Combinatorics of Partial Sums with Applications to Limit Theorems

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Some arguments in the proof of the ergodic limit theorems have always been understood to be of a combinatorial nature. Besides the celebrated maximal ergodic theorem a certain smoothing procedure due to Chacon and Ornstein is an extremely important device of this type.

Something like these theorems is presented here in purely combinatorial terms. The applications given in the second part of the paper should give an idea how the results can be used in measure theory. General ergodic limit theorems will be derived in a forthcoming paper using this combinatorics together with a theory of decomposition of joint distributions, developed by H. Rost and the author.

I. Combinatorics. Let Ω be a space and let $s_{i+1}^i, s_{i+2}^{i+1}, \dots$ be a sequence of real valued functions on Ω .

Denote (with max and min defined pointwise)

- 1) $s_k^i = s_{i+1}^i + s_{i+2}^{i+1} + \dots + s_k^{k-1}$ for $i < k$, $s_i^i = 0$ for all i .
- 2) $*h_k^i = \max \{s_{i+1}^i, s_{i+2}^{i+1}, \dots, s_k^i\}$ for $i < k$.
- 3) $*l_k^i = -\min \{s_k^{k-1}, s_k^{k-2}, \dots, s_k^i\}$ for $i < k$.
- 4) $h_k^i = (*h_k^i)^+ = \max \{s_i^i, s_{i+1}^i, \dots, s_k^i\}$, $h_i^i = 0$.
- 5) $l_k^i = (*l_k^i)^+ = \max \{-s_k^k, -s_k^{k-1}, \dots, -s_k^i\}$, $l_k^k = 0$.
- 6) $d_k^i = (*h_k^i)^-$ for $i < k$.
- 7) $g_k^i = (*l_k^i)^-$ for $i < k$.
- 8) $\tilde{l}_k^i = \max \{s_k^k, s_k^{k-1}, \dots, s_k^i\}$ for $i \leq k$.
- 9) $\tilde{g}_k^i = \min \{s_i^i, s_{i+1}^i, \dots, s_{i-1}^i\} - \min \{s_i^i, \dots, s_k^i\}$ for $i < k$.

Remark 1. The functions $h_k^i, l_k^i, g_k^i, d_k^i$ are nonnegative. $\tilde{l}_k^i$ and $\tilde{g}_k^i$ are obtained from the sequence $\{s_{i+1}^i = -s_{i+1}^i\}$ in the same way as l_k^i and g_k^i are obtained from $\{s_{i+1}^i\}$.

Remark 2. It is quite useful to associate a "path" with every point $\omega \in \Omega$ as follows: The path is a graph in the plane R^2 which is linear between successive integer abscissa i and $i+1$ and has there the slope s_{i+1}^i . The values of the functions just defined can easily be read off this path.

s_k^i is the increment between i and k .

h_k^i is the maximal height above the vertex i in the interval between i and k .

l_k^i is the maximal height above the final vertex k or the loss from the maximal height.

g_k^i is the gain in k for the maximum since i .

d_k^i is the deficit which remained ever since the path left the vertex i .

Here are two typical paths:

Lemma 1.

$$\begin{aligned} \text{a) } s_{i+1}^i &= *h_{i+1}^i = -*l_{i+1}^i, \\ (s_{i+1}^i)^+ &= h_{i+1}^i = g_{i+1}^i = l_{i+1}^i, \\ (s_{i+1}^i)^- &= l_{i+1}^i = d_{i+1}^i = \tilde{g}_{i+1}^i. \end{aligned}$$

$$\text{b) } s_k^i = h_k^i - l_k^i \text{ for } i \leq k.$$

Lemma 2.

$$\begin{aligned} \text{a) } g_k^i &= h_k^i - h_{k-1}^i. \\ \text{b) } d_k^i &= l_k^i - l_k^{i+1} = h_k^i - h_k^{i+1} - s_{i+1}^i. \\ \text{c) } \tilde{g}_k^i &= [\max \{s_k^{k-1}, \dots, s_k^i\}]^-. \end{aligned}$$

Proof of c) (as an example): $\min \{s_i^i, s_{i+1}^i, \dots, s_{k-1}^i\} - \min \{s_i^i, \dots, s_k^i\}$ vanishes unless

$$s_k^i < \min \{s_i^i, \dots, s_{k-1}^i\}.$$

Hence

$$\tilde{g}_k^i = (s_k^i - \min \{s_i^i, s_{i+1}^i, \dots, s_{k-1}^i\})^-$$

$$\begin{aligned}
&= (\max (s_k^i - s_i^i, s_k^i - s_{i+1}^i, \dots, s_k^i - s_{k-1}^i))^- \\
&= (\max (s_k^i, s_k^{i+1}, \dots, s_k^{k-1}))^-.
\end{aligned}$$

The second equality in b) follows from a) and Lemma 1.b).

Lemma 3.

- a) $*h_k^i = s_{i+1}^i + h_k^{i+1}$ for $i < k$.
b) $*l_k^i = -s_k^{k-1} + l_{k-1}^i$ for $i < k$.

Remark. If $i + 1 < k$ then $h_k^{i+1} = (*h_k^{i+1})^+$ and a) is a recursion formula for the $*h_k^i$:

$$*h_{i+1}^i = s_{i+1}^i; \quad *h_k^i = s_{i+1}^i + (*h_k^{i+1})^+ \quad \text{for } i + 1 < k.$$

b) can be considered as a recursion formula for the $*l_k^i$ in the same way.

Proof of a).

$$\begin{aligned}
\max \{s_{i+1}^i, s_{i+2}^i, \dots, s_k^i\} \\
&= \max \{s_{i+1}^i + s_{i+1}^{i+1}, s_{i+1}^i + s_{i+2}^{i+1}, \dots, s_{i+1}^i + s_k^{i+1}\} \\
&= s_{i+1}^i + \max \{s_{i+1}^{i+1}, s_{i+2}^{i+1}, \dots, s_k^{i+1}\} = s_{i+1}^i + h_k^{i+1}.
\end{aligned}$$

Lemma 4.

- a) $s_{i+1}^i = (h_{k-1}^i - h_k^{i+1}) + g_k^i - d_k^i$ for $i < k$.
b) $s_k^{k-1} = (l_{k-1}^i - l_k^{i+1}) + g_k^i - d_k^i$ for $i < k$.
c) $-s_{i+1}^i = (s_k^{k-1} - s_{i+1}^i) + (l_{k-1}^i - l_k^{i+1}) + \tilde{g}_k^i - (\tilde{l}_k^i - \tilde{l}_k^{i+1})$ for $i < k$.

