*The Discrete Analog of a Minimum Problem in Conformal Mapping**

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Communicated by P. R. GARABEDIAN

1. Introduction In this paper it is shown that the solution of a minimum problem in the theory of conformal mapping can be approximated by discrete functions. Using these discrete functions, the Riemann mapping function can be approximated for a certain class of simply-connected regions (domains). It is also shown that the Bergman kernel function can be approximated, for certain regions, by using these discrete functions.

Discrete methods have been employed by Ferrand [3] to approximate the conformal mapping of a simply-connected region onto the interior of a triangle and by Lelong-Ferrand [4] to approximate the conformal mapping of a simply-connected region onto the interior of a rectangle. In both cases the mapping function was normalized by specifying the correspondence between sets of boundary points. Lelong-Ferrand also considered the problem of mapping multiply-connected regions.

The conformal mapping of a bounded simply-connected region onto the unit disk can also be obtained by solving a Dirichlet problem. The solution of the Dirichlet problem can be approximated by discrete functions provided the boundary of the region is sufficiently smooth.

There does not appear to be any previously known discrete method for approximating the Bergman kernel function. The Bergman kernel function can be used to obtain solutions to certain interpolation problems.

The main result in this paper is the development of a sequence of discrete functions converging to the solution of a minimum problem. Additionally, there are results which bear on two other concepts: the convergence of discrete functions and the approximation of rational functions by discrete functions.

2. Properties of discrete analytic and discrete harmonic functions. As an introduction to the theory of discrete harmonic and discrete analytic functions,

* This work was partially supported by Texas Christian University Research Foundation Grant No. M6891.

some basic properties of these functions will be summarized. A detailed development, as well as the proofs of the results stated here, may be found in the papers of Duffin [2], Ferrand [3], and Lelong-Ferrand [4].

Let Π denote the complex plane and, for each nonnegative integer n , let

$$\Pi_n = \{z \in \Pi : z = 2^{-n}(p + iq), p, q \text{ integers}\}.$$

A nonempty subset of Π_n will be called a *net* and the points of the subset will be called *nodes*. A node is called an *even node* if $p + q$ is even and an *odd node* if $p + q$ is odd. A *net square* of Π_n is a closed convex subset of Π whose boundary is a square with side of length 2^{-n} and whose vertices are nodes of Π_n .

Let D be a net. A net square of Π_n is said to be a *net square* of D if all its vertices are nodes of D . Let $\mathfrak{D}$ be the union of the net squares of D . A node is an *interior node* of D if it is an interior point of $\mathfrak{D}$. Similarly, a node is a *boundary node* of D if it is a boundary point of $\mathfrak{D}$. The nodes of D which are not contained in $\mathfrak{D}$ are *isolated nodes*. Let D' and D'' denote the even and odd nodes of D , respectively. We will call 2^{-n} the *net width* of D and will denote it by h .

A complex-valued function f defined on D is said to be *discrete analytic* on D , if on each net square of D , the difference equation

$$(2.1) \quad f(z) + if(z+h) - f(z+h+ih) - if(z+ih) = 0,$$

is satisfied, where $z, z+h, z+h+ih$, and $z+ih$ denote the vertices of the net square. A real or complex-valued function f defined on $D'(D'')$ is said to be *discrete harmonic* on $D'(D'')$ if the difference equation

$$(2.2) \quad f(z) = \frac{1}{4}[f(z+h+ih) + f(z-h+ih) + f(z-h-ih) + f(z+h-ih)],$$

is satisfied for each z of $D'(D'')$ which is an interior node of D . If f is discrete analytic on D , then f and its real and imaginary parts are discrete harmonic on both D' and D'' . If $z, w \in D$, w is called a *D-neighbor* of z if $|z-w| = h$. If $z, w \in D'(D'')$, w is called a *D'(D'')-neighbor* of z if $|z-w| = 2^{1/2}h$.

A net D is called a *discrete region* if it has no isolated nodes and if the interior of $\mathfrak{D}$ is a region. If D is a discrete region and the boundary of $\mathfrak{D}$ is a simple closed curve, then D is said to be a *simple discrete region*.

Let D be a discrete region. A *path in D'(D'')* is a sequence of nodes $z_0, z_1, \dots, z_m$ in $D'(D'')$ such that for $i = 1, 2, \dots, m$, z_i and z_{i-1} are opposite vertices of a net square of D . A *path in D* is a sequence of nodes $z_0, z_1, \dots, z_m$ in D such that for $i = 1, 2, \dots, m$, z_i is a *D-neighbor* of z_{i-1} . A *closed path* is a path where $z_0 = z_m$. The following line integrals are defined over a path $z_0, z_1, \dots, z_m$ in D for functions f and g defined on D .

