Uniformly Bounded Cosine Functions in Hilbert Space*

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1. Introduction. Let H design an arbitrary Hilbert space. It has been shown by B. Sz. Nagy ([15]) that if $\{T(t); -\infty < t < \infty\}$ is a family of bounded linear operators in H satisfying the "exponential equations" $T(0) = I$, $T(s+t) = T(s)T(t)$ and if $|T(t)| \leq K$ for $-\infty < t < \infty$ then there exists a self adjoint operator Q in H ($K^{-1}I \leq Q \leq KI$) such that $QT(t)Q^{-1}$ is *unitary* for all t . We prove in the present note a result of the same sort for operator-valued functions $\{C(t), -\infty < t < \infty\}$ satisfying the "cosine equations"

$$(1.1) \quad C(0) = I, \quad 2C(s)C(t) = C(s+t) + C(s-t)$$

that are uniformly bounded in $]-\infty, \infty[$, namely, the existence of an operator Q of the same character as the one above and such that $QC(t)Q^{-1}$ is *self adjoint* for all t (Theorem 3.1). The proof is a combination of the ideas of [15] with some methods borrowed from the theory of almost periodic functions. A discrete version is also proved. (Theorem 3.6). Under additional "smoothness" conditions on the t -dependence of $C(t)$ a more precise representation is obtained in Section 4; a representation for the almost periodic case appears in Section 5. Finally, the results are applied to second order operational differential equations.

The version of the Banach integral constructed in Section 2 will only be used in the case where no hypotheses on t -dependence of $C(\cdot)$ are made and can thus be skipped by anyone interested only in the results in Section 6.

Abridging slightly the nomenclature of [16] we will call a *cosine function* (c.f.) in H any function $t \rightarrow C(t)$ defined in $]-\infty, \infty[$ with values bounded operators in H satisfying (1.1) for all s, t . A few properties of the family $\{C(t)\}$ can be immediately derived from (1.1). For instance, setting $s = 0$ and subtracting $C(t)$ from the equality thus obtained we get $C(t) = C(-t)$, that is, $C(\cdot)$ is an even function. But then $2C(s)C(t) = C(s+t) + C(s-t) = C(t+s) + C(t-s) = 2C(t)C(s)$, i.e. $C(s)$ and $C(t)$ commute with each other for any value of s, t .

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We make use in Sections 4, 5 and 6 of a few elementary results on cosine functions and the abstract Cauchy problem for second order operational differential equations, all of which can be found in [5].

2. Banach Integrals. Let Φ be the linear space of all complex-valued functions $s \rightarrow f(s)$ defined and bounded in $]-\infty, \infty[$, Π the subspace of Φ consisting of all periodic functions with period 1. We shall need in what follows a family $f \rightarrow \int_{\alpha}^{\beta} f(s) ds$ of linear functionals in Φ depending on two real parameters α, β ($\alpha \leq \beta$) and satisfying the following conditions:

$$(2.1) \quad \int_{\alpha}^{\beta} f(s) ds = \beta - \alpha$$

if $f(s) = 1$ for all s .

$$(2.2) \quad \int_{\alpha}^{\beta} f(s) ds \leq \int_{\alpha}^{\beta} g(s) ds$$

if $f(s) \leq g(s)$ in $[\alpha, \beta]$.

$$(2.3) \quad \int_{\alpha}^{\gamma} f(s) ds + \int_{\gamma}^{\beta} f(s) ds = \int_{\alpha}^{\beta} f(s) ds$$

if $\alpha < \gamma < \beta$.

$$(2.4) \quad \int_{\alpha+h}^{\beta+h} f(s) ds = \int_{\alpha}^{\beta} f(s+h) ds$$

for any real h and any $f \in \Phi$.

$$(2.5) \quad \int_{\alpha/2}^{\beta/2} f(2s) ds = \frac{1}{2} \int_{\alpha}^{\beta} f(s) ds.$$

It can be constructed by means of Banach limits as follows. Let f be any function in Φ , α, β two real numbers such that $\alpha \leq \beta$ and let $\omega > 0$. Assume for the moment that

$$(2.6) \quad -\omega < 2\alpha, \quad 2\beta < \omega$$

Define

$$(2.7) \quad f_{\alpha, \beta, \omega}(s) = 0$$

for $-\frac{1}{2} \leq s < \alpha/\omega$ and for $\beta/\omega \leq s < \frac{1}{2}$,

$$f_{\alpha, \beta, \omega}(s) = \omega f(\omega s)$$

for $\alpha/\omega \leq s < \beta/\omega$, and extend it to the real axis in such a way that the resulting function is 1-periodic (that is, belongs to Π). As for the case in which one of the two equalities (2.6) fails to hold, define $f_{\alpha, \beta, \omega}$ by means of (2.7) but now for all real s . Define, for any $f \in \Phi$