Proof of a). Lemma 3.a) yields:

$$\begin{aligned}
(*h_k^i)^+ - (*h_k^i)^- &= s_{i+1}^i + h_k^{i+1}, \\
s_{i+1}^i &= h_k^i - h_k^{i+1} - d_k^i = (h_{k-1}^i - h_k^{i+1}) + (h_k^i - h_{k-1}^i) - d_k^i.
\end{aligned}$$

Lemma 2.a) yields a). b) follows from a) by Lemma 1.b). c) is just b) applied to the sequence $\{s_{i+1}^i\}$ with $\tilde{s}_{i+1}^i = -s_{i+1}^i$ where Lemma 2.b) is used.

Theorem 1. Denote by ψ_k^i the indicator function of

$$\{\max \{s_{i+1}^i, s_{i+2}^i, \dots, s_k^i\} \geq 0\} = \{h_k^i \geq 0\}, \quad \psi_\infty^i = \lim_k \psi_k^i,$$

and by $\tilde{\psi}_k^i$ the indicator function of

$$\{\min \{s_{i+1}^i, s_{i+2}^i, \dots, s_k^i\} \leq 0\}, \quad \tilde{\psi}_\infty^i = \lim_k \tilde{\psi}_k^i,$$

Then for $i < k$,

$$\text{a) } s_{i+1}^i \cdot \psi_k^i \geq (h_{k-1}^i - h_k^{i+1}) + g_k^i,$$

$$\text{b)} \quad -s_{i+1}^i \cdot \psi_k^i \geq (s_k^{k-1} - s_{i+1}^i) + (\bar{l}_{k-1}^i - \bar{l}_k^{i+1}) + \tilde{g}_k^i,$$

$$\text{c)} \quad s_{i+1}^i \cdot \psi_k^i + (s_{i+1}^i + d_k^i)(1 - \psi_\infty^i) \geq h_{k-1}^i - h_k^{i+1} + g_k^i.$$

Proof of a) and c). From Lemma 4.a) we get, since $d_k^i \cdot \psi_k^i = 0$,

$$s_{i+1}^i \psi_k^i + (s_{i+1}^i + d_k^i)(1 - \psi_k^i) = h_{k-1}^i + h_k^{i+1} + g_k^i.$$

Since $\psi_\infty^i \geq \psi_k^i$ and $(s_{i+1}^i + d_k^i)(1 - \psi_k^i) \leq 0$, we get a) and c).

Proof of b). By reduction to a): Put $\bar{s}_k^i = -s_k^i$.

$$1) \quad \bar{\psi}_k^i = \{-\min\{s_{i+1}^i, \dots, s_k^i\} \geq 0\} = \{\max\{\bar{s}_{i+1}^i, \dots, \bar{s}_k^i\} \geq 0\}, \text{ while } \psi_k^i = \{\max\{s_{i+1}^i, \dots, s_k^i\} \geq 0\}.$$

2) According to Lemma 2.c)

$$\begin{aligned} \bar{g}_k^i &= [\max\{s_k^{k-1}, s_k^{k-2}, \dots, s_k^i\}]^- = [\max\{-\bar{s}_k^{k-1}, -\bar{s}_k^{k-2}, \dots, -\bar{s}_k^i\}]^- \\ &= [-\min\{\bar{s}_k^{k-1}, \dots, \bar{s}_k^i\}]^-, \end{aligned}$$

while

$$g_k^i = (*\bar{l}_k^i)^- = [-\min\{s_k^{k-1}, \dots, s_k^i\}]^-.$$

$$\begin{aligned} 3) \quad \bar{l}_{k-1}^i - \bar{l}_k^{i+1} + s_k^{k-1} - s_{i+1}^i &= \max\{s_{k-1}^{k-1}, \dots, s_{k-1}^i\} - \max\{s_k^k, \dots, s_k^{i+1}\} + s_k^{i+1} - s_{k-1}^i \\ &= \max\{-(s_{k-1}^i - s_{k-1}^{k-1}), -(s_{k-1}^i - s_{k-1}^{k-2}), \dots, -(s_{k-1}^i - s_{k-1}^i)\} \\ &\quad - \max\{-(s_k^{i+1} - s_k^k), -(s_k^{i+1} - s_k^{k-1}), \dots, -(s_k^{i+1} - s_k^{i+1})\} \\ &= \max\{\bar{s}_{k-1}^i, \bar{s}_{k-2}^i, \dots, \bar{s}_i^i\} - \max\{\bar{s}_k^{i+1}, \bar{s}_{k-1}^{i+1}, \dots, \bar{s}_{i+1}^{i+1}\}, \end{aligned}$$

while

$$h_{k-1}^i - h_k^{i+1} = \max\{s_i^i, s_{i+1}^i, \dots, s_{k-1}^i\} - \max\{s_{i+1}^{i+1}, s_{i+2}^{i+1}, \dots, s_k^{i+1}\}.$$

4) If a) is applied to the sequence $\{\bar{s}_{i+1}^i\}$, then one gets b).

Lemma 5.

a) $\min(g_k^i, d_k^i) = 0$ for $i < k$.

b) $\dots \leq g_k^{k-3} \leq g_k^{k-2} \leq g_k^{k-1} = (s_k^{k-1})^+$ for every k .

c) $(s_{i+1}^i)^- = d_{i+1}^i \geq d_{i+2}^i \geq d_{i+3}^i \geq \dots \geq d_\infty^i$ for every i .