$$(2.3) \quad \int_{z_0}^{z_m} f(z) \, \delta z = \frac{1}{2} \sum_{i=1}^m [f(z_i) + f(z_{i-1})](z_i - z_{i-1}).$$

$$(2.4) \quad \int_{z_0}^{z_m} f(z) : g(z) \, \delta z = \frac{1}{4} \sum_{i=1}^m [f(z_i) + f(z_{i-1})][g(z_i) + g(z_{i-1})](z_i - z_{i-1}).$$

$$(2.5) \quad \int_{z_0}^{z_m} f'(z) : g'(z) \delta z = \sum_{i=1}^m [f(z_i) - f(z_{i-1})][g(z_i) - g(z_{i-1})](z_i - z_{i-1})^{-1}.$$

In case the above path, denoted by B , is a closed path, the last two integrals may be expressed as follows:

$$(2.6) \quad \int_B f(z) : g(z) \delta z = \frac{1}{4} \sum_{i=1}^m f(z_i)[g(z_{i-1})(z_i - z_{i-1}) + g(z_i)(z_{i+1} - z_{i-1}) + g(z_{i+1})(z_{i+1} - z_i)];$$

$$(2.7) \quad \int_B f'(z) : g'(z) \delta z = - \sum_{i=1}^m f(z_i) \left[\frac{g(z_{i+1}) - g(z_i)}{z_{i+1} - z_i} - \frac{g(z_i) - g(z_{i-1})}{z_i - z_{i-1}} \right].$$

Here z_{m+1} is understood to be z_1 . For a closed path, B , the integral (2.7) will be denoted by $-\int_B f(z) : g''(z) \delta z$.

If D is a simple discrete region and if the functions f and g are discrete analytic on D , then each of the line integrals is independent of the path of integration.

Let S be a net square of D . Let z_0, z_1, z_2, z_3 be the nodes of S beginning with an even node z_0 and indexed by the order in which they are encountered by proceeding around ∂S in the positive direction from z_0 . If $f = P + iQ$ is defined on D , equation (2.1) may be replaced by the two equations

$$(2.8) \quad P(z_2) - P(z_0) = Q(z_3) - Q(z_1),$$

$$(2.9) \quad P(z_3) - P(z_1) = Q(z_0) - Q(z_2).$$

Thus, if f is discrete analytic on D , then by (2.8) the values of Q on D'' are determined by the values of P on D' and the value of Q at an arbitrary node of D'' . Similarly, by (2.9), the values of Q on D' are determined by the values of P on D'' and the value of Q at an arbitrary node of D' . The functions P and Q are both discrete harmonic on D' and D'' .

Suppose P' is a function which is discrete harmonic on D' . Let L denote an arbitrary path $z_0, z_1, \dots, z_m$ in D'' . For $1 \leq i \leq m$, let u_i and v_i denote the nodes of D' such that z_{i-1}, u_i, z_i, v_i are the nodes of a net square in the order in which they are encountered by proceeding around its boundary in the positive direction from z_{i-1} . A necessary and sufficient condition for the existence of a single-valued function Q'' defined on D'' such that P' and Q'' satisfy equation (2.8) on each net square of D is that

$$(2.10) \quad V_L = \sum_{i=1}^m (P'(v_i) - P'(u_i)),$$

vanishes for every closed path L in D'' . Now Q'' is discrete harmonic and any two such functions differ by a constant. Thus, if $z'' \in D''$ and w'' is a real number, then there is only one Q'' with $Q''(z'') = w''$. A sufficient condition for V_L to vanish is that D be a simple discrete region. The function Q'' is called the *discrete harmonic conjugate* of P' .

A *discrete analytic demi-function* on D is defined to be a pair of functions (P', Q'') , P' defined and discrete harmonic on D' , Q'' defined and discrete harmonic on D'' , such that P' and Q'' satisfy (2.8) on each net square of D .

Let I denote the interior nodes of D . When $I \neq \emptyset$, using P' and Q'' , we define the functions $\bar{P}'$ and $\bar{Q}''$ on I by

$$\bar{P}'(z) = \begin{cases} P'(z), & \text{if } z \in D' \cap I, \\ \frac{1}{4}[P'(z+h) + P'(z+ih) + P'(z-h) + P'(z-ih)], & \text{if } z \in D'' \cap I, \end{cases}$$

$$\bar{Q}''(z) = \begin{cases} Q''(z), & \text{if } z \in D'' \cap I, \\ \frac{1}{4}[Q''(z+h) + Q''(z+ih) + Q''(z-h) + Q''(z-ih)], & \text{if } z \in D' \cap I. \end{cases}$$

On any net square of I , the function $f = \bar{P}' + i\bar{Q}''$ clearly satisfies equation (2.8) and a simple computation will show that f also satisfies equation (2.9). Therefore, f is discrete analytic on I and will be called the discrete analytic function determined by (P', Q'') .

Discrete analytic functions satisfy a maximum principle. A discrete analytic function does not assume its maximum modulus at an interior node. Both a maximum and a minimum principle hold for real-valued discrete harmonic functions. A real-valued discrete harmonic function does not assume its maximum or minimum at an interior node.