$$(2.8) \quad \int_{\alpha}^{\beta} f(s) ds = \text{LIM}_{n \rightarrow \infty} F(f_{\alpha, \beta, 2^n})$$

where

$$F(f_{\alpha, \beta, 2^n}) = \text{LIM}_{s \rightarrow \infty} f_{\alpha, \beta, 2^n}(s) \quad \text{and} \quad \text{LIM}_{n \rightarrow \infty} \text{ (resp. LIM}_{s \rightarrow \infty} \text{)}$$

indicates a fixed Banach limit of bounded sequences (resp. of bounded functions); see [2], Ch. II, §3 or [8], Ch. 2, 2.10 for full details about these limits. A routine check of (2.8) shows that the properties (2.1)–(2.5) are in fact satisfied. We note in passing a simple consequence of (2.1) and (2.2) that will be useful in the sequel: if $|f(s)| \leq M$ in $\alpha \leq s \leq \beta$, then

$$(2.9) \quad \left| \int_{\alpha}^{\beta} f(s) ds \right| \leq M(\beta - \alpha)$$

It can be easily shown on the basis of (2.1), (2.2) and (2.3) that the “integrals” just defined coincide with the ordinary Riemann integrals whenever the latter exist. This might not necessarily be the case for Lebesgue integrals. (See [2], Ch. II, §3 for comments on a similar situation).

3. The Non-Measurable Case.

Theorem 3.1. *Let C be a cosine function in H . Assume that*

$$(3.1) \quad |C(t)| \leq K, \quad -\infty < t < \infty$$

Then there exists a self adjoint operator Q , $((\sqrt{3}(2K+1))^{-1}I \leq Q \leq KI)$ such that

$$(3.2) \quad QC(t)Q^{-1}$$

is self adjoint for every t .

For the proof, we shall need two preliminary results.

Lemma 3.2. *Let $f \in \Phi$. Then*

$$(3.3) \quad \text{LIM}_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(s) ds$$

exists and equals

$$(3.4) \quad \text{LIM}_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(s+h) ds$$

for any real h .

Proof. It follows from (2.9) that the function of T in (3.3) is bounded, thus its Banach limit as $T \rightarrow \infty$ exists. Observe next that, by virtue of (2.3) and (2.4), the difference between the integral in (3.4) and the one in (3.3) for a given value of T reduces to

$$(3.5) \quad \frac{1}{T} \int_T^{T+h} f(s) ds - \frac{1}{T} \int_0^h f(s) ds.$$

A new application of (2.9) shows that (3.5) converges to zero as $T \rightarrow \infty$ (thus has Banach limit zero). Linearity of $\text{LIM}_{T \rightarrow \infty}$ completes the proof.

Lemma 3.3. Let $T > 0$, $\epsilon \leq (2K + 1)^{-1}$, $u \in E$,

$$e = (e(\epsilon, T)) = \{t; 0 \leq t \leq T, |C(t)u| < \epsilon |u|\}.$$

Then

$$(3.6) \quad \int_0^T \chi_\epsilon(s) ds \leq \frac{2T}{3},$$

χ_ϵ the characteristic function of e .

Proof. Set $t = s = \eta/2$ in (1.1). We obtain

$$2C(\eta/2)^2 = C(\eta) + I$$

Thus, if $\eta \in e$,

$$2K |C(\eta/2)u| \geq 2 |C(\eta/2)^2 u| \geq (1 - \epsilon) |u|.$$

Accordingly, if $\epsilon \leq (2K + 1)^{-1}$,

$$|C(\eta/2)u| \geq \epsilon |u|$$

that is, $\eta/2 \notin e$. The functions $\chi_\epsilon(s)$, $\chi_\epsilon(2s)$, have then disjoint support. Hence

$$(3.7) \quad \int_0^T \chi_\epsilon(s) ds + \int_0^T \chi_\epsilon(2s) ds \leq T.$$

However, taking (2.5) into consideration,

$$\int_0^T \chi_\epsilon(2s) ds = \frac{1}{2} \int_0^{2T} \chi_\epsilon(s) ds = \frac{1}{2} \int_0^T \chi_\epsilon(s) ds,$$

which, together with (3.7) implies (3.6).

