat{h}$ a.e. in G then $u \geq \hat{u}$ a.e. in Ω .*

Proof. Observe first that it suffices to prove the theorem under the additional assumption that h and $\hat{h}$ satisfy (iv). For $h, \hat{h}$ can be approximated by a sequence of functions $h_i, \hat{h}_i \in C^\infty(\bar{G})$ with $h_i \geq \hat{h}_i$ such that

$$\|h_i - h\|_{L^1(G)} \rightarrow 0, \quad \|\hat{h}_i - \hat{h}\|_{L^1(G)} \rightarrow 0,$$

as $j \rightarrow \infty$. Then the corresponding weak solutions are such that $u_i \geq \hat{u}_i$ a.e. by the theorem, since $h_i, \hat{h}_i$ satisfy (iv). But the stability theorem implies that $u_i \rightarrow u$ and $\hat{u}_i \rightarrow \hat{u}$ in $L^2(\Omega)$. Taking subsequences which converge almost everywhere, we conclude that $u \geq \hat{u}$ a.e. in Ω .

In order to prove the theorem for $h, \hat{h} \in H^1(G)$, we recall the techniques used to prove theorem 4: Introduce smooth approximations $a_m, b_m, h_m, \hat{h}_m$ to $a, b, h, \hat{h}$, respectively with $h_m \geq \hat{h}_m$. With c_m as in (4.5)–(4.7), consider the solutions v_m of

$$(6.1) \quad \frac{\partial v_m}{\partial t} = c_m(v_m) \Delta v_m + \frac{1}{m} \quad \text{in } \Omega,$$

$$(6.2) \quad v_m = b_m \left(h_m + \frac{1}{m} \right) \quad \text{in } G(0),$$

$$(6.3) \quad \frac{\partial}{\partial n} v_m = \frac{1}{m} \quad \text{on } S,$$

and the solutions ϑ_m of

$$(6.4) \quad \frac{\partial \vartheta_m}{\partial t} = c_m(\vartheta_m) \Delta \vartheta_m \quad \text{in } \Omega,$$

$$(6.5) \quad \vartheta_m = b_m(\hat{h}_m) \quad \text{in } G(0),$$

$$(6.6) \quad \frac{\partial}{\partial n} \vartheta_m = 0 \quad \text{on } S.$$

The lemma of Westphal–Prodi [7; pp. 52–53] implies that $v_m > \vartheta_m$ in Ω . Therefore $u_m > \hat{u}_m$ in Ω since $\zeta'_m > 0$.

It remains to show that $u_m \rightarrow u$ a.e. and $\hat{u}_m \rightarrow \hat{u}$ a.e. in Ω . For $\hat{u}_m$ we use exactly the same proof as was used in the proof of Theorem 4 to conclude that a subsequence $\hat{u}_{m'}$ converges almost everywhere to a weak solution $\hat{u}'$. But $\hat{u} = \hat{u}'$ a.e. by Theorem 3 and it follows that $\hat{u}_m \rightarrow \hat{u}$ a.e. For $u_m \rightarrow u$ a.e. the proof involves only slight alterations: Lemmas 6–8 remain valid with enlarged constants. The important thing to note here is that an estimate of the type (4.8) also holds in this case. Thus we obtain a subsequence $u_{m'}$ converging almost everywhere to a function u' . The proof that u' is a weak solution is changed only

by addition of the terms

$$J_m^4 \equiv - \int_S \frac{1}{m} \phi \, dS_x \, dt$$

and

$$J_m^5 \equiv - \iint_\Omega \frac{1}{m} \phi \, dx \, dt.$$

It is clear that $J_m^4 \rightarrow 0$ and $J_m^5 \rightarrow 0$ as $m \rightarrow \infty$. Thus, as before, $u = u'$ a.e. and $u_m \rightarrow u$ a.e. in Ω . The proof of Theorem 6 is complete.

