Locally Compact Spaces Associated with Abstract Convolutions

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1. The subject of this paper was suggested by a reading of the paper [3] by C. Ionescu Tulcea who proved that the space of characters of a certain algebra of measures was locally compact for the topology of uniform convergence on compacta. This result was subsequently generalized in [4], [5], and [2]. The present theorem further extends this work.

2. Let X be a locally convex topological vector space. Denote by $\mathcal{K}(X)$ the set of linear mappings, T , of X into X such that if B is a bounded subset of X , then $T(B)$ is relatively compact in X . Let Y be a vector space in duality with X by a bilinear mapping $(x, y) \rightarrow \langle x, y \rangle$ of $X \times Y$ into $\mathbb{C}$. We suppose for each y in Y the mapping $x \rightarrow \langle x, y \rangle$ is continuous.

3. **Left convolutions.** Let L be a subspace of Y such that the mappings $x \rightarrow \langle x, \nu \rangle$ for $\nu \in L$ separates the points of X . A mapping $(\mu, \nu) \rightarrow \mu\nu$ of $Y \times L$ into Y is called a *left convolution* on L if:

- (i) For each ν in L the mapping $\mu \rightarrow \mu\nu$ is linear and $\sigma(Y, X)$ continuous on Y ;
- (ii) For each ν in L the operator ν' defined by the equations $\langle \nu'(x), \mu \rangle = \langle x, \mu\nu \rangle$ for each x in X and μ in Y is a member of $\mathcal{K}(X)$.

Below we always assume that L^* (the algebraic dual of L) is endowed with the topology $\sigma(L^*, L)$. Denote by E the set of all non-zero x in X such that $\langle x, \mu\nu \rangle = \langle x, \mu \rangle \langle x, \nu \rangle$ for all μ in Y and ν in L . Suppose that B is a subset of E which is bounded in X and let $W = \bar{B} - \{0\}$. Let h be the mapping with domain W and range contained in L^* defined by $\langle \mu, h(x) \rangle = \langle x, \mu \rangle$ for each x in W and μ in L .

4. Theorem. *The mapping h is a homeomorphism and W is locally compact.*

Proof. Clearly h is a continuous injection. We will show that $h(W) \cup \{0\}$ is closed, that h^{-1} is continuous and finally that $h(W) \cup \{0\}$ is compact. Suppose y is a non-zero element of $\overline{h(W)}$. Then there is ν in L such that $\langle \nu, y \rangle \neq 0$. Let V be the neighborhood of y defined by

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$$V = \{z: |\langle \nu, z - y \rangle| \leq \frac{1}{2} |\langle \nu, y \rangle|\}.$$

If $\mathcal{F}$ is a filter base on $h(W) \cap V$ converging to y , we have for each x in $\cup h^{-1}(\mathcal{F})$,

$$|\langle x, \nu \rangle| \geq \frac{1}{2} |\langle \nu, y \rangle|,$$

hence $\cup h^{-1}(\mathcal{F})$ is contained in the complement of a neighborhood of zero in X . Since for each x in E and μ in Y we have

$$\langle x, \mu \rangle \langle x, \nu \rangle = \langle x, \mu\nu \rangle = \langle \nu^t(x), \mu \rangle,$$

we conclude that

$$x = \langle x, \nu \rangle^{-1} \nu^t(x),$$

for every x in $\cup h^{-1}(\mathcal{F})$. Since $\cup h^{-1}(\mathcal{F})$ is bounded in X we deduce that $\cup h^{-1}(\mathcal{F})$ is relatively compact in X , hence that the filter base $h^{-1}(\mathcal{F})$ has an adherent point x_0 in W . Take a filter base $\mathcal{G}$ which is finer than $h^{-1}(\mathcal{F})$ and which converges to x_0 . By the continuity of h , $h(\mathcal{G})$ converges to $h(x_0)$, but since $h(\mathcal{G})$ is finer than $\mathcal{F}$, $h(\mathcal{G})$ converges to y ; therefore $h(x_0) = y$. Since h is injective we conclude that x_0 is the only adherent point of $h^{-1}(\mathcal{F})$, that is, $h^{-1}(\mathcal{F})$ actually converges to x_0 . Since y was arbitrary in $h(W)$ except for being non-zero, we conclude that $h(W) \cup \{0\}$ is closed, and that h^{-1} is continuous. Thus h is a homeomorphism. Since L^* may be regarded as the dual of a barreled space (L with the strongest locally convex topology) we may use Theorem 1, Chap. IV, Section 2 of [1] to obtain compactness of the bounded set $h(W) \cup \{0\}$. This completes the proof.

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