Theorem 2. For every pair $i < k$

$$\begin{aligned} (s_{i+1}^i + d_\infty^i) + (s_{i+2}^{i+1} + d_\infty^{i+1}) + \dots + (s_k^{k-1} + d_\infty^{k-1}) &= s_k^i \cdot \chi_k^i + (h_\infty^i - h_k^k)(1 - \chi_k^i) \\ &= [\bar{s}_{i+1}^i(\chi_i^i - \chi_{i+1}^i) + s_{i+2}^i(\chi_{i+1}^i - \chi_{i+2}^i) + \dots \\ &\quad + s_k^i(\chi_{k-1}^i - \chi_k^i)] + s_k^i \cdot \chi_k^i - h_\infty^k(1 - \chi_k^i), \end{aligned}$$

where χ_k^i denotes the indicator function of the set

$$\{\chi_k^i = 1\} \equiv \{\omega: s_m^i(\omega) > h_k^i(\omega) \text{ for some } m = m(\omega) > k\}.$$

Remark.

$$\{\sup_{k \geq i} s_k^i > 0\} = \chi_i^i \geq \chi_{i+1}^i \geq \chi_{i+2}^i \geq \dots$$

such that the limit $\chi_\omega^i = \lim_{k \rightarrow \infty} \chi_k^i$ is the indicator function of $\{\chi_\omega^i = 1\} = A^i = \{\omega: s_i^i(\omega), s_{i+1}^i(\omega), s_{i+2}^i(\omega), \dots \text{ does not assume its supremum}\}$.

Proof.

1) A number j is called a ladder index for $\omega \in \Omega$, if $h_\omega^j(\omega) = 0$. Clearly j is a ladder index for ω iff $\chi_j^j(\omega) = 0$. For an ω , for which the sequence

$$s_i^i(\omega), s_{i+1}^i(\omega), \dots$$

does not assume its supremum, no ladder index j with $j \geq i$ exists. Otherwise the smallest ladder index j with $j = j(i, \omega) > i$ is determined by the property

$$\max \{s_i^i(\omega), \dots, s_{j-1}^i(\omega)\} < s_j^i(\omega) = \sup_{n \geq i} \{s_n^i(\omega)\},$$

or equivalently by

$$\chi_{j-1}^i(\omega) - \chi_j^i(\omega) = 1.$$

Clearly for $j > i$

$$s_i^i(\chi_{j-1}^i - \chi_j^i) = h_\omega^i(\chi_{j-1}^i - \chi_j^i) \geq 0.$$

2) If i is not a ladder index for ω , then $d_\omega^i(\omega) = 0$. If i is a ladder index for ω and $j = j(i, \omega) > i$ is the next ladder index for ω then

$$\begin{aligned} d_\omega^i(\omega) + d_\omega^{i+1}(\omega) + \dots + d_\omega^{j-1}(\omega) &= d_\omega^i(\omega) = -s_j^i(\omega) \\ &= -(s_{i+1}^i(\omega) + \dots + s_j^{i-1}(\omega)). \end{aligned}$$

3) If $\chi_k^i(\omega) = 1$ then there exists no ladder index for ω with the property $i \leq j < k$. Hence $d_\omega^i(\omega) = 0$ for all j with $i \leq j < k$ and

$$s_{i+1}^i(\omega) + d_\omega^i(\omega) + \dots + s_k^{k-1}(\omega) + d_\omega^{k-1}(\omega) = s_k^i(\omega).$$

4) If $\chi_k^i(\omega) = 0$ then there exist ladder indices between i and k . If $j = j(i, \omega)$ $i \leq j \leq l \leq k$ is the smallest and $l = l(k, \omega)$ is the largest among them, then for such an ω

$$\begin{aligned} &(s_{i+1}^i + d_\omega^i + \dots + s_j^{j-1} + d_\omega^{j-1}) \\ &+ (s_{j+1}^j + d_\omega^j + \dots + s_l^{l-1} + d_\omega^{l-1}) + (s_{l+1}^l + d_\omega^l + \dots + s_k^{k-1} + d_\omega^{k-1}) \\ &= h_\omega^i + 0 + s_k^i + d_\omega^i = h_\omega^i - h_\omega^k, \end{aligned}$$

Remark. The functions ${}_m s_k^i$ have a simple interpretation in the path picture. Denote for every j by $j(m, \omega)$ the vertex of the path where the maximum between j and $j + m$ is attained then

$${}_m s_k^i(\omega) = s_{k(m, \omega)}^{i(m, \omega)}(\omega).$$

Proof of Lemma 7. a) and b) are trivial. By the remark following Lemma 6:

$$\begin{aligned} g_{i+1+m}^i &= g_{i+1+m}^{i+1} - d_{i+m}^i + d_{i+1+m}^i. \\ {}_m s_k^i &= (g_{i+1+m}^i + \cdots + g_{k+m}^{k-1}) - (d_{i+1+m}^i + \cdots + d_{k+m}^{k-1}) \\ &= (g_{i+1+m}^{i+1} + \cdots + g_{k+m}^k) - (d_{i+m}^i + \cdots + d_{k-1+m}^{k-1}) \\ &= {}_{m-1} s_{k+1}^{i+1} - d_{i+m}^i + d_{k+m}^k \quad \text{or also} \\ &= {}_{m-1} s_k^i - g_{i+m}^i + g_{k+m}^k. \end{aligned}$$

To prove d) we add equalities from c) with different m

$$\begin{aligned} {}_m s_k^i &= {}_{m-1} s_{k+1}^{i+1} - d_{i+m}^i + d_{k+m}^k \\ {}_{m-1} s_{k+1}^{i+1} &= {}_{m-2} s_{k+2}^{i+2} - d_{i+m}^{i+1} + d_{k+m}^{k+1} \\ &\vdots \\ {}_1 s_{k+m-1}^{i+m-1} &= {}_0 s_{k+m}^{i+m} - d_{i+m}^{i+m-1} + d_{k+m}^{k+m-1}. \\ {}_m s_k^i &= {}_0 s_{k+m}^{i+m} - (d_{i+m}^i + d_{i+m}^{i+1} + \cdots + d_{i+m}^{i+m-1}) + (d_{k+m}^k + d_{k+m}^{k+1} + \cdots + d_{k+m}^{k+m-1}) \\ &= {}_0 s_{k+m}^{i+m} - l_{i+m}^i + l_{k+m}^k, \end{aligned}$$

using Lemma 2.b).

The second equality of d) is proved in the same way using Lemma 2.a) and equalities of the second type in c).