Proof of Theorem 3.1. Define, for $u, v \in H$

$$(3.8) \quad [u, v] = \text{LIM}_{T \rightarrow \infty} \frac{1}{T} \int_0^T (C(s)u, C(s)v) ds,$$

the existence of the Banach limit being assured by Lemma 3.2. (Here $(\cdot, \cdot)$ is the scalar product in H .) Plainly the map $u \rightarrow [u, v]$ is linear in u for v fixed, $[v, u] = \overline{[u, v]}$, $|[u, v]| \leq K^2 |u| |v|$. Then ([5], Ch. V, §84) there exists a self adjoint operator P , $|P| \leq K^2$ such that

$$(3.9) \quad [u, v] = (Pu, v).$$

By virtue of (2.2) and of Lemma 3.3, if $T > 0$

$$\begin{aligned} \int_0^T |C(s)u|^2 ds &\geq \frac{1}{(2K + 1)^2} \int_0^T (1 - \chi_\epsilon(s)) ds \\ &\geq \frac{T}{3(2K + 1)^2} |u|^2. \end{aligned}$$

Accordingly, it follows from the definition (3.8) of $[u, v]$ and from (3.9) that

$$(Pu, u) \geq \frac{1}{3(2K + 1)^2} |u|^2.$$

Let now Q be the self adjoint, positive square root of P . Plainly,

$$(\sqrt{3}(2K + 1))^{-1}I \leq Q \leq KI.$$

Let t be any real number, $u, v \in H$. We obtain, making use of the cosine equations (1.1) and of Lemma 3.2 that

$$\begin{aligned} (3.10) \quad (PC(t)u, v) &= \text{LIM}_{T \rightarrow \infty} \frac{1}{T} \int_0^T (C(s)C(t)u, C(s)v) ds \\ &= \frac{1}{2} \text{LIM}_{T \rightarrow \infty} \frac{1}{T} \int_0^T (C(s + t)u, C(s)v) ds \\ &\quad + \frac{1}{2} \text{LIM}_{T \rightarrow \infty} \frac{1}{T} \int_0^T (C(s - t)u, C(s)v) ds \\ &= \frac{1}{2} \text{LIM}_{T \rightarrow \infty} \frac{1}{T} \int_0^T (C(s)u, C(s - t)v) ds \\ &\quad + \frac{1}{2} \text{LIM}_{T \rightarrow \infty} \frac{1}{T} \int_0^T (C(s)u, C(s + t)v) ds \\ &= \text{LIM}_{T \rightarrow \infty} \frac{1}{T} \int_0^T (C(s)u, C(s)C(t)v) ds \\ &= (Pu, C(t)v). \end{aligned}$$

Consequently,

$$PC(t) = C^*(t)P.$$

Pre- and post-multiplying by Q^{-1} and taking into account that $(QC(t)Q^{-1})^* = Q^{-1}C^*(t)Q$ we see that (3.2) is self adjoint for all t , as claimed.

Remark 3.4. The argument in Lemma 3.3 is an adaptation of one used by E. Giusti in [7] (Lemma 2.4) in connection with periodic cosine functions.

Remark 3.5. There is a discrete analogue of Theorem 3.1, namely

Theorem 3.6. Let $\{C_n, n = \dots - 1, 0, 1, \dots\}$ be a family of bounded operators in H such that

$$(3.11) \quad C_0 = I, \quad 2C_m C_n = C_{m+n} + C_{m-n} \\ |C_n| \leq K$$

for $m, n = \dots, -1, 0, 1, \dots$. Then there exists a self adjoint operator Q , $(2(2K + 1))^{-1}I \leq Q \leq KI$ and a unitary operator U such that

$$(3.12) \quad C_n = \frac{1}{2}Q^{-1}(U^n + U^{-n})Q$$

$$n = \dots, -1, 0, 1, \dots$$

The proof is rather similar to the one for the continuous case and will thus only be sketched. The operator P is now defined by

$$(3.13) \quad (Pu, v) = \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n (C_n u, C_n v).$$

Again P is a self adjoint operator in H with $|P| \leq K^2$. Proceeding as in the proof of Lemma 3.3 one can show that, if $|C_m u| < \epsilon |u|$ for some integer m , $\epsilon \leq (2K+1)^{-1}$ and $u \in E$ then $|C_{2m} u| \geq \epsilon |u|$; hence the number of integers between 0 and n for which $|C_m u| < \epsilon |u|$ is at most equal to the number of even integers between $n/2$ and n . But this number is asymptotic to $n/4$ as $n \rightarrow \infty$, thus

$$(Pu, u) \geq \frac{1}{4(2K+1)^2} |u|^2.$$

Denoting now by Q the positive self adjoint square root of P we see, after a short computation entirely similar to (3.10) and following lines that

$$(3.14) \quad F_n = QC_n Q^{-1}$$

is self adjoint for all n . Observe next that as a consequence of (3.14) F_n satisfies as well (3.11). In particular,

$$F_{2n} = 2F_n^2 - I.$$

By the Spectral Mapping Theorem ([15], Ch. IX, p. 421), if $\lambda \in \sigma(F_1)$ and we define, inductively, $\lambda_1 = \lambda$, $\lambda_m = 2\lambda_m^2 - 1$, then $\lambda_m \in \sigma(F_{2^{m-1}})$. But it is not difficult to see that $\lambda_m \rightarrow \infty$ if $|\lambda| > 1$. On the other hand, it follows from uniform boundedness of $\{C_n\}$ and from (3.14) that the operators F_n (hence the sets, $\sigma(F_n)$) are uniformly bounded, which shows that $\sigma(F_1) \subseteq [-1, 1]$. Let now $\arccos \lambda$ be the function inverse to $\cos \lambda$, say, in the interval $[-\pi/2, \pi/2]$ and define