3. A minimum problem for a class of discrete analytic functions. Let $L^2(\mathfrak{R})$ denote the set of functions analytic in a bounded region $\mathfrak{R}$ which satisfy $\int_{\mathfrak{R}} |f(z)|^2 dx dy < \infty$. If the inner product of two functions f and g is defined by

$$(f, g)_{\mathfrak{R}} = \int_{\mathfrak{R}} f(z) \overline{g(z)} dx dy,$$

then $L^2(\mathfrak{R})$ is a Hilbert space. If $a \in \mathfrak{R}$, there is a unique function f in $L^2(\mathfrak{R})$ such that $f(a) = 1$ and

$$\|f\|_{\mathfrak{R}} = \inf \{ \|g\|_{\mathfrak{R}} : g(a) = 1, g \in L^2(\mathfrak{R}) \}, \quad \text{where} \quad \|f\|_{\mathfrak{R}}^2 = (f, f)_{\mathfrak{R}}.$$

Thus, there is a unique solution to the problem of finding the function with minimum norm among all functions g in $L^2(\mathfrak{R})$ with $g(a) = 1$. This function is related to the Bergman kernel function of $\mathfrak{R}$, $K(z, t)$, by the equation $K(z, a) = f(z)/\|f\|_{\mathfrak{R}}^2$. In case $\mathfrak{R}$ is simply-connected, the conformal mapping F of $\mathfrak{R}$ onto the unit disk with $F(a) = 0$ and $F'(a) > 0$ is given by the equation

$$F(z) = \frac{\pi^{1/2}}{\|f\|_{\mathfrak{R}}} \int_a^z f(t) dt.$$

We will develop an analogous minimum problem for a class of discrete analytic functions. This class of functions will not be a vector space over the field of com-

plex numbers, and as a result some of the properties of the function f above do not carry over to the discrete case.

Let $\mathfrak{D}$ be the union of the net squares of a bounded discrete region of Π_N . Suppose $n \geq N + 3$ and let $D = \mathfrak{D} \cap \Pi_n$. Let I , D' , and D'' denote the interior, even, and odd nodes of D , respectively.

We note that if (P'_1, Q'_1) and (P'_2, Q'_2) are discrete analytic demi-functions on D which determines the discrete analytic functions f and g , then $f = g$ on I implies $P'_1 = P'_2$ on D' and $Q'_1 = Q'_2$ on D'' . That is, there is a one-to-one correspondence between the set of discrete analytic demi-functions on D and a subset of the set of discrete analytic functions on I . Some properties of this subset will be given.

Let $H(I)$ be the set of all functions discrete analytic on I . Bolen [1] has shown that $H(I)$ is a Hilbert space with the inner product of two functions f and g defined by

$$(f, g)_I = h^2 \sum_{z \in I} f(z) \overline{g(z)}.$$

Let $B(I)$ be the subset of functions in $H(I)$ determined by the discrete analytic demi-functions on D .

Lemma 3.1. *The set $B(I)$ is a closed subset of $H(I)$ with respect to the norm topology.*

Proof. We note, as was proved by Bolen [1], that convergence in the norm of $H(I)$ is equivalent to pointwise convergence. Let f be a limit point of $B(I)$. Then there is a sequence $\{f_k\}$ of functions in $B(I)$ which converges to f . Suppose f_k is determined by (P'_k, Q'_k) . It can be shown that $P'_k(z)$ converges for every z in D' and $Q'_k(z)$ converges for every z in D'' as $k \rightarrow \infty$. Thus P'_k converges to a function P' on D' and Q'_k converges to a function Q'' on D'' . Clearly (P', Q'') is a discrete analytic demi-function which determines f . Hence f belongs to $B(I)$ and $B(I)$ is closed.

If f and g are in $B(I)$ and r is a real number, then $f + g$ and rf belong to $B(I)$.

Theorem 3.2. *The set $B(I)$ with the induced norm is a Banach space over the field of real numbers.*

Since the inner product of $H(I)$ is, in general, not real-valued, $B(I)$ is not a real Hilbert space with respect to this inner product. Furthermore, $B(I)$ is not closed under multiplication by complex constants, and hence, is not a vector space over the field of complex numbers.

Let a belong to I and let Φ be the set of functions in $B(I)$ with the value 1 at a . It can be easily shown that Φ is a closed convex subset of $B(I)$. Thus there is a unique function f in Φ with minimum norm. The function f can be characterized as that function in Φ such that

$$(3.1) \quad \operatorname{Re} \sum_{z \in I} k(z) \overline{f(z)} = 0,$$

for every $k \in B(I)$ with $k(a) = 0$.

We now have the existence and uniqueness of the solution to the problem of finding the function with minimum norm among all functions in $B(I)$ with the value 1 at a . It is desired to obtain a method for the computation of the solution of this minimum problem. In the following we shall assume that D is a simple discrete region.