Theorem 3. For $i < k$ and $m = 0, 1, 2, \dots$ the inequalities hold

$$\text{a) } (s_k^i)^+ \leq h_{i+m}^i + \frac{1}{m+1} (h_{i+1+m}^i + h_{i+2+m}^{i+1} + \cdots + h_{k+m}^{k-1}).$$

$$\text{b) } (s_{k+m}^{i+m})^- \leq l_{k+m}^k + \frac{1}{m+1} (l_{i+1+m}^i + l_{i+2+m}^{i+1} + \cdots + l_{k+m}^{k-1}).$$

Proof of a). By Lemma 7.d)

$${}_m s_k^i = s_k^i - h_{i+m}^i + h_{k+m}^k, \quad ({}_m s_k^i)^+ + h_{i+m}^i \geq (s_k^i)^+.$$

If this is applied for $m = 0, 1, \dots, m$ and the inequalities are added

$$\begin{aligned} (s_k^i)^+ &\leq ({}_0 s_k^i)^+ + h_i^i \leq g_{i+1}^i + \dots + g_k^{k-1} + h_i^i \\ (s_k^i)^+ &\leq ({}_1 s_k^i)^+ + h_{i+1}^i \leq g_{i+2}^i + \dots + g_{k+1}^{k-1} + h_{i+1}^i \\ &\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ (s_k^i)^+ &\leq ({}_m s_k^i)^+ + h_{i+m}^i \leq g_{i+m+1}^i + \dots + g_{k+m}^{k-1} + h_{i+m}^i, \end{aligned}$$

then using Lemma 2.a) and the monotonicity of h_i^i for fixed i

$$(m+1)(s_k^i)^+ \leq (h_{i+1+m}^i + \dots + h_{k+m}^{k-1}) + (m+1)h_{i+m}^i.$$

Proof of b). By the second equality of 7.d)

$$(s_{k+m}^{i+m})^- \leq ({}_m s_k^i)^- + l_{k+m}^k \quad \text{for } m = 0, 1, 2, \dots \text{ and } i < k.$$

Adding the following inequalities

$$\begin{aligned} (s_{k+m}^{i+m})^- &\leq ({}_0 s_{k+m}^{i+m})^- + l_{k+m}^{k+m} \leq d_{i+m+1}^{i+m} + \dots + d_{k+m}^{k+m-1} + l_{k+m}^{k+m} \\ (s_{k+m}^{i+m})^- &\leq ({}_1 s_{k+m-1}^{i+m-1})^- + l_{k+m}^{k+m-1} \leq d_{i+m+1}^{i+m-1} + \dots + d_{k+m}^{k+m-2} + l_{k+m}^{k+m-1} \\ &\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ (s_{k+m}^{i+m})^- &\leq ({}_m s_k^i)^- + l_{k+m}^k \leq d_{i+m+1}^i + \dots + d_{k+m}^{k-1} + l_{k+m}^k \end{aligned}$$

and using Lemma 2.b) and Lemma 5.c) the assertion b) follows.

II. Application to measure theory. Let $(\Omega, \mathcal{L}, \mu)$ be a measure space and $s_1^0, s_2^1, s_3^2, \dots$ a sequence of summable real functions. The constructions in I then yield summable functions

$$s_k^i, h_k^i, l_k^i, g_k^i, d_k^i, l_k^i \quad \text{and} \quad \tilde{g}_k^i,$$

and $\mathcal{L}$ -measurable sets

$$\begin{aligned} \{\psi_k^i = 1\} &= \{\max\{0, s_{i+1}^i, \dots, s_k^i\} \geq 0\}, \\ \{\chi_k^i = 1\} &= \{h_k^i < \sup\{h_j^i : j > i\}\}, \\ A_\infty^i &= \bigcup_{k>i} \{\psi_k^i = 1\} = \{\omega : s_k^i(\omega) \geq 0 \text{ for some } k > i\}, \\ A^i &= \bigcap_{k \geq i} \{\chi_k^i = 1\} = \{\omega : s_k^i(\omega) < \sup\{s_j^i : j \geq i\} \text{ for all } k\}. \end{aligned}$$

We shall use the following notations

$$\begin{aligned} 1) \quad C_k^i &= \limsup_{m \rightarrow \infty} \int (h_{i+m}^i - h_{k+m}^k) d\mu, \\ 2) \quad \hat{C}_k^i &= \limsup_{m \rightarrow \infty} \int (\tilde{h}_{i+m}^i - \tilde{h}_{k+m}^k) d\mu, \end{aligned}$$

with

$$\tilde{h}_k^i = \max \{0, -s_{i+1}^i, -s_{i+2}^i, \dots, -s_k^i\}.$$

Theorem 2.1 (*Maximal Ergodic Lemma*). *If $\int |s_{i+1}^i| d\mu$ is bounded, then*

$$\text{a) } \int_{A_{\infty}^i} s_{i+1}^i d\mu \geq C_{i+1}^i + \liminf_{m \rightarrow \infty} \int g_{i+m}^i d\mu \geq C_{i+1}^i.$$

$$\text{b) } \int_{A_{\infty}^i} s_{i+1}^i d\mu \leq -\tilde{C}_{i+1}^i - \liminf_{m \rightarrow \infty} \int \tilde{g}_{i+m}^i d\mu \leq -\tilde{C}_{i+1}^i.$$

Proof.

1) Integration of the Combinatorial Inequality Theorem 1.a) yields for every k

$$\int s_{i+1}^i \psi_k^i d\mu \geq \int (h_{k-1}^i - h_k^{i+1}) d\mu + \int g_k^i d\mu$$

and $\int g_k^i d\mu \geq 0$ is bounded by $\sup_i \int (s_{i+1}^i)^+ d\mu$.

2) ψ_k^i is monotone in k for fixed i and

$$A_{\infty}^i = \bigcup_{k > i} \{\psi_k^i = 1\} = \{\lim_k \psi_k^i = 1\}.$$

3) s_{i+1}^i is negative on the complement of $\{\psi_{i+1}^i = 1\}$. Hence

$$\lim_{k \rightarrow \infty} \int s_{i+1}^i \psi_k^i d\mu = \int_{A_{\infty}^i} s_{i+1}^i d\mu.$$

4) $\int_{A_{\infty}^i} s_{i+1}^i d\mu$ is bounded below by the upper limit of

$$\int (h_{i+m}^i - h_{i+m+1}^{i+1}) d\mu + \int g_{i+m+1}^i d\mu.$$

The assertions b) are obtained if a) is applied to the sequence $s_{i+1}^i = -s_{i+1}^i$ (compare Theorem 1).

Remark. For a) we only used that $\int (s_{i+1}^i)^+ d\mu$ is bounded, for b), that $\int (s_{i+1}^i)^- d\mu$ is bounded.

