An Application of the Holomorphic Functional Calculus to Sharpen a Classical Matrix Commutator Theorem

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1. In [3] (Theorem 1), M. Marcus and R. C. Thompson prove the following fact.

Theorem I. *Suppose that A and B are operators on a finite dimensional Hilbert space. Assume that*

- (i) $0 \notin \sigma(A)$ (spectrum A) and $0 \notin$ numerical range B ;*
- (ii) $ABA^{-1}B^{-1}$ commutes with A ;*
- (iii) A and $ABA^{-1}B^{-1}$ are normal.*

Then B commutes with A .

As the authors point out, this is an improvement of a classical theorem.

On the other hand, in [2] (page 21), I. Kaplansky has shown the following fact.

Theorem II. *Suppose that A and C are operators on a Hilbert space. Assume that*

- (ii)' $CA - AC$ commutes with A ;*
- (iii)' A is normal.*

Then C commutes with A .

In this note we shall use Theorem II and the holomorphic functional calculus (see [1], sections 5.2, 5.3, 5.4) to prove the following result.

Theorem III. *Suppose that A and B are operators on a Hilbert space. Assume that*

- (i)' $0 \notin$ convex hull $\sigma(B)$;*
- (ii) BAB^{-1} commutes with A ;*
- (iii)' A is normal.*

Then B commutes with A .

As a consequence, hypotheses (i) and (iii) of Theorem I may be weakened to (i)' and (iii)' of Theorem III.

Whereas Theorem I is proved by a simple matrix computation, Theorem III

does not seem to be any easier to prove if the Hilbert space is finite dimensional.

All operators in this article are linear and bounded, and all Hilbert spaces are complex (that is, the scalar field is $\mathbb{C}$).

2. Lemma. *Suppose that B is an operator on a Hilbert space H satisfying (i)' of Theorem III. Denote by Φ_B the map $T \mapsto BTB^{-1}$ (a bounded linear map on the Banach space of all operators on H). Then $\sigma(\Phi_B)$ is disjoint from the negative real axis.*

Proof. Denote by L_B (resp. $R_{B^{-1}}$) the map $T \mapsto BT$ (resp. $T \mapsto TB^{-1}$). Then $\Phi_B = L_B R_{B^{-1}} = R_{B^{-1}} L_B$; hence $\sigma(\Phi_B) \subset \sigma(L_B)\sigma(R_{B^{-1}})$.

It is clear that $\sigma(L_B) \subset \sigma(B)$ and that $\sigma(R_{B^{-1}}) \subset \sigma(B^{-1}) = \sigma(B)^{-1}$.

From (i)' we have in particular: $0 \notin \text{convex hull } \{\lambda_1, \lambda_2\}$, for any $\lambda_1, \lambda_2 \in \sigma(B)$; equivalently: $\sigma(B)\sigma(B)^{-1}$ is disjoint from the negative real axis. The Lemma now follows.

3. Denote by $\log$ the principle value of the logarithm function on the complement of the negative real axis. If B satisfies (i)' of Theorem III, then by Lemma 2 $\log$ is holomorphic on an open set containing $\sigma(\Phi_B)$. If, moreover, $\sigma(B)$ is contained in the open right half plane, then $\log$ is holomorphic on an open set containing $\sigma(B)$.

Lemma. *Suppose that B is an operator on a Hilbert space such that $\sigma(B)$ is contained in the open right half plane (so that in particular B satisfies (i)' of Theorem III). Then*

$$\log \Phi_B = \text{ad } \log B.$$

(Here $\text{ad } S$ for any operator S means the map $T \mapsto ST - TS$.)

Proof. If S is any operator, then $\exp \text{ad } S = \Phi_{\exp S}$. (This can be proved by expanding $\exp \text{ad } S$ and $\exp S$ as power series, and comparing coefficients.) Taking $S = \log B$ we get $\exp \text{ad } \log B = \Phi_{\exp \log B}$.

Since $\exp \log z = z$ on an open set containing $\sigma(B)$, $\exp \log B = B$.

Thus $\exp \text{ad } \log B = \Phi_B$, whence $\log \Phi_B = \log \exp \text{ad } \log B$.

We have $\sigma(\log B) = \log(\sigma(B)) \subset \{\mu \in \mathbb{C} \mid -\pi/2 < \text{Im } \mu < \pi/2\}$. Hence by an argument similar to that of Lemma 2, $\sigma(\text{ad } \log B) \subset \{\mu \in \mathbb{C} \mid -\pi < \text{Im } \mu < \pi\}$. Since on this open set $\log \exp z = z$, we have $\log \exp \text{ad } \log B = \text{ad } \log B$.

4. Proof of Theorem III. Since $\log \Phi_B$ is the limit of rational functions in Φ_B , hypothesis (ii) implies that $(\log \Phi_B)(A)$ commutes with A .

Replacing B by $\lambda^{-1}B$, where λ is the point closest to 0 in the convex hull of $\sigma(B)$ (which is compact), does not change either the assumptions or the conclusion of Theorem III, so we may suppose that $\sigma(B)$ is contained in the open right half plane.

Then, by Lemma 3, $\log \Phi_B = \text{ad } \log B$.

Thus, we obtain that $(\text{ad } \log B)(A)$ commutes with A . The hypotheses of Theorem II are now satisfied, with $C = \log B$, and we conclude that $\log B$ commutes with A .

Since $B = \exp \log B$ is the limit of polynomials in $\log B$, B commutes with A .

REFERENCES

- [1] E. HILLE & R. S. PHILLIPS, Functional analysis and semigroups, revised edition, *Amer. Math. Soc. Colloq. Publ.* 31 1957.
- [2] I. KAPLANSKY, Functional analysis, survey article in *Some aspects of Analysis and probability*, John Wiley and Sons, Inc., New York, 1958.
- [3] M. MARCUS & R. C. THOMPSON, On a classical commutator result, *J. Math. Mech.*, **16** (1966) 583–588.

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