Let w be an arbitrary node of D'' , and let $z_1, \dots, z_r$ denote the boundary nodes of D' . Let f_0 be the discrete analytic function determined by (P'_0, Q'_0) where $P'_0 = 0$ and $Q'_0 = 1$. For $i = 1, \dots, r$, let f_i be the discrete analytic function determined by (P'_i, Q'_i) where P'_i is the solution of the discrete Dirichlet problem

$$P'_i(z_j) = \begin{cases} 0, & \text{if } z_j \neq z_i \\ 1, & \text{if } z_j = z_i \end{cases},$$

and $Q'_i(w) = 0$. Note that $\{f_0, \dots, f_r\}$ is a basis for $B(I)$. Using the characterization of the solution of the minimum problem by equation (3.1), the function with minimum norm can be expressed as the solution of a system of equations. The function $f = c'_0 f_0 + \dots + c'_r f_r$ is the unique function in Φ with minimum norm if and only if $c'_0, \dots, c'_r$ is the unique solution of the following system of $r + 2$ equations with $r + 1$ unknowns:

$$\sum_{m=0}^r (\alpha_{mk} + \beta_{mk} - \mu_k \gamma_m - \nu_k \lambda_m) c_m = 0, \quad k = 1, \dots, r,$$

$$\sum_{m=0}^r \mu_m c_m = 1, \quad \sum_{m=0}^r \nu_m c_m = 0,$$

where $\alpha_{mk} = \sum_{z \in I} \operatorname{Re} f_m(z) \operatorname{Re} f_k(z)$, $\beta_{mk} = \sum_{z \in I} \operatorname{Im} f_m(z) \operatorname{Im} f_k(z)$, $\gamma_m = \sum_{z \in I} \operatorname{Re} f_m(z)$, $\lambda_m = \sum_{z \in I} \operatorname{Im} f_m(z)$, $\mu_k = \operatorname{Re} f_k(a)$, and $\nu_k = \operatorname{Im} f_k(a)$.

4. Sequences of discrete analytic functions. Much of the remainder of this paper will deal with sequences of discrete functions defined on nets with successively smaller net width. This section provides sufficient conditions for determining whether a given sequence of discrete functions has a convergent subsequence. As might be expected, the notion of equicontinuity is of primary interest.

Lelong-Ferrand [4, p. 157] provides the following criteria for the equicontinuity of sequences of discrete analytic functions determined by discrete analytic demi-functions. Let D_N be a bounded discrete region in Π_N , and let $\mathfrak{D}$ be the union of the net squares of D_N . For $n \geq N$, note that $D_n = \mathfrak{D} \cap \Pi_n$ is a discrete region. Let I_n , D'_n , and D''_n denote the interior, even, and odd nodes of D_n , respectively. Suppose $\{(P'_n, Q'_n)\}$ is a sequence of discrete analytic demi-functions with $n \geq N$ and (P'_n, Q'_n) defined on D_n . Let

$$\alpha_n(P'_n) = \sum_{z \in A_n} |P'_n(z) - P'_n(z + h + ih)|^2 + \sum_{z \in B_n} |P'_n(z) - P'_n(z - h + ih)|^2,$$

where $h = 2^{-n}$ and

$$A_n = \{z \in D'_n : z + h + ih \in D'_n\},$$

$$B_n = \{z \in D'_n : z - h + ih \in D'_n\}.$$

Theorem 4.1. *If f_n is the discrete analytic function on I_n determined by (P'_n, Q'_n) and if there exists a positive number K such that $\alpha_n(P'_n) < K$ for every $n \geq N$, then the sequence $\{f_n\}$ is equicontinuous on every compact subset of the interior of $\mathfrak{D}$, and for each infinite sequence of integers, there is an increasing subsequence $\{n_p\}$ such that $\{f_{n_p}\}$ either approaches infinity at every point of $\bigcup_{n \geq N} \Pi_n$ interior to $\mathfrak{D}$ or converges uniformly on every compact subset of the interior of $\mathfrak{D}$ to a function analytic on the interior of $\mathfrak{D}$.*

We now state a lemma due to Verblunsky [5] which leads to the next theorem.

Lemma 4.2. *If there exists a positive number M such that*

$$\max_{z \in D'_n} |P'_n(z)| < M,$$

for all $n \geq N$, then there exists a positive number K such that for each $n \geq N$, if $z, w \in D'_n$ and $|z - w| = 2^{1/2}h$, then

$$|P'_n(z) - P'_n(w)| < \frac{KM}{j},$$

where j is the least number of nodes of D'_n needed to construct a path in D'_n from either of the points z or w to the boundary of $\mathfrak{D}$.

A sequence $\{P'_n\}$ which satisfies the hypothesis of Lemma 4.2 is said to be uniformly bounded on $\mathfrak{D}$.

Theorem 4.3. *If $\{P'_n\}$ is uniformly bounded on $\mathfrak{D}$ and if f_n is the discrete analytic function on I_n determined by (P'_n, Q'_n) , then the conclusion of Theorem 4.1 holds.*

Lemma 4.2 was a conjecture of Ferrand [3, p. 168] who noted that Theorem 4.3 would follow from this conjecture.

If $\{f_n\}$ is uniformly bounded on compact subsets of the interior of $\mathfrak{D}$, then the same is true for $\{P'_n\}$. It will be shown that for $\{f_n\}$ to be uniformly bounded on compact subsets of the interior of $\mathfrak{D}$, it is sufficient that $\{\|f_n\|_{I_n}\}$ be bounded.

