Toeplitz Operators with Locally Sectorial Symbols

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Let L^2 denote the complex Lebesgue space for normalized Lebesgue measure on the unit circle $\mathbf{T}$, H^2 the associated Hardy space of functions with zero negative Fourier coefficients, and P the projection of L^2 onto H^2 . For φ in L^∞ we define the Toeplitz operator T_φ on H^2 by $T_\varphi f = P(\varphi f)$ for f in H^2 . The operator T_φ is clearly bounded and the mapping $\varphi \rightarrow T_\varphi$ is linear, preserves adjoints, and is isometric. The mapping is not, in general, multiplicative, and it is this fact which makes the problem of deciding which T_φ are invertible both interesting and nontrivial. In this note we give a purely local condition on the function φ from which one can deduce that the Toeplitz operator T_φ is a Fredholm operator and can determine by inspection whether it is invertible.

Our results generalize those previously obtained by Widom [5] and Devinatz [2] for the case of piecewise continuous functions.

A measurable function is called *sectorial* if its essential values lie in an open sector of opening less than π having its vertex at the origin and *locally sectorial* if every point in the domain has a neighborhood on which the function is sectorial. We shall show that if φ is bounded away from zero and is locally sectorial, then T_φ is a Fredholm operator. We shall also determine the index of T_φ (that is, the dimension of its null space minus the codimension of its range).

The requirement that φ be bounded away from zero is necessary for T_φ to be a Fredholm operator. (This is essentially due to Hartman and Wintner [3].) The condition that φ be locally sectorial, although not necessary, seems to be the most general local sufficient condition one can find.

The crucial fact is that it is possible to assign an index (the winding number about the origin for a non-vanishing continuous function) to any locally sectorial

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function. Recall that for the real-valued function f one defines

$$f^+(t_0) = \limsup_{t \rightarrow t_0} f(t) \quad \text{and} \quad f^-(t_0) = \liminf_{t \rightarrow t_0} f(t).$$

The difference $f^+(t) - f^-(t)$ is the *oscillation* of f at t .

Lemma. *Suppose φ is locally sectorial on $\mathbf{T}$. Then there is a real-valued measurable function θ defined on the real line which satisfies*

$$\varphi(e^{it})e^{-i\theta(t)} > 0 \quad \text{a.e. and} \quad \theta^+(t) - \theta^-(t) < \pi.$$

Moreover, any two such functions differ by a constant integral multiple of 2π .

Proof. Let $I = (t_1, t_2)$ be a maximal open interval on which there is defined a function θ satisfying both conditions. We shall show that $I = (-\infty, \infty)$.

Suppose for example that $t_1 > -\infty$. Then since φ is locally sectorial there is an open interval N containing t_1 , and a function θ' defined on N satisfying the conditions. On $I \cap N$ the function $\theta - \theta'$ takes values which are integral multiples of 2π and yet has oscillation less than 2π . This implies $\theta - \theta'$ is a constant of the form $2\pi n$ (n integral) on $I \cap N$. But then

$$\theta''(t) = \begin{cases} \theta(t), & t \in I \\ \theta'(t) + 2\pi n, & t \in N, \end{cases}$$

defines a function on the larger interval $I \cup N$ satisfying both conditions. This contradicts the maximality of I .

Thus the existence of θ is established. The argument for the “moreover” statement of the Lemma has already been given. Any other function θ' with the requisite properties differs from θ by a function taking values which are all integral multiples of 2π and having oscillation less than 2π at each point. This difference is consequently a constant integral multiple of 2π .

It is now easy to define the index of φ . Along with $\theta(t)$ the function $\theta(t + 2\pi)$ satisfies the conditions of the Lemma. Then there exists an integer n such that

$$\theta(t + 2\pi) - \theta(t) = 2\pi n,$$

and this integer is the index of φ .

Theorem. *If φ is a locally sectorial function on $\mathbf{T}$ that is bounded away from zero, then T_φ is a Fredholm operator whose index is minus the index of φ . Moreover, T_φ is invertible if and only if the index of φ is zero.*

Proof. Assume first that φ has index zero. Then the function $\theta(t)$ has period 2π and hence there is a function f defined on $\mathbf{T}$ satisfying

$$f(e^{it}) = \theta(t) \quad \text{and} \quad f^+(z) - f^-(z) < \pi.$$

This last inequality may be written

$$f^+(z) - \frac{\pi}{2} < f^-(z) + \frac{\pi}{2}.$$

The right side of this inequality is lower semi-continuous and the left side upper semi-continuous. The right side minus the left is positive and lower semi-continuous and therefore (since it achieves its lower bound) is always greater than or equal to some $\delta > 0$. Thus

$$f^+(z) - \frac{\pi}{2} + \frac{\delta}{2} \leq f^-(z) + \frac{\pi}{2} - \frac{\delta}{2},$$

and by a theorem of H. Hahn [4, p. 72] there exists a continuous function g on $\mathbf{T}$ satisfying

$$f^+(z) - \frac{\pi}{2} + \frac{\delta}{2} \leq g(z) \leq f^-(z) + \frac{\pi}{2} - \frac{\delta}{2}.$$

An application of Fejér's Theorem yields a polynomial p in z and $\bar{z}$ satisfying

$$f^+(z) - \frac{\pi}{2} + \frac{\delta}{4} \leq p(z) \leq f^-(z) + \frac{\pi}{2} - \frac{\delta}{4}.$$

Now since

$$\varphi(z)e^{-if(z)} > 0$$

for each z , the function

$$\psi(z) = \varphi(z)e^{-ip(z)},$$

is bounded away from zero and has values lying in the open right half-plane. Therefore, by a result of Brown and Halmos [1, p. 99], T_ψ is invertible. Moreover, if we write $p = p_1 + p_2$, where p_1 and p_2 are polynomials in z and $\bar{z}$, respectively, and set $q_i = \exp \{ip_i\}$, then

$$T_\varphi = T_{q_2} T_\psi T_{q_1}$$

and hence T_φ is invertible.

Returning now to the general case, define $e_n(z) = z^n$ and let φ have index n . We easily see that the function $e_{-n}\varphi$ is also locally sectorial and has index zero; hence $T_{e_{-n}\varphi}$ is invertible. If $n > 0$, then

$$T_\varphi = T_{e_{-n}\varphi} T_{e_n}.$$

Since T_{e_n} is clearly Fredholm with index $-n$, the same is true of T_φ . A similar argument holds with $n < 0$, and the theorem is established.

We conclude with two comments. If φ is a function on $\mathbf{T}$ which is locally sectorial and bounded away from zero, we proved that there exists a continuous function h on $\mathbf{T}$ such that $\operatorname{Re}(h\varphi) \geq 0$. A similar result is undoubtedly true for functions on any reasonable space and might be of interest in establishing the ellipticity of operators with discontinuous symbols. Further, an analogous result for matrix valued Toeplitz operators would be of interest. A possible definition of sectorial for a matrix function is that the closed convex hull of its essential values consist only of invertible matrices. An immediate question which arises in trying to

make use of this definition is the following: if $\mathcal{K}$ is a compact convex set of invertible matrices, do there exist invertible matrices S and T such that

$$SKT + (SKT)^* \geq \epsilon I > 0 \quad \text{for } K \text{ in } \mathcal{K}?$$

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