7. Regularity. Let $0 < \alpha < 1$ and $\Omega' \subset \Omega$ be any subdomain. Consider the Banach space $C^{\alpha, \alpha/2}(\Omega')$ of Holder continuous functions in Ω' with exponent α in the x -variables and exponent $\alpha/2$ in the t -variable. The norm $\| \cdot \|_{\Omega'}^\alpha$ in $C^{\alpha, \alpha/2}(\Omega')$ is given by

$$\|u\|_{\Omega'}^\alpha \equiv \sup_{\Omega'} |u| + \langle u \rangle_{x, \Omega'}^\alpha + \langle u \rangle_{t, \Omega'}^{\alpha/2},$$

where

$$\langle u \rangle_{x, \Omega'}^\alpha \equiv \sup_{(x', t), (x'', t) \in \Omega'} \frac{|u(x', t) - u(x'', t)|}{|x' - x''|^\alpha},$$

$$\langle u \rangle_{t, \Omega'}^{\alpha/2} \equiv \sup_{(x, t'), (x, t'') \in \Omega'} \frac{|u(x, t') - u(x, t'')|}{|t' - t''|^{\alpha/2}}.$$

In the following theorem we assume that Ω' is separated from $S \cup \overline{G(0)}$ by a positive distance d .

Theorem 7. *The weak solution u of (1.1)–(1.5) in Ω is Holder continuous; $u \in C^{\alpha, \alpha/2}(\Omega')$ for some α .*

Proof. Recall the smooth approximations a_m, b_m, h_m to a, b, h and the corresponding solutions u_m of (4.2)–(4.4). Set $w_m(x, t) = a_m(u_m(x, t))$ and let η_m be the C^∞ inverse function, $u_m = \eta_m(w_m)$, of a_m . It is easy to see, because of (4.1), that w_m is the solution of

$$(7.1) \quad \frac{\partial w_m}{\partial t} = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left\{ D_m(w_m) \frac{\partial w_m}{\partial x_i} \right\} \quad \text{in } \Omega,$$

$$(7.2) \quad w_m = a_m(h_m) \quad \text{in } G(0),$$

$$(7.3) \quad \frac{\partial w_m}{\partial n} = 0 \quad \text{on } S,$$

where $D_m(w_m)$ is given by

$$D_m(w_m) = \frac{b'_m(\eta_m(w_m))}{a'_m(\eta_m(w_m))}.$$

Set $\nu_1 = \bar{\gamma}_0/\bar{\gamma}_1$ and $\nu_2 = \bar{\gamma}_1/\bar{\gamma}_0$. From (4.1) it then follows that

$$(7.4) \quad 0 < \nu_1 \leq D_m(w_m) \leq \nu_2,$$

independent of m .

Equation (7.1) is in divergence form. Hence, because of (4.8) and (7.4), the Holder estimate of Ladyzhenskaja and Ural'tseva [12; p 204] can be applied. It follows that there is an exponent $\alpha = \alpha(n, \nu_1, \nu_2)$ and a constant $C_{14} = C_{14}(n, C_9, \nu_1, \nu_2, d)$ such that

$$|w_m|_{\Omega'}^\alpha \leq C_{14},$$

for $m = 1, 2, \dots$. Hence according to (4.1) there exists a constant $C_{15}(d)$ such that

$$|u_m|_{\Omega'}^\alpha \leq C_{15}(d), \quad (m = 1, 2, \dots).$$

This means that the sequence $\{u_m\}$ is equicontinuous and uniformly bounded in Ω' . Therefore a subsequence (but really the whole sequence by uniqueness) of u_m converges uniformly to u in Ω' . Thus u is Holder continuous in Ω' and, in fact,

$$|u|_{\Omega'}^\alpha \leq C_{15}(d).$$

The proof of Theorem 7 is complete,

Remark 1. A uniform Holder estimate for u on $\bar{\Omega}$ can be obtained under the additional assumption that the initial function h is Holder continuous on $\overline{(G0)}$.

Remark 2. The continuity of u implies that the sets

$$E^+ = \{(x, t) \in \Omega \mid u(x, t) > 0\}$$

and $E^- = \{(x, t) \in \Omega \mid u(x, t) < 0\}$ are open. The smoothness of u in $E^+(E^-)$ is limited, therefore, only by the smoothness of the functions $\alpha_i(u)$, $k_i(u)$, ($i = 1, 2$) in $[0, \infty)(-\infty, 0]$. Hence, for example, the weak solution u will actually be a C^∞ classical solution of the appropriate equation in each of the sets E^+ and E^- , provided that $\alpha_i(u)$, $k_i(u)$, ($i = 1, 2$) are C^∞ for $u \neq 0$. In general one should not expect the derivatives of u to be continuous across the set $\{u = 0\}$.