Let $z = (j + ik)h$ belong to Π_n , and define a function d_n for each $n \geq 0$ by

$$d_n(z) = (4\pi^2 h)^{-1} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{iz + ikv} w^{-1}(x, y) dx dy,$$

$$w(x, y) = 1 + ie^{ix} - ie^{iy} - e^{iz + iy}.$$

This function was developed by Duffin [2] who also gave a simple relation between it and the discrete Green's function g_n defined by

$$g_n(z) = (8\pi^2 h)^{-1} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (1 - e^{iz + ikv})(1 + (-1)^{i+k}) |w(x, y)|^{-2} dx dy.$$

This relation is

$$d_n(z) = -g_n(z) + ig_n(z - h) + g_n(z - h - ih) - g_n(z - ih).$$

If z is an even node of Π_n , an asymptotic expression for g_n is

$$2\pi h g_n(z) = \log 4r + \gamma + (\cos 4\theta)/6r^2 + O(r^{-4}),$$

where $r^2 = j^2 + k^2$, $\theta = \tan^{-1}(k/j)$, and γ is Euler's constant.

Let m and M be positive numbers such that $m < M$ and let $z = u + iv$ belong to Π_n with $m \leq |z| \leq M$. By the above expression for g_n , if z is an even node of Π_n , then

$$d_n(z) = -\frac{u+v}{2\pi(u^2+v^2)} + O(h),$$

and if z is an odd node of Π_n , then

$$d_n(z) = -\frac{i(u-v)}{2\pi(u^2+v^2)} + O(h).$$

Thus the sequence $\{d_n\}$ is uniformly bounded on compact subsets of Π which do not contain 0.

For each $n \geq 0$, define the functions s_n and t_n on Π_n by

$$s_n(z) = \frac{1+i}{4} [d_n(z) + d_n(z+h) + d_n(z+h+ih) + d_n(z+ih)],$$

$$t_n(z) = \frac{1-i}{16} [d_n(z) - d_n(z+h) + d_n(z+h+ih) - d_n(z+ih)].$$

Suppose g is discrete analytic on the bounded discrete region D of Π_n , and let z_0 be an interior node of D . From the results of Duffin [2, pp. 342, 346], we have

$$g(z_0) = \int_{\partial D} g(z) : [s_n(z - z_0) + ih^2 t''(z - z_0)] \delta z.$$

Note that both the sequences $\{s_n\}$ and $\{t_n\}$ are uniformly bounded on compact subsets of Π which do not contain 0. We can now state the following lemma and the resulting theorem.

Lemma 4.4. *Let $\mathcal{C}$ be a compact subset of Π and let m and M be positive numbers such that $m < M$. There exists a positive number K such that for any bounded discrete region D_N of Π_N with $\mathcal{C}$ a subset of $\mathcal{D}$, the union of the net squares of D_N , and*

$$\partial \mathcal{D} \subset \{z \in \Pi : m \leq \text{dist}(z, \mathcal{C}) \leq M\},$$

and for any function g discrete analytic on $D_n = \mathcal{D} \cap \Pi_n$, $n \geq N$, we have

$$|g(z_0)| \leq Kh \sum_{z \in B} |g(z)|,$$

for every z_0 in $\mathcal{C} \cap \Pi_n$, where $B = \partial \mathcal{D} \cap \Pi_n$.

Theorem 4.5. *Let $\mathcal{C}$ be a compact subset of the interior of $\mathcal{D}$, the union of the net squares of some bounded discrete region D_N of Π_N . There exists a positive number*

M such that if $n \geq N$ and g is a discrete analytic function on $D_n = \mathfrak{D} \cap \Pi_n$, then $|g(z_0)| \leq M \|g\|_{D_n}$ for every z_0 in $\mathfrak{C} \cap \Pi_n$.

To summarize our results, we may state the following theorem. The proof follows directly from Theorem 4.3 and Theorem 4.5.

Theorem 4.6. *Let D_N be a bounded discrete region in Π_N and let $\mathfrak{D}$ be the union of the net squares of D_N . Suppose f_n is the discrete analytic function on I_n determined by (P'_n, Q'_n) defined on $D_n = \mathfrak{D} \cap \Pi_n$, $n \geq N$, and the sequence $\{\|f_n\|_{I_n}\}$ is bounded. Then the sequence $\{f_n\}$ is equicontinuous on every compact subset of the interior of $\mathfrak{D}$, and for each infinite sequence of integers, there is an increasing subsequence $\{n_p\}$ such that $\{f_{n_p}\}$ converges uniformly on every compact subset of the interior of $\mathfrak{D}$ to a function analytic on the interior of $\mathfrak{D}$.*

5. Approximation of rational functions by discrete analytic functions. Discrete analogs of the powers of z were defined by Ferrand [3] and further developed by Duffin [2]. The use of these analogs in approximating analytic functions was considered by Bolen [1].