$$(3.15) \quad G_n = \cos(nB) = \frac{1}{2}(U^n + U^{-n}),$$

where $B = \arccos(F_1)$ (see [15], Ch. IX for the necessary notions of functional calculus) and U is the unitary operator e^{iB} . Plainly $G_0 = I$, $G_1 = F_1$. It follows from the equality

$$F_{n+1} = 2F_n F_1 - F_{n-1}$$

(which is of course a particular case of (3.11)) and from the fact that the family $\{G_n\}$ satisfies exactly the same equation (consequence of its definition (3.15) and of the functional calculus) that $F_n = G_n$ for all n , which ends the proof of Theorem 3.6.

Remark 3.7. The result of Sz.-Nagy mentioned in Section 1 (see [15]) has been extended to group representations, under adequate restrictions either on

the group or on the representation (see for instance [4], [10]). It is not clear whether a similar extension could be carried out in the cosine function case.

4. The Measurable Case.

Theorem 4.1. *Assume $C(\cdot)$ satisfies all the hypotheses of Theorem 3.1 and that, in addition, the function $t \rightarrow C(t)u$ is strongly measurable in H for all $u \in H$ ([9], Ch. III, §1, 3.5). Then*

$$(4.1) \quad C(t) = Q^{-1} \cos(tS)Q$$

S a nonnegative (and possibly unbounded) self adjoint operator, Q as in Theorem 3.1.

Proof. We begin by observing that, by virtue of the hypothesis of strong measurability and of Lemma 5.2 of [5] $C(\cdot)u$ is continuous for all $u \in H$. Then, if Q is the operator provided by Theorem 3.1, $F(\cdot) = QC(\cdot)Q^{-1}$ is a cosine function in H (all of whose values are self adjoint operators) uniformly bounded in $]-\infty, \infty[$ and such that $F(\cdot)u$ is continuous in H for all $u \in H$. By virtue of [5], Lemmas 5.4 and 5.6 there exists a closed operator B (the infinitesimal generator of F) such that $R(\lambda; B) = (\lambda I - B)^{-1}$ exists for all $\lambda > 0$ and is given there by the Laplace transform formula

$$(4.2) \quad (R(\lambda^2; B)u, v) = \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} (F(t)u, v) dt$$

for any $u, v \in H$. Observe now that, since $F(t)$ is self adjoint the integrand in (4.2) can be written $e^{-\lambda t}(u, F(t)v) = e^{-\lambda t}(\overline{F(t)v}, u)$; a new application of (4.2) then yields

$$R(\lambda; B)^* = R(\lambda; B).$$

But this implies, in view of [5], Ch. VIII, §117 that B itself is self adjoint. Since $\sigma(B)$ is contained in $]-\infty, 0]$, $-B \geq 0$. Let S be the nonnegative self adjoint square root of $-B$, and define

$$(4.3) \quad G(t) = \cos(tS)$$

If $u, v \in H$ it follows from the functional calculus for self adjoint operators that, for $\lambda > 0$

$$(4.4) \quad (R(\lambda^2; -S^2)u, v) = \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} (G(t)u, v) dt.$$

Comparing with (4.2), keeping in mind that $-S^2 = B$ and making use of the well-known uniqueness property of Laplace transforms, we get $(F(t)u, v) = (G(t)u, v)$ for all $t \in [0, \infty[$, $u, v \in H$. This implies

$$(4.5) \quad F(t) = G(t),$$

for $t \geq 0$ and, since $F(-t) = F(t)$, $G(-t) = G(t)$ for all t . Combining (4.5) with (4.3) the result follows.

Remark 4.2. Observe that, under the present hypotheses, all the functions involved in the proof of Theorem 3.1 are measurable—in fact, continuous, after [5], Lemma 5.2—and then the Banach integrals in its proof can be replaced by ordinary Riemann integrals.