8. Persistence of the concentration relations. According to Remark 2 at the end of the proof of Theorem 4, our weak solution u exists in Ω_∞ (for all time). The discussion of this section is connected with the following intuitive considerations: Suppose the total amount of substance 1 initially present is more than is needed to completely react with the total amount of substance 2 initially present. Then substance 1 should never be completely used up but substance 2 ought to disappear after a finite length of time. After that there is no longer any free boundary. Suppose, on the other hand, that there is an exact balance in the total amount of the two substances present initially. Then neither substance can

be depleted unless the other is also. Hence in that case the free boundary should exist for all time.

Lemma 9. For $0 \leq t_0 \leq T$ and any test function ϕ satisfying (2.1),

$$(8.1) \quad \int_{t_0}^T \int_{G(t)} \left[a(u) \frac{\partial \phi}{\partial t} + b(u) \Delta \phi \right] dx dt + \int_{G(t_0)} a(u) \phi dx = 0.$$

Proof. Recall the approximations a_m, b_m, h_m, u_m used in (4.2)–(4.4) to construct u . Multiply (4.2) by ϕ and integrate by parts in the region $G \times [\tau, T]$:

$$\int_{\tau}^T \int_G \left[a_m(u_m) \frac{\partial \phi}{\partial t} + b_m(u_m) \Delta \phi \right] dx dt + \int_{G(\tau)} a_m(u_m) \phi dx = 0.$$

If the above expression is integrated on τ from 0 to t_0 , then the same arguments as those used for J_m^k in the proof of Theorem 4 can be applied. Letting $m \rightarrow \infty$ we obtain the result

$$\int_0^{t_0} \left\{ \int_{\tau}^T \int_G \left[a(u) \frac{\partial \phi}{\partial t} + b(u) \Delta \phi \right] dx dt + \int_{G(\tau)} a(u) \phi dx \right\} d\tau = 0.$$

The above relation implies that (8.1) holds for almost all t_0 . Since u is continuous, (8.1) is valid for all $t_0, 0 \leq t_0 \leq T$.

Remark. If u is the weak solution in $\Omega = G \times (0, T)$ corresponding to the initial data h in $G(0)$, then it follows from (8.1) that in $G \times (t_0, T)$ u is the weak solution corresponding to the initial data $u(x, t_0)$ on $G(t_0)$. Hence if there is some t_0 for which $u \equiv 0$ in $G(t_0)$, then $u \equiv 0$ in $G \times [t_0, \infty)$ by the uniqueness theorem.

The next lemma is an easy consequence of (2.2) and Lemma 9.

Lemma 10. For $0 \leq t_0 < \infty$ and any test function ϕ such that $\partial \phi / \partial n = 0$ on S ,

$$(8.2) \quad \int_{G(t_0)} a(u) \phi dx = \int_{G(0)} a(h) \phi dx + \int_0^{t_0} \int_G \left[a(u) \frac{\partial \phi}{\partial t} + b(u) \Delta \phi \right] dx dt.$$

In what follows μ_x denotes n -dimensional Lebesgue measure in G (or $G(t)$). Define the sets

$$E^+(t) = \{u > 0\} \cap G(t),$$

$$E^-(t) = \{u < 0\} \cap G(t),$$

and consider the constant

$$H \equiv \int_{G(0)} a(h) dx.$$

Setting $\phi \equiv 1$ in (8.2) we obtain

$$(8.3) \quad \int_{G(t)} a(u) \, dx = H \quad \text{for } t \geq 0.$$

Theorem 8.

- (a) If $H > 0$ then $\mu_x\{E^+(t)\} > 0$ for all $t \geq 0$.
 (b) If $H < 0$ then $\mu_x\{E^-(t)\} > 0$ for all $t \geq 0$.
 (c) If $H = 0$ and, for some t_0 , $\mu_x\{E^-(t_0)\} = 0$ ($\mu_x\{E^+(t_0)\} = 0$) then $\mu_x\{E^+(t_0)\} = 0$ ($\mu_x\{E^-(t_0)\} = 0$) also.