For each $n \geq 0$, we define the functions $u_n^0, u_n^1, \dots$ on Π_n inductively by

$$u_n^0(z) = 1, \quad u_n^{k+1}(z) = (k+1) \int_0^z u_n^k(z) \delta z.$$

For all $k = 0, 1, \dots$, u_n^k is discrete analytic on Π_n . If k is fixed, u_n^k converges uniformly on compact subsets of Π to z^k as $n \rightarrow \infty$. Thus, if $f(z) = a_0 + a_1 z + \dots + a_m z^m$ is a polynomial, and $f_n(z) = a_0 u_n^0(z) + a_1 u_n^1(z) + \dots + a_m u_n^m(z)$, then f_n converges uniformly to f on compact subsets of Π as $n \rightarrow \infty$. A somewhat similar result will be established for certain rational functions.

Let r and R be positive numbers such that $r < R$, $r = j2^{-N}$, and $R = k2^{-N}$ for some positive integers j, k , and N . Let

$$\mathfrak{D} = \{z \in \Pi : r \leq \max \{|\operatorname{Re} z|, |\operatorname{Im} z|\} \leq R\},$$

and $D_n = \mathfrak{D} \cap \Pi_n$ for $n \geq N$. Note that D_n is a discrete region and $\mathfrak{D}$ is the union of the net squares of D_n . Let I_n , D'_n , and D''_n denote the interior, even, and odd nodes of D_n , respectively.

Let m be an odd positive integer, and define the function P by $P(z) = \operatorname{Re}(z^{-m})$ for z in $\mathfrak{D}$. For $n > N$, let P'_n be the discrete harmonic function on D'_n which assumes the values of $P(z)$ at the boundary nodes of D'_n . Since m is odd, at the boundary nodes of D'_n ,

$$P'_n(-\bar{z}) = P(-\bar{z}) = -P(z) = -P'_n(z).$$

The function P_n^* , defined by $P_n^*(z) = P'_n(-\bar{z})$ for z in D'_n , is discrete harmonic on D'_n . Thus $P_n^*(-\bar{z}) = -P'_n(z)$ for all z in D'_n . Let L be the closed path $z_0, \dots, z_k$ in D''_n such that the set of points $\{z_1, \dots, z_k\}$ are distinct, and such that for each j , $1 \leq j \leq k$, the points z_{j-1} and z_j lie on a net square of D_n which contains

a point of

$$\{z \in \Pi : \max \{|\operatorname{Re} z|, |\operatorname{Im} z|\} = r\}.$$

Let V_L be as defined in (2.10). Since $P'_n(-\bar{z}) = -P'_n(z)$ for all z in D'_n , $V_L = 0$. This implies that $V_L = 0$ for every closed path L in D''_n . Let a be an interior node of D'_n , and let Q''_n be the discrete harmonic conjugate of P'_n with $Q''_n(a+h) = \operatorname{Im}((a+h)^{-m})$. By the maximum and minimum principles for discrete harmonic functions, the sequence $\{P'_n\}$ is uniformly bounded on $\mathfrak{D}$. Let f_n be the discrete analytic function on I_n determined by (P'_n, Q''_n) . It is well-known that P'_n converges to P uniformly on $\mathfrak{D}$. Theorem 4.3 can now be used in a straightforward argument to show that f_n converges to z^{-m} uniformly on compact subsets of the interior of $\mathfrak{D}$.

The sequence of discrete analytic functions converging to z^{-m} will be used to obtain another sequence of discrete analytic functions which converges to z^{-m+1} . For $n > N$, define the operator Z_n of Duffin [2] by

$$4Z_nf_n(z) = zS_nf_n(z) - i\bar{z}T_nf_n(z),$$

$$S_nf_n(z) = f_n(z) + f_n(z+h) + f_n(z+h+ih) + f_n(z+ih),$$

$$T_nf_n(z) = f_n(z) - f_n(z+h) + f_n(z+h+ih) - f_n(z+ih).$$

Note that Z_nf_n is not defined on certain boundary nodes on I_n . However, Z_nf_n is discrete analytic on the net consisting of the interior nodes of I_n . The function Z_nf_n converges to z^{-m+1} uniformly on compact subsets of the interior of $\mathfrak{D}$.

The above results may be summarized in the following statement. If $\mathfrak{C}$ is a compact subset of Π which does not contain 0, and if m is a positive integer, then there is a sequence $\{f_n\}$, with f_n discrete analytic on $\mathfrak{C} \cap \Pi_n$, such that f_n converges to z^{-m} uniformly on $\mathfrak{C}$. This statement gives the desired result.

Theorem 5.1. *Let g be a rational function whose poles are nodes of some net. Let $\mathcal{K}$ be a compact set containing no poles of g . Then there is a sequence $\{g_n\}$ such that g_n is discrete analytic on $\mathcal{K} \cap \Pi_n$ and g_n converges to g uniformly on $\mathcal{K}$.*

6. Convergence of the solution of the discrete minimum problem. Let $\mathfrak{R}$ be a bounded region of finite connectivity which is bounded by simple closed curves. The problem of finding the function with minimum norm from among all functions in $L^2(\mathfrak{R})$ having the value 1 at a given fixed point will be considered. It will be proven that the solution of this minimum problem can be expressed as the limit of a sequence of discrete functions which are solutions of discrete minimum problems. To this end, a sequence of nets will be defined on $\mathfrak{R}$ in such a way that, on each net, our discrete minimum problem will have a unique solution.