Remark 4.3. It does not seem likely that Theorem 4.1 remains true if the strong measurability hypothesis is replaced by a weak one, *i.e.* measurability of $(C(\cdot)u, v)$ for all $u, v \in H$ —except of course when H is separable, for in this case both notions of measurability coincide ([9], Theorem 3.5.3). There is, however, a curious result in this direction, similar to (part of) an argument of Sz.-Nagy concerning self adjoint semigroups (see [9], Lemma 22.3.1); if $F(\cdot)$ is a cosine function in H such that $F(t)$ is self adjoint for all t and $(F(\cdot)u, v)$ is continuous for all $u, v \in H$ then $F(\cdot)u$ is continuous in H for all u . The proof follows from the identity

$$2 |F(t)u - F(s)u|^2 = ((F(2t) + I)u, u) - 2((F(s + t) + F(s - t))u, u) + ((F(2s) + I)u, u)$$

which, in turn, is a simple consequence of (1.1) and of the self adjointness of F . This result could be applied to our case as follows: if $C(\cdot)$ is such that $(C(\cdot)u, v)$ is continuous for all $u, v \in H$ then the same is true of the cosine function $F(t) = QC(t)Q^{-1}$ provided by Theorem 3.1; by the preceding result $F(\cdot)u$ is continuous in H for all $u \in H$ and then we are in conditions of applying Theorem 4.1.

Observe, finally, that in the case H is separable the portion of the proof concerning identification of $F(\cdot)$ is contained in a result of S. Kurepa ([11], Theorem 2).

Remark 4.4. Theorem 4.1 generalizes Theorem 3 in [12]; there the same conclusion is obtained under the hypothesis of measurability of $C(\cdot)$ in the uniform operator topology plus a condition on the (bounded) infinitesimal generator of C . It also generalizes Theorem 2.5 in [6]. We will comment on this in the forthcoming Remark 6.3.

Remark 4.5. The converse of Theorem 4.1 is true, that is every family $C(\cdot)$ of bounded operators in H admitting the representation (4.1) is a c.f. having all the properties assumed in Theorem 4.1. The proof is simple.

5. The Almost Periodic Case.

Theorem 5.1. Let $C(\cdot)$ be a cosine function in H . Assume that, for every $u, v \in H$ the function $t \rightarrow (C(t)u, v)$ is almost periodic (in the sense of H. Bohr, see [3], Chapter I). Then C admits the representation (4.1), where S is a self adjoint operator of the form

$$S = \sum_{\lambda \geq 0} \lambda P_{\lambda},$$

$\{P_{\lambda}, \lambda \geq 0\}$ a family of mutually orthogonal (self adjoint) projections.

Proof. It follows from the fact that almost periodic functions are bounded in $]-\infty, \infty[$ ([3], Theorem 1.2) and from a routine application of the Banach-Steinhaus theorem that

$$(5.2) \quad |C(t)| \leq K, \quad -\infty < t < \infty.$$

Although (as 5.2 suggests) Theorem 5.1 can be proved by further refinement of Theorem 4.1 we shall prefer a direct proof that exhibits in a fairly explicit way the spectral family of S . For each $\lambda \geq 0$ define a bounded operator N_λ in H by the formula

$$(5.3) \quad \begin{aligned} (N_\lambda u, v) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \cos \lambda t (C(t)u, v) dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{i\lambda t} (C(t)u, v) dt. \end{aligned}$$

For existence of the limit in the right-hand side see [3], Theorem 1.13; incidentally, $(N_\lambda u, v) = 0$ except for (at most) a countable number of λ 's. We have, for any λ and t real, $u, v \in H$

$$\begin{aligned} (N_\lambda C(t)u, v) &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{i\lambda s} (C(s)C(t)u, v) ds \\ &= \frac{e^{-i\lambda t}}{2} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{i\lambda(s+t)} (C(s+t)u, v) ds \\ &\quad + \frac{e^{i\lambda t}}{2} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{i\lambda(s-t)} (C(s-t)u, v) ds \\ (5.4) \quad &= \cos \lambda t (N_\lambda u, v). \end{aligned}$$

Observe next that N_λ and $C(t)$ commute (this is a consequence of the fact that $C(s)$ and $C(t)$ commute). Multiply then (5.4) by $e^{i\mu t}/2T$, integrate in $(-T, T)$ and let $T \rightarrow \infty$. Since the expression

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{i\mu t} \cos \lambda t dt,$$

equals 1 if $\lambda = \mu = 0$, $\frac{1}{2}$ if $|\lambda| = |\mu| \neq 0$ and 0 otherwise, $N_\lambda N_\mu = N_\lambda$ if $\lambda = \mu = 0$, $N_\lambda N_\mu = (\frac{1}{2})N_\lambda$ if $|\lambda| = |\mu| > 0$, $N_\lambda N_\mu = 0$ otherwise. Then, if we define $M_0 = N_0$, $M_\lambda = 2N_\lambda$ for $\lambda > 0$, $\{M_\lambda; \lambda \geq 0\}$ is a family of orthogonal projections.