Corollary 2.1. *Let $s_{i+1}^i = x^i - p^i$ with $\int |x^i| d\mu$ bounded, $p^i \geq 0$. If $A = \{\omega: \sup_{k > i} s_k^i > 0\}$, then*

$$\int_A p^i d\mu \leq -C_{i+1}^i + \int_{\Omega} |x^i| d\mu.$$

Proof. $A \subset \{\omega: \sup s_k^i > 0\} \subset A_{\infty}^i$ and $\int_{A_{\infty}^i} (x^i - p^i) d\mu \geq C_{i+1}^i$.
Hence

$$\int_A p^i d\mu \leq \int_{A_{\infty}^i} p^i d\mu \leq -C_{i+1}^i + \int_{A_{\infty}^i} x^i d\mu \leq -C_{i+1}^i + \int_{\Omega} |x^i| d\mu.$$

Theorem 2.2. (*Exact Maximal Lemma*). Let $\{s_{i+1}^i\}$ be a sequence of real functions with $\int (s_{i+1}^i)^+ d\mu$ bounded. Denote by $\tau(i, \omega)$ the smallest natural number $j = \tau(i, \omega) \geq i$ for which

$$s_i^i(\omega) = \sup \{s_k^i(\omega) : i \leq k\} = h_\infty^i(\omega);$$

for $\omega \in A^i = \{\omega : h_k^i(\omega) < h_\infty^i(\omega) \text{ for all } k\}$ we put $\tau(i, \omega) = +\infty$; for every number $k > i$ $\tau(i) \wedge k$ denotes the minimum of k and $\tau(i, \omega)$. Then

$$\int s_{\tau(i) \wedge k}^i d\mu = C_k^i + G_k^i + \int_{\{\tau(i) < k\}} h_\infty^k d\mu,$$

with the abbreviation

$$G_k^i = \liminf_{m \rightarrow \infty} \int (g_{i+m}^i + g_{i+1+m}^{i+1} + \cdots + g_{k-1+m}^{k-1}) d\mu.$$

Proof. By Lemma 4.a)

$$\begin{aligned} s_{i+1}^i + d_{i+m}^i &= h_{i+m-1}^i - h_{i+m}^{i+1} + g_{i+m}^i \\ s_{i+2}^{i+1} + d_{i+1+m}^{i+1} &= h_{i+m}^{i+1} - h_{i+m+1}^{i+2} + g_{i+1+m}^{i+1} \\ &\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ s_k^{k-1} + d_{k-1+m}^{k-1} &= h_{k+m-2}^{k-1} - h_{k+m-1}^k + g_{k-1+m}^{k-1}. \end{aligned}$$

Addition and integration yields

$$\begin{aligned} \int (s_{i+1}^i + d_{i+m}^i + \cdots + s_k^{k-1} + d_{k-1+m}^{k-1}) d\mu \\ = \int (h_{i+m-1}^i - h_{k+m-1}^k) d\mu + \int (g_{i+m}^i + \cdots + g_{k-1+m}^{k-1}) d\mu. \end{aligned}$$

For $m \rightarrow \infty$ the integral to the left converges. Hence

$$\begin{aligned} \int (s_{i+1}^i + d_\infty^i + \cdots + s_k^{k-1} + d_\infty^{k-1}) d\mu \\ = \limsup_{m \rightarrow \infty} \int (h_{i+m-1}^i - h_{k+m-1}^k) d\mu + \liminf_{m \rightarrow \infty} \int (g_{i+m}^i + \cdots + g_{k-1+m}^{k-1}) d\mu \\ = C_k^i + G_k^i. \end{aligned}$$

The Combinatorial Theorem 2 gives

$$\begin{aligned} \int (s_{i+1}^i + d_\infty^i + \cdots + s_k^{k-1} + d_\infty^{k-1}) d\mu \\ = \int [s_k^i \cdot \chi_k^i + s_{i+1}^i (\chi_i^i - \chi_{i+1}^i) + \cdots + s_k^i (\chi_{k-1}^i - \chi_k^i)] d\mu - \int h_\infty^k (1 - \chi_k^i) d\mu \\ = \int s_{\tau(i) \wedge k}^i d\mu - \int_{\{\tau(i) < k\}} h_\infty^k d\mu. \end{aligned}$$

In fact, $\tau(i, \omega) = j$ is equivalent with $1 = \chi_{j-1}^i(\omega) > \chi_j^i(\omega) = 0$.

Corollary 2.2. *If $\int (s_{i+1}^i)^+ d\mu$ is bounded, then*

$$\int_{\{\sup_{k>i} s_k^i \geq 0\}} s_{i+1}^i d\mu \geq C_{i+1}^i + G_{i+1}^i .$$

Proof.

1) It should be remarked that the corollary is an improvement on the estimate of Theorem 2.1, since

$$\{\sup_{k>i} s_k^i \geq 0\} \supset A_{\infty}^i ,$$

and s_{i+1}^i is negative on the complement of A_{∞}^i .

$$2) \{\tau(i) \neq i\} = \{h_{\infty}^i > 0\} = \{\sup_{k>i} s_k^i > 0\},$$

$$\{\tau(i) = i\} = \{\tau(i) < i+1\} = \{\sup_{k>i} s_k^i = 0\} \cup \{\sup_{k>i} s_k^i < 0\}.$$

3) On $\{\sup_{k>i} s_k^i = 0\}$ the functions h_{∞}^{i+1} and $-s_{i+1}^i$ coincide, furthermore $h_{\infty}^i = 0$.