Let $a \in \mathfrak{R}$ be such that a is a node of some net. For each nonnegative integer k , let $\mathfrak{G}_k$ be the largest set such that the following conditions hold:

- (1) $\mathfrak{G}_k$ is the union of net squares of Π_k contained in $\mathfrak{R}$,
- (2) the interior of $\mathfrak{G}_k$ is a region, and
- (3) a is an interior node of $\mathfrak{G}_k \cap \Pi_{k+3}$.

If for some k , there does not exist a set satisfying these conditions, we set $\mathcal{G}_k = \emptyset$.

For each integer n with $n \geq 3$, let $\mathfrak{D}_n = \mathcal{G}_{n-3}$. Assume n is large enough so that $\mathfrak{D}_n \neq \emptyset$. Let I_n denote the interior nodes of $\mathfrak{D}_n \cap \Pi_n$. Let $\mathcal{K}_n$ denote the union of the net squares of I_n . If Φ_n is the class of all discrete analytic functions on I_n determined by discrete analytic demi-functions such that the value of each function in Φ_n at a is 1, then, as was proved in Section 3, there is a unique function f_n in Φ_n with minimum norm. Note that $\mathcal{R} = \bigcup_{n \geq N} K_n$ for all $N \geq 3$. A subsequence of $\{f_n\}$ will be denoted by $\{f_p\}$ where p ranges over an ordered subset of the nonnegative integers. Similar notation will be used to denote subsequences of $\{f_p\}$.

Theorem 6.1. *The function f_n converges uniformly on compact subsets of $\mathcal{R}$ to a function f in $L^2(\mathcal{R})$ with $f(a) = 1$. Furthermore, f minimizes $\|g\|_{\mathcal{R}}$ for all functions g in $L^2(\mathcal{R})$ having $g(a) = 1$.*

Proof. Let $g(z) = P(z) + iQ(z)$ be a rational function such that $g(a) = 1$, and such that the poles of g are nodes of $\Pi_n - \mathcal{R}$ for some n . By Theorem 5.1, there exists a sequence $\{g_n\}$, with g_n discrete analytic on $\overline{\mathcal{R}} \cap \Pi_n$, such that g_n converges to g uniformly on $\mathcal{R}$. Let $g'_n = P'_n + iQ'_n(g''_n = P''_n + iQ''_n)$ be the restriction of g_n to the even (odd) nodes of $D_n = \mathfrak{D}_n \cap \Pi_n$. Assume n is sufficiently large so that a is an even node of I_n . Let $U'_n(z) = P'_n(z) + (1 - P'_n(a))$ and $V''_n(z) = Q''_n(z) - Q''_n(a)$. Since g_n is discrete analytic on D_n , (P'_n, Q'_n) , and hence (U'_n, V''_n) , is a discrete analytic demi-function on D_n . By the uniform continuity of g , $\bar{P}'_n$ converges to P and $\bar{Q}''_n$ converges to Q uniformly on $\mathcal{R}$. Let $r_n = U'_n + i\bar{V}''_n$. The function r_n is discrete analytic on I_n and is determined by (U'_n, V''_n) . Since $r_n(a) = 1$, $\|f_n\|_{I_n} \leq \|r_n\|_{I_n}$ for each n . Since r_n converges to g uniformly on $\mathcal{R}$, $\|r_n\|_{I_n}$ converges to $\|g\|_{\mathcal{R}} < \infty$. Thus there exists an N such that $\|f_n\|_{I_n} < N$ for every n . Let m be any positive integer for which $\mathcal{K}_m$ is defined. For $n \geq m$, let $\|f_n\|_{I_n \cap \mathcal{K}_m} = \|f_n\|_{J_m}$. Then $\|f_n\|_{I_n} \geq \|f_n\|_{J_m}$ for $n \geq m$. Thus $\{\|f_n\|_{J_m}\}$ is bounded. By Theorem 4.6, there is a subsequence of $\{f_n\}$ which converges uniformly on every compact subset of the interior of $\mathcal{K}_m$ to an analytic function. Therefore, there is a subsequence $\{f_p\}$ of $\{f_n\}$ which converges to an analytic function f uniformly on compact subsets of $\mathcal{R}$.

Since f_p converges to f uniformly on $\mathcal{K}_m$ for each m as $p \rightarrow \infty$, for any fixed m there exists an $M > m$ such that $p > M$ implies

$$|\|f_p\|_{J_m} - \|f\|_{\mathcal{K}_m}| \leq 1.$$

Thus if $p > M$, then

$$\|f\|_{\mathcal{K}_m} \leq \|f_p\|_{J_m} + 1 \leq \|f_p\|_{I_p} + 1 < N + 1.$$

This implies that $\|f\|_{\mathcal{K}_m} \leq N + 1$ for every m and $\|f\|_{\mathcal{R}} = \lim_{m \rightarrow \infty} \|f\|_{\mathcal{K}_m}$ is finite. We further note that $f(a) = \lim_{p \rightarrow \infty} f_p(a) = 1$.