Let now P be the self adjoint operator defined in the proof of Theorem 3.1, that is

$$(5.5) \quad (Pu, v) = \text{LIM}_{T \rightarrow \infty} \frac{1}{T} \int_0^T (C(s)u, C(s)v) ds,$$

and let Q be the positive, self adjoint square root of P . Making use of (5.3) and (5.4) we obtain

$$\begin{aligned}
 (5.6) \quad (PN_\lambda u, v) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \cos \lambda t (N_\lambda u, C(t)v) \, ds \\
 &= (N_\lambda u, N_\lambda v) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \cos \lambda t (C(t)u, N_\lambda v) \, dt \\
 &= (Pu, N_\lambda v),
 \end{aligned}$$

or $PN_\lambda = N_\lambda^*P$. But then

$$PM_\lambda = M_\lambda^*P,$$

which, as in a similar case in the proof of Theorem 3.1 shows that $P_\lambda = QM_\lambda Q^{-1}$ is a self adjoint operator for all λ . Let now S be the (self adjoint) operator in H defined by

$$\begin{aligned}
 S &= \sum_{\lambda \geq 0} \lambda P_\lambda, \\
 U(t) &= Q^{-1} \cos(tS)Q = \sum_{\lambda \geq 0} \cos(\lambda t) M_\lambda,
 \end{aligned}$$

A simple computation shows

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{i\lambda t} (U(t)u, v) \, dt = (N_\lambda u, v).$$

Comparing this with (5.3), we see that the almost periodic functions $(C(t)u, v)$, $(U(t)u, v)$ have the same Fourier series (since both functions are even, we only need to consider $\lambda \geq 0$). Then, by [3], Theorem 1.19 they must coincide, which ends the proof of Theorem 5.1.

Remark 5.2. If we strengthen slightly one of the hypotheses in Theorem 5.1 we can get rid of the last vestiges of Banach limits in its proof. In fact, assume that $C(\cdot)u$ is almost periodic (as an H -valued function, [3], Chapter VI) for all $u \in H$. Then the numerical function $(C(\cdot)u, C(\cdot)v)$ is almost periodic for all $u, v \in H$ and the Banach limit in (5.5) can be replaced by an ordinary limit.

Remark 5.3. If H is separable, then at most a countable number of the projections P_λ can be different from zero.

Remark 5.4. The converse of Theorem (5.1) is true, even in a slightly strengthened form: if C admits the representation (4.1) with S a self adjoint operator given by (5.1), then $C(\cdot)u$ is an almost periodic function in H for all u . The proof resembles very much that of the forthcoming Theorem 6.5 and is therefore omitted.

Remark 5.5. Theorem 5.1 was proved by E. Giusti in the particular case $C(\cdot)$ is *periodic* in [6]. The present proof is an adaptation of his.

6. Some Applications to Operational Differential Equations. Let A be a linear, (possibly unbounded) operator in H such that $R(\lambda; A)$ exists for some λ .

A H -valued function $u(\cdot)$ defined in $]-\infty, \infty[$ is said to be a (strong) solution of the second-order operational differential equation

$$(6.1) \quad u''(t) = Au(t),$$

if and only if $u(\cdot)$ is twice continuously differentiable in $]-\infty, \infty[$, $u(t) \in D(A)$, the domain of A , and (6.1) is satisfied for all t .

Theorem 6.1. Assume that (a) there exists a dense subspace D of E such that, if $u_0, u_1 \in D$ then there exists a solution $u(\cdot)$ of (6.1) with $u(0) = u_0$, $u'(0) = u_1$. Assume, further,

(b) There exists a constant $K > 0$ and a function $M(t)$, positive and bounded on compacts of $]-\infty, \infty[$ such that

$$(6.2) \quad |u(t)| \leq K |u(0)| + M(t) |u'(0)|, \quad -\infty < t < \infty$$

for all $t \geq 0$, where $u(\cdot)$ is any solution of (6.1). Then

$$(6.3) \quad A = Q^{-1}BQ,$$

Q as in Theorem 3.1, B a self adjoint operator such that $-B \geq 0$.

Theorem 6.2. Same hypotheses as in Theorem 6.1, except that the function $M(\cdot)$ in (b) is assumed to be constant in $]-\infty, \infty[$. Then A admits the representation (6.3), with B self adjoint, $-B \geq \epsilon I$ for some $\epsilon > 0$.

Proof. For $u \in D$ define

$$C(t)u = u(t),$$

where $u(\cdot)$ is the solution of (6.1) with $u(0) = u$, $u'(0) = 0$ (note that (b) implies uniqueness of solutions with given initial data). For each $t \in]-\infty, \infty[$ $C(t)$ is well defined and—again by condition (b)—bounded, thus it can be extended to all of H by continuity. We shall denote the extension by the same symbol. Plainly, $C(0) = I$, $C(\cdot)u$ is continuous in H for all $u \in H$ and

$$|C(t)| \leq K, \quad -\infty < t < \infty.$$

It is not difficult to see that if $u \in D$ then $C(-t)u$ is a solution of (6.1) with the same initial data as $C(t)u$, thus by uniqueness $C(t) = C(-t)$, that is $C(\cdot)$ is an even function. Observe next that if $u(\cdot)$ is any solution of (6.1) with $u'(0) = 0$ then

$$(6.4) \quad u(t) = C(t)u(0)$$

for all t . This is a consequence of the very definition of $C(t)$ when $u(0) \in D$ and it follows in the general case from an approximation argument based in (6.2) and denseness of D . Let now $u \in D$, t fixed,

$$u(s) = C(s+t)u + C(s-t)u.$$

Due to the s -invariance of (6.1), $u(\cdot)$ is a solution. Since $u(0) = C(t)u +$