4) Theorem 2.2. gives

$$\begin{aligned} \int_{\{\tau(i) \neq i\}} s_{i+1}^i d\mu &= \int s_{\tau(i) \wedge (i+1)}^i d\mu \\ &= C_{i+1}^i + G_{i+1}^i + \int_{\{\sup_{k>i} s_k^i = 0\}} -s_{i+1}^i d\mu + \int_{\{\sup_{k>i} s_k^i < 0\}} h_{\infty}^{i+1} d\mu. \end{aligned}$$

$$\begin{aligned} 5) \int_{A_{\infty}^i} s_{i+1}^i d\mu &\geq \int_{\{\sup_{k>i} s_k^i \geq 0\}} s_{i+1}^i d\mu \\ &= C_{i+1}^i + G_{i+1}^i + \int_{\{\sup_{k>i} s_k^i < 0\}} h_{\infty}^{i+1} d\mu \geq C_{i+1}^i + G_{i+1}^i . \end{aligned}$$

Corollary 2.3. *If $\int (s_{i+1}^i)^+ d\mu$ is bounded then for every $k > i$*

$$\int h_k^i d\mu = C_k^i + G_k^i + \int \min(l_k^i, h_{\infty}^k) d\mu.$$

Proof.

$$1) \text{ On } \{\tau(i) \leq k\} \quad s_{\tau(i) \wedge k}^i(\omega) = h_k^i(\omega) = h_{\infty}^i(\omega).$$

$$\text{On } \{\tau(i) \geq k\} \quad s_{\tau(i) \wedge k}^i(\omega) = h_k^i(\omega) - l_k^i(\omega) = s_k^i(\omega).$$

$$2) \text{ On } \{\tau(i) \leq k\} \quad l_k^i(\omega) \geq h_{\infty}^k(\omega).$$

$$\text{On } \{\tau(i) \geq k\} \quad l_k^i(\omega) \leq h_{\infty}^k(\omega).$$

3) By Theorem 2.2,

$$C_k^i + G_k^i + \int_{\{\tau(i) < k\}} h_\infty^k d\mu = \int s_{\tau(i) \wedge k}^i d\mu = \int h_k^i d\mu - \int_{\{\tau(i) \geq k\}} l_k^i d\mu,$$

$$\int h_k^i d\mu - C_k^i - G_k^i = \int_{\{\tau(i) < k\}} h_\infty^k d\mu + \int_{\{\tau(i) \geq k\}} l_k^i d\mu = \int \min(h_\infty^k, l_k^i) d\mu.$$

Theorem 2.3. For a sequence $\{s_{i+1}^i\}$ with $\int (s_{i+1}^i)^- d\mu$ bounded holds for $k \geq i$

a) $\limsup_{m \rightarrow \infty} \frac{1}{m} \int (s_m^0)^+ d\mu \leq \frac{1}{k} \limsup_{m \rightarrow \infty} \frac{1}{m+1} \int (h_k^0 + h_{1+k}^1 + \dots + h_{m+k}^m) d\mu.$

b) $\limsup_{m \rightarrow \infty} \frac{1}{m} \int (s_m^0)^- d\mu \leq \frac{1}{k} \limsup_{m \rightarrow \infty} \frac{1}{m+1} \int (l_k^0 + l_{1+k}^1 + \dots + l_{m+k}^m) d\mu.$

Proof of a). Integration of the combinatorial inequality Theorem 3.a) gives

$$\frac{1}{m} \int (s_{i+m}^i)^+ d\mu \leq \frac{1}{m} \int h_{i+k}^i d\mu + \frac{1}{m(k+1)} \int (h_{i+1+k}^i + \dots + h_{i+m+k}^{i+m-1}) d\mu.$$

Since

$$\int h_{i+k}^i d\mu \leq k \cdot \int (|s_{i+1}^i| + \dots + |s_{i+k}^{i+k-1}|) d\mu,$$

the inequality implies the existence of $\epsilon(m)$ with $\epsilon(m) \rightarrow 0$ and

$$\frac{1}{m} \int (s_{i+m}^i)^+ d\mu \leq \frac{1}{k+1} \limsup_{m \rightarrow \infty} \frac{1}{m} \int (h_k^0 + \dots + h_{m-1+k}^{m-1}) d\mu + \epsilon(m).$$

Proof of b). Integration of Theorem 3.b) gives

$$\begin{aligned} \frac{1}{m} \int (s_{i+m+k}^{i+k})^- d\mu &\leq \frac{1}{m(k+1)} \int (l_{i+1+k}^i + \dots + l_{i+m+k}^{i+m-1}) d\mu + \frac{1}{m} \int l_{i+m+k}^{i+m} d\mu \\ &\leq \frac{1}{k+1} \limsup_m \frac{1}{m} \int (l_{1+k}^0 + \dots + l_{m+k}^{m-1}) d\mu + \epsilon(m), \end{aligned}$$

with $\epsilon(m) \rightarrow 0$, since

$$\frac{1}{m} \int l_{i+m+k}^{i+m} d\mu \leq \frac{1}{m} \cdot k \cdot \sup_i \int (s_{i+1}^i)^- d\mu.$$

Corollary 2.4 If $\{s_{i+1}^i\}$ is such that $\int (s_{i+1}^i)^- d\mu$ is bounded and

$$\frac{1}{m+1} \int (l_k^0 + l_{1+k}^1 + \dots + l_{m+k}^m) d\mu$$

converges for every k . Then for $k \geq 1$

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \int (s_m^0)^- d\mu \leq \liminf_{m \rightarrow \infty} \frac{1}{m+1} \int (d_k^0 + d_{1+k}^1 + \dots + d_{m+k}^m) d\mu.$$

If in particular there exists $\epsilon(k)$ with $\epsilon(k) \rightarrow 0$ and $\int d_{m+k}^m d\mu \leq \epsilon(k)$ for all m , then

$$\frac{1}{m} \int (s_m^0)^- d\mu \rightarrow 0.$$

Proof.

1) By Lemma 2.6)

$$d_k^i = l_k^i - l_k^{i+1}.$$

Hence

$$\begin{aligned} \frac{1}{m+1} \int (d_k^0 + \dots + d_{m+k}^m) d\mu \\ = \frac{1}{m+1} \int (l_k^0 + \dots + l_{m+k}^m) d\mu - \frac{1}{m+1} \int (l_k^1 + \dots + l_{m+k}^{m+1}) d\mu \end{aligned}$$

converges for $m \rightarrow \infty$.