Proof. First consider (a): If there were some t_0 at which $\mu_x\{E^+(t_0)\} = 0$ then, by (8.3),

$$0 < H = \int_{E^-(t_0)} a(u) \, dx \leq 0.$$

This is a contradiction; hence $\mu_x\{E^+(t)\} > 0$ for all t . A similar argument proves (b). For (c) we have

$$0 = H = \int_{E^+(t_0)} a(u) \, dx$$

if, say, $\mu_x\{E^-(t_0)\} = 0$. But that is impossible unless $\mu_x\{E^+(t_0)\} = 0$. The proof of Theorem 8 is complete.

The remainder of this section is based on the assumption that, uniformly for $x \in G$, we have

$$(8.4) \quad \lim_{t \rightarrow \infty} u(x, t) = c.$$

In order to determine the constant c , observe that

$$a(c)\mu_x\{G\} = \int_{G(\infty)} a(c) \, dx = H.$$

Therefore,

$$c = a^{-1}(\{H/\mu_x\{G\}\}).$$

It follows that $c = 0$ if and only if $H = 0$, $c > 0$ if and only if $H > 0$, and $c < 0$ if and only if $H < 0$. As a corollary of these considerations, we obtain:

Theorem 9.

- (a) If $H > 0$ then $E^+(t) \equiv G(t)$, for sufficiently large t .
 (b) If $H < 0$ then $E^-(t) \equiv G(t)$ for sufficiently large t .

9. A numerical procedure. In order to approximate u , it suffices to approximate the solution u_m of (4.2)–(4.4). Recalling Section 7, an approximation of u_m can be obtained from an approximation of the solution $w_m = a_m(u_m)$ of (7.1)–(7.3). This can be done by finite difference techniques. In addition the divergence structure and boundary conditions in (7.1)–(7.3) indicate that the Galerkin

procedures of Jim Douglas and T. Dupont for non-linear parabolic equations [6] would be worth considering. Since [6] has not yet appeared, it seems reasonable to briefly describe one of their techniques as an example of a possible method of approximating w_m .

Let

$$(9.1) \quad \langle f, g \rangle = \int_G f(x)g(x) \, dx$$

and

$$(9.2) \quad a(f; g, v) = \int_G D_m(f) \sum_{i=1}^N \frac{\partial g}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx.$$

Suppose that $v_1, \dots, v_N \in H^1(G)$ and are linearly independent. Let $\mathfrak{M}$ denote the subspace spanned by $v_1, \dots, v_N$. An approximation

$$(9.3) \quad W_i(x) = \sum_{k=1}^N \alpha_{k,i} v_k(x)$$

for w_m at $t = t_j = j \Delta t$, $j = 1, \dots, M$, $\Delta t = T/M$, can be obtained by solving the following systems of linear equations for the $\alpha_j = (\alpha_{1,j}, \dots, \alpha_{N,j})$:

$$(9.4) \quad \left\langle \frac{Z_{i+1} - W_i}{\Delta t}, V \right\rangle + a(W_i; \tfrac{1}{2}Z_{i+1} + \tfrac{1}{2}W_i, V) = 0,$$

$$\left\langle \frac{W_{i+1} - W_i}{\Delta t}, V \right\rangle + a(\tfrac{1}{2}Z_{i+1} + \tfrac{1}{2}W_i, \tfrac{1}{2}W_{i+1} + \tfrac{1}{2}W_i, V) = 0,$$

for $j \geq 0, \quad V \in \mathfrak{M},$

$$\langle W_0, V \rangle = \langle a_m(h_m), V \rangle, \quad V \in \mathfrak{M},$$

where the Z_{i+1} denotes a linear combination of the form (9.3) which is determined first as a prediction of W_{i+1} and is then used in the second (correction) equation to actually determine the function W_{i+1} . It is important to mention again that (9.4) is reducible to a sequence of linear systems of equations for the α_j , $j = 0, 1, 2, \dots, M$. Douglas and Dupont obtain estimates of second order correctness in Δt and spatial correctness depending upon $\mathfrak{M}$.

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Date communicated: DECEMBER 23, 1969.