$C(-t)u = 2C(t)u$, we obtain from (6.4) and from the denseness of D that

$$2C(s)C(t) = C(s+t) + C(s-t).$$

We have, at this stage, verified all of the properties of C necessary to apply Theorem 4.1; according to it,

$$(6.5) \quad C(t) = Q^{-1} \cos(tS)Q, \quad -\infty < t < \infty,$$

where S is self adjoint, $S \geq 0$, and where Q is described in the statement of Theorem 3.1. We make use now again of formula (4.2)—together with the observation that B , the infinitesimal generator of $C(\cdot)$ coincides with A ([5], Theorem 5.9; we note that assumptions (a) and (b) in Theorem 2 imply that the Cauchy problem for (6.1) is uniformly well posed in $]-\infty, \infty[$ in the terminology of [5]). Consequently, we have, for $\lambda > 0$, $u, v \in H$,

$$(6.6) \quad (R(\lambda^2; A)u, v) = \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} (C(t)u, v) dt.$$

On the other hand, it follows from the representation (6.5) that

$$\begin{aligned} (R(\lambda^2; -Q^{-1}S^2Q)u, v) &= (Q^{-1}R(\lambda^2; -S^2)Qu, v) \\ &= \frac{1}{\lambda} Q^{-1} \left(\int_0^\infty e^{-\lambda t} (\cos(tS)u, v) dt \right) Q \\ &= \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} (C(t)u, v) dt, \quad \lambda > 0. \end{aligned}$$

Consequently $R(\lambda; A) = R(\lambda; -Q^{-1}S^2Q)$, for $\lambda > 0$ thus $A = Q^{-1}(-S^2)Q$. This is formula (6.3) with $B = -S^2$.

As for Theorem (6.2), we only need to observe that if, say, $u_0 \in D(A)$, $u_1 \in D(Q^{-1}(-B)^{1/2}Q)$

$$(6.7) \quad u(t) = Q^{-1} \cos(t(-B)^{1/2})Qu_0 + Q^{-1}(-B)^{1/2} \sin(t(-B)^{1/2})Qu_1,$$

is a solution of (6.1) with $u(0) = u_0$, $u'(0) = u_1$ (the verification, based on the functional calculus for B is simple). Letting now $u_0 = 0$ and applying (6.2) we obtain

$$(6.8) \quad |(-B)^{1/2} \sin(t(-B)^{1/2})| \leq |Q| |Q^{-1}| |Q^{-1}(-B)^{1/2} \sin t(-B)^{1/2}Q| \leq M$$

for all t . But $|(-B)^{-1/2} \sin(t(-B)^{1/2})|$ equals

$$\text{ess. sup} \left| \frac{\sin t\lambda}{\lambda} \right|,$$

(the essential supremum taken with respect to all the measures $\mu(d\lambda) = (P(d\lambda)u, v)$, $u, v \in H$, $\{P(d\lambda)\}$, the spectral family of $(-B)^{1/2}$; see [15], Ch. IX, §128). Noting that

$$\left| \frac{\sin t\lambda}{\lambda} \right| = \frac{1}{\lambda} \sin 1,$$

for $\lambda > 0$, $t = 1/\lambda$ we see that if (6.8) is to be verified all the measures $(P(d\lambda)u, v)$ must vanish outside some fixed interval $[\delta, \infty[$, $\delta > 0$. This ends the proof of Theorem 6.2.

We observe at this point that the converses of Theorems (6.1) and (6.2) are as well true. This can be established directly or with the help of [5], Lemma 5.8 and Theorem 5.9, the role of the cosine function there played by $C(t) = Q^{-1} \cos(t(-QAQ^{-1})^{1/2})Q$.

Remark 6.3. Theorem 6.1 was proved in [6] with the additional condition that the “energy” (see definition there) of certain solutions of (6.1) should be bounded. As shown here, this condition is unnecessary.

Remark 6.4. Theorems 6.1 and 6.2 can still be proved if we only assume existence of *weak* solutions of (6.1) in one sense or another—as long as our weak solutions are weakly continuous (see Remark 4.3).

We pass now to the case of abstract second order differential equations with almost periodic solutions. It was proved by C. F. Muckenhoupt ([13]) that the solutions of the mixed problem for the vibrating string equation in a finite interval are almost periodic (as functions of time with values in a suitable “energy” space). This result was later generalized by various authors (see [1] or [3] for references). As made clear, for instance, in [1], Chapter II, these results are consequences of a theorem in abstract differential equations that we state below in a form convenient for our needs.