2) These limits are decreasing with $k \rightarrow \infty$ and

$$\begin{aligned} \frac{1}{k} \cdot \lim_{m \rightarrow \infty} \frac{1}{m+1} \int (l_k^0 + \dots + l_{m+k}^m) d\mu \\ = \frac{1}{k} \cdot \left[\lim_{m \rightarrow \infty} \frac{1}{m+1} \int (d_1^0 + \dots + d_{m+1}^m) d\mu + \dots \right. \\ \left. + \lim_{m \rightarrow \infty} \frac{1}{m+1} \int (d_k^0 + \dots + d_{m+k}^m) d\mu \right], \end{aligned}$$

which tends with $k \rightarrow \infty$ to

$$\inf_k \lim_{m \rightarrow \infty} \frac{1}{m+1} \int (d_m^0 + \dots + d_{m+k}^m) d\mu.$$

III. Processes with nearly stationary increments.

Theorem 3.1'. Let $x^0, x^1, x^2, \dots; p^0, p^1, p^2, \dots$ be integrable and $p^i \geq 0$. Denote for constants α and β

$$\begin{aligned} {}^\alpha h_{k+1}^i &= \max \{0, x^i - \alpha p^i, \dots, x^i + \dots + x^k - \alpha(p^i + \dots + p^k)\} \\ {}^\beta l_{k+1}^i &= \max \{0, x^k - \beta p^k, \dots, x^i + \dots + x^k - \beta(p^i + \dots + p^k)\}. \end{aligned}$$

Assume that for some $\alpha > \beta$

- 1) $\int (x^i - \beta p^i) d\mu$ is independent of i ;
- 2) $\int {}^\alpha h_{k+1}^i d\mu$ and $\int {}^\beta l_{k+1}^i d\mu$ are independent of i for every k ;
- 3) For μ -almost all ω there exist $j = j(\omega) \geq 0$ and $k = k(\omega) \geq 0$ with

$$\begin{aligned} x^i(\omega) + \dots + x^{i+j}(\omega) &< \beta(p^i(\omega) + \dots + p^{i+j}(\omega)), \\ x^i(\omega) + \dots + x^{i+k}(\omega) &> \alpha(p^i(\omega) + \dots + p^{i+k}(\omega)). \end{aligned}$$

Then $p^i = 0$ μ -a.e. and $\int x^i d\mu = 0$.

Theorem 3.2'. If $s_1^0, s_2^1, \dots$ are integrable with

- 1) $\int (s_{i+1}^i)^- d\mu$ is bounded,
- 2) $\int l_{i+k}^i d\mu$ is independent of i for every k ,
- 3) $\sup_{k>0} s_k^0(\omega) \geq 0$ μ -a.e.

then

$$\frac{1}{m} \int (s_m^0)^- d\mu \rightarrow 0.$$

Theorem 3.3'. If a sequence $s_1^0, s_2^1, s_3^2, \dots$ of summable functions satisfies

- 1) $\int h_{i+k}^i d\mu$ and $\int \tilde{h}_{i+k}^i d\mu$ is independent of i for $k \geq 1$,
- 2) $s_k^0(\omega)$ is neither bounded above nor bounded below for almost all ω ;

then

$$\frac{1}{m} \int \max \{ |s_k^0(\omega)| : 0 \leq k \leq m \} d\mu \rightarrow 0, \text{ as } m \rightarrow \infty.$$

We shall prove generalizations of these theorems: The first is of the type used in the common proofs of Birkhoff's Theorem, the second and the third are weak laws of large numbers for nearly stationary sequences.

Theorem 3.1. Let $x^0, x^1, \dots, p^0, p^1, \dots$ be summable with $p^i \geq 0$ and let α, β be real numbers with $\alpha > \beta$. If

$$1) S = \lim_{m \rightarrow \infty} \int (x^m - \beta p^m) d\mu \text{ exists,}$$

$$2) \limsup_{m \rightarrow \infty} \int ({}^\alpha h_{m-1}^0 - {}^\alpha h_m^1) d\mu \geq 0,$$

$$\limsup_{m \rightarrow \infty} \int ({}^\beta \tilde{h}_{m-1}^0 - {}^\beta \tilde{h}_m^1) d\mu$$

$$= \limsup_{m \rightarrow \infty} \int ({}^\beta \tilde{l}_{m-1}^0 - {}^\beta \tilde{l}_m^1) d\mu + S - \int (x^0 - \beta p^0) d\mu$$

$$\geq S - \int (x^0 - \beta p^0) d\mu,$$

$$3) \sup_n \frac{x^0 + \dots + x^n}{p^0 + \dots + p^n} > \alpha > \beta > \inf_n \frac{x^0 + \dots + x^n}{p^0 + \dots + p^n} \text{ } \mu\text{-almost everywhere,}$$

then

$$\int p^0 d\mu \leq \frac{1}{\alpha - \beta} \left(\int (x^0 - \beta p^0) d\mu - S \right) \text{ and } S \leq 0.$$

Proof. The Maximal Lemma Theorem 2.1 applied to the sequences $\{^{\alpha}s_{i+1}^i = x^i - \alpha p^i\}$ respectively $\{^{\beta}s_{i+1}^i = x^i - \beta p^i\}$ yields

- a) $\int (x^0 - \alpha p^0) d\mu \geq ^{\alpha}C_1^0 \geq 0$,
 b) $\int (x^0 - \beta p^0) d\mu \leq -^{\beta}\tilde{C}_1^0 \leq \int (x^0 - \beta p^0) d\mu - S$.

In fact:

$$^{\alpha}A_{\infty}^0 = \{\omega: x^0(\omega) + \dots + x^n(\omega) - \alpha(p^0(\omega) + \dots + p^n(\omega)) > 0 \text{ for some } n \geq 0\} = \Omega;$$

$$^{\beta}\tilde{A}_{\infty}^0 = \{\omega: x^0(\omega) + \dots + x^n(\omega) - \beta(p^0(\omega) + \dots + p^n(\omega)) < 0, \text{ for some } n \geq 0\} = \Omega$$

modulo nullsets by assumption 3).

The difference of these inequalities gives

$$\int (-\alpha + \beta)p^0 d\mu \geq S - \int (x^0 - \beta p^0) d\mu.$$

Since $\alpha > \beta$ the assertion is proved.

Remark. Theorem 3.1 is applicable to the following classical situations:

I) T is a positive contraction of the real $L^1(\Omega, \mathcal{L}, \mu)$; $x, p \in L^1$ and $p \geq 0$; $x^i = T^i x$, $p^i = T^i p$; α, β arbitrary real. In fact a simple calculation shows

$$\int h(x^i, x^{i+1}, \dots; p^i, p^{i+1}, \dots) d\mu \geq \int h(x^{i+1}, x^{i+2}, \dots; p^{i+1}, p^{i+2}, \dots) d\mu$$

for every nonnegative h of the form

$$h(\xi^0, \xi^1, \dots; \eta^0, \eta^1, \dots) = \max_i \left\{ \sum_i (\alpha_{i,i} \xi^i - \beta_{i,i} \eta^i) \right\}$$

with $\alpha_{i,i}, \beta_{i,i}$ arbitrary real.

II) T is a contraction of the complex $L^1(\Omega, \mathcal{L}, \mu)$; $x \in L^1$, $x^i = |T^i x|$; $p^0, p^1, \dots$ are such that for every $y \in L^1$ with $|y| \leq p^i$ holds $|Ty| \leq p^{i+1}$ $\alpha > \beta > 0$.

In this case the inequality of the integrals above holds for those nonnegative h for which $\alpha_{i,i}, \beta_{i,i} \geq 0$, (Compare the explicit form of the $^{\alpha}h_{k+1}^i, ^{\beta}\tilde{l}_{k+1}^i$ given in Theorem 3.1').

Theorem 3.2' is an easy consequence of Corollary 2.4. In fact, condition 3) can be written

$$[\max(s_1^0, \dots, s_k^0)]^- = d_k^0 \rightarrow 0 \text{ almost every where as } k \rightarrow \infty \text{ } d_{i+k}^i = l_{i+k}^i - l_{i+k}^{i+1} \text{ (Lemma 2.b) together with condition 2) shows } \int d_{i+k}^i d\mu \rightarrow 0 \text{ uniformly in } i \text{ for } k \rightarrow \infty.$$

The theorem is not easily applied to the situations described above.

Theorem 3.3. If a sequence of summable functions $\{s_{i+1}^i\}$ satisfies

1) $\int |s_{i+1}^i| d\mu$ is bounded,

2) $\liminf_{k \rightarrow \infty} \frac{1}{k} (B_k^0 + \tilde{B}_k^0) \geq 0$ where the notations are used

$$B_k^0 = \liminf_{m \rightarrow \infty} \int (h_m^0 - h_{k+m}^k) d\mu,$$

$$\tilde{B}_k^0 = \liminf_{m \rightarrow \infty} \int (\tilde{h}_m^0 - \tilde{h}_{k+m}^k) d\mu,$$

3) $s_n^0(\omega)$ is neither bounded above nor bounded below, μ -a.e. then

$$\frac{1}{m} \int \max \{ |s_k^0| : 0 \leq k \leq m \} d\mu \rightarrow 0.$$

Proof.

1) The method of proving the Exact Maximal Lemma can be applied to prove

$$\int s_{\tau(0) \wedge k}^0 d\mu \geq B_k^0 + \limsup_m \int (g_m^0 + g_{1+m}^1 + \cdots + g_{k-1+m}^{k-1}) d\mu$$

of after replacing $\{s_{i+1}^i\}$ by $\{\tilde{s}_{i+1}^i = -s_{i+1}^i\}$

$$-\int s_{\tau(0) \wedge k}^0 d\mu \geq \tilde{B}_k^0 + \limsup_m \int (\tilde{g}_m^0 + \tilde{g}_{1+m}^1 + \cdots + \tilde{g}_{k-1+m}^{k-1}) d\mu$$

where $\tau(0)$ denotes the first index where $s_n^0(\omega)$ assumes its infimum.

2) By the assumption 3) $\tau(0) = +\infty = \tilde{\tau}(0)$ and therefore

$$\int s_{\tau(0) \wedge k}^0 d\mu = \int s_k^0 d\mu; \quad -\int s_{\tilde{\tau}(0) \wedge k}^0 d\mu = -\int s_k^0 d\mu.$$

Addition of the formulas gives

$$\begin{aligned} -(B_k^0 + \tilde{B}_k^0) &\geq \limsup \int (g_m^0 + \cdots + g_{k-1+m}^{k-1}) d\mu \\ &\quad + \limsup \int (\tilde{g}_m^0 + \cdots + \tilde{g}_{k-1+m}^{k-1}) d\mu. \end{aligned}$$

Since the right side is nonnegative, assumption 2) implies

$$\frac{1}{k} (B_k^0 + \tilde{B}_k^0) \rightarrow 0,$$

$$\frac{1}{k} \limsup_m \int (g_m^0 + \cdots + g_{k-1+m}^{k-1}) d\mu \rightarrow \infty$$

and correspondingly for the $\tilde{g}_m^0$.

3) For $\epsilon > 0$ there exists k and $m(k, \epsilon)$ such that for $m \geq m(k, \epsilon)$

$$\begin{aligned}
\epsilon &\geq \frac{1}{m \cdot k} \int [(g_1^0 + \cdots + g_k^{k-1}) \\
&\quad + (g_2^0 + \cdots + g_{k+1}^{k-1}) + \cdots + (g_m^0 + \cdots + g_{k-1+m}^{k-1})] d\mu \\
&= \frac{1}{m \cdot k} \int (h_m^0 + h_{1+m}^1 + \cdots + h_{k-1+m}^{k-1}) d\mu.
\end{aligned}$$

4) If we denote $\sigma = \sup \int |s_{i+1}^i| d\mu$ then

$$\begin{aligned}
\left| \int h_{i+m}^i d\mu - \int h_m^0 d\mu \right| &\leq \int |s_i^0| d\mu + \int |s_{m+i}^m| d\mu \leq 2 \cdot j \cdot \sigma. \\
\int (h_m^0 + h_{1+m}^1 + \cdots + h_{k-1+m}^{k-1}) d\mu &\geq k \cdot \int h_m^0 d\mu - 2\sigma \cdot \frac{(k-1)(k-2)}{2}. \\
5) \quad \epsilon &\geq \frac{1}{m \cdot k} \int (h_m^0 + \cdots + h_{k-1+m}^{k-1}) d\mu \geq \frac{1}{m} \int h_m^0 d\mu - \sigma \cdot \frac{(k-1)(k-2)}{m \cdot k}.
\end{aligned}$$

This holds for all sufficiently large m ; hence

$$\limsup_m \frac{1}{m} \int (h_m^0) d\mu \leq 2\epsilon.$$

Since $\epsilon > 0$ was chosen arbitrarily small we have proved

$$\frac{1}{m} \int h_m^0 d\mu \rightarrow 0.$$

6) By the same method follows

$$\frac{1}{m} \int \tilde{h}_m^0 d\mu \rightarrow 0.$$

The proof is concluded by the remark

$$\max \{ |s_k^0| : 0 \leq k \leq m \} \leq h_m^0 + \tilde{h}_m^0.$$

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