Theorem 6.5. *Let B be a self adjoint operator in H of the form*

$$(6.9) \quad B = - \sum_{\lambda \geq \lambda_0} \lambda^2 P_\lambda ,$$

$\lambda_0 > 0$, $\{P_\lambda ; \lambda \geq \lambda_0\}$ a family of self adjoint orthogonal projections. Let E (the energy space of the equation

$$(6.10) \quad u''(t) = Bu(t)$$

be the orthogonal sum of $D((-B)^{1/2})$ and H , $D((-B)^{1/2})$ endowed with the “graph” scalar product $u, v \rightarrow ((-B)^{1/2}u, (-B)^{1/2}v)$. Then all (generalized) solutions of (6.9) with $(u(0), u'(0)) \in E$ are almost periodic as E -valued functions.

For the notion of Hilbert space valued almost periodic functions see [3], Chapter VI or [1], Chapter I. Recall that (by definition) a *generalized solution* of (6.10) is any H -valued function of the form

$$(6.11) \quad u(t) = \cos(t(-B)^{1/2})u_0 + (-B)^{1/2} \sin(t(-B)^{1/2})u_1 ,$$

$u_0, u_1 \in H$; if $u_0 \in D(B)$, $u_1 \in D((-B)^{1/2})$ then (6.11) is an actual (strong) solution of (6.10) (this was already observed in the proof of Theorem 6.2) and it is easy to see that all the assumptions in Theorem 6.2 are satisfied for (6.10). On the other hand, it is also easily seen that if the pair (u_0, u_1) in (6.11) belongs to E ,

then $u(\cdot)$ is continuously differentiable for all t (in fact,

$$(6.12) \quad u'(t) = -\sin(t(-B)^{1/2})(-B)^{1/2}u_0 + \cos(t(-B)^{1/2})u_1,$$

and $(u(t), u'(t)) \in E$. Thus we can think of any generalized solution of (6.10) with $u(0) \in D((-B)^{1/2})$ as an E -valued function. Taking now advantage of the particular form of B we obtain

$$(6.13) \quad u(t) = \sum_{\lambda \geq \lambda_0} \cos \lambda t P_\lambda u_0 + \sum_{\lambda \geq \lambda_0} \frac{\sin \lambda t}{\lambda} P_\lambda u_1$$

where, in view of the relation

$$(6.14) \quad |u|^2 = \sum_{\lambda \geq \lambda_0} |P_\lambda u|^2,$$

at most a countable number of the "coefficients" $P_\lambda u_{0,1}$ are different from zero. Applying $(-B)^{1/2}$ to both sides of (6.13) and observing that (again by (6.14)) the series so obtained converges in H uniformly in $]-\infty, \infty[$ we see that ([3], Chapter VI) $u(\cdot)$ is almost periodic as a function with values in $D((-B)^{1/2})$. Operating in a similar way with (6.12)—but now in the space H —we likewise obtain that $u'(\cdot)$ is almost periodic in H , and thus so is $(u(\cdot), u'(\cdot))$ in E .

Remark 6.6. The conclusion of Theorem 6.5 is not altered if B is replaced by $Q^{-1}BQ$, Q a bounded, invertible operator.

We apply now the results of Section 5 to get a converse of Theorem 6.5—as extended by Remark 6.6.

Theorem 6.7. Assume the hypotheses of Theorem (3.2) are satisfied. Assume, moreover, that for any solution $u(\cdot)$ of (6.1) and for any $v \in H$ the numerical function

$$(u(t), v),$$

is almost periodic. Then A admits the representation (6.3) where B is given by (6.9).

Proof. If $C(\cdot)$ is the cosine function in the proof of Theorem 6.2, the present hypotheses imply that

$$(6.15) \quad (C(t)u, v),$$

is almost periodic for all $u \in D$, $v \in H$. But if $u \in E$, $\{u_n\} \subseteq D$, $u_n \rightarrow u$ then $(C(\cdot)u_n, v) \rightarrow (C(\cdot)u, v)$ uniformly in $]-\infty, \infty[$ (consequence of the uniform boundedness of $C(\cdot)$) and thus (6.15) is almost periodic for all $u, v \in H$. Proceeding now as in the proof of Theorem (6.1) but applying Theorem 5.1 (instead of Theorem 4.1) to $C(\cdot)$ and keeping in mind that the assumptions of Theorem 6.2 are as well satisfied, we obtain $A = Q^{-1}(-S^2)Q$, where $S = \sum_{\lambda \geq \mu_0} \lambda P_\lambda$, $\mu_0 > 0$, $\{P_\lambda\}$ an orthogonal family of self adjoint projections. This ends the proof of Theorem 6.7.

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