Let g be the rational function given above and let $\{r_n\}$ be the sequence which was constructed. We will now show that $\|g\|_{\mathcal{R}} \geq \|f\|_{\mathcal{R}}$. For each positive integer

m , there exists a $\mathcal{K}_i$ such that

$$| \|f\|_{\mathcal{K}_i} - \|f\|_{\mathfrak{A}} | < \frac{1}{2m},$$

and there exists an f_p such that $p > j$ and

$$| \|f_p\|_{J_i} - \|f\|_{\mathcal{K}_i} | < \frac{1}{2m}.$$

Thus

$$| \|f_p\|_{J_i} - \|f\|_{\mathfrak{A}} | < \frac{1}{m}.$$

There is a sequence $\{\|f_q\|_{J_i}\}$, where the value of q and i in each term depends on m and $q > i$, such that $\|f_q\|_{J_i}$ converges to $\|f\|_{\mathfrak{A}}$ as $m \rightarrow \infty$. Now

$$\|r_q\|_{I_q} \geq \|f_q\|_{I_q} \geq \|f_q\|_{J_i},$$

and since $\|r_q\|_{I_q}$ converges to $\|g\|_{\mathfrak{A}}$ as $m \rightarrow \infty$, we have $\|g\|_{\mathfrak{A}} \geq \|f\|_{\mathfrak{A}}$.

Since $\mathfrak{A}$ is bounded by simple closed curves, there is a set of rational functions, whose poles are nodes of $\Pi_n - \overline{\mathfrak{A}}$ for some n , which forms a complete orthonormal system in $L^2(\mathfrak{A})$. If $k \in L^2(\mathfrak{A})$ and $k(a) = 1$, then for any $\epsilon > 0$, there is a rational function g , whose poles are nodes of $\Pi_n - \overline{\mathfrak{A}}$, such that $g(a) = 1$ and $\|k - g\|_{\mathfrak{A}} < \epsilon$. Thus $|\|k\|_{\mathfrak{A}} - \|g\|_{\mathfrak{A}}| \leq \|k - g\|_{\mathfrak{A}} < \epsilon$. We can now conclude that $\|f\|_{\mathfrak{A}} \leq \|k\|_{\mathfrak{A}}$ for all k in $L^2(\mathfrak{A})$ with $k(a) = 1$.

Since $\{f_n\}$ is an equicontinuous sequence on each compact subset of $\mathfrak{A}$, any other convergent subsequence of $\{f_n\}$ will converge uniformly on compact subsets of $\mathfrak{A}$ and hence will converge to f . Therefore, $\{f_n\}$ converges to f uniformly on compact subsets of $\mathfrak{A}$.

Corollary 6.2. *When $n \rightarrow \infty$, $\|f_n\|_{I_n}$ converges to $\|f\|_{\mathfrak{A}}$.*

Proof. Let $\epsilon > 0$ be arbitrary. There exists a rational function g , whose poles are nodes of $\Pi_n - \overline{\mathfrak{A}}$ for some n , such that $g(a) = 1$ and $\|g - f\|_{\mathfrak{A}} < \epsilon/2$. Let $\{r_n\}$ be the sequence of discrete analytic functions constructed in the proof of Theorem 6.1. Now $|\|g\|_{\mathfrak{A}} - \|f\|_{\mathfrak{A}}| \leq \|g - f\|_{\mathfrak{A}} < \epsilon/2$. There exists an N_1 such that if $n > N_1$, then $|\|r_n\|_{I_n} - \|g\|_{\mathfrak{A}}| < \epsilon/2$ which implies that

$$\|f_n\|_{I_n} \leq \|r_n\|_{I_n} < \|g\|_{\mathfrak{A}} + \frac{\epsilon}{2} < \|f\|_{\mathfrak{A}} + \epsilon.$$

Choose m such that $|\|f\|_{\mathcal{K}_m} - \|f\|_{\mathfrak{A}}| < \epsilon/2$. There exists an $N_2 \geq m$ such that $n > N_2$ implies that $|\|f_n\|_{J_m} - \|f\|_{\mathcal{K}_m}| < \epsilon/2$ where $\|f_n\|_{J_m} = \|f_n\|_{I_n \cap \mathcal{K}_m}$. Thus, if $n > N_2$, then

$$\|f_n\|_{I_n} \geq \|f_n\|_{J_m} \geq \|f\|_{\mathcal{K}_m} - \frac{\epsilon}{2} > \|f\|_{\mathfrak{A}} - \epsilon.$$

If we let $N = \max\{N_1, N_2\}$, then $n > N$ implies that

$$||f_n||_{I_n} - ||f||_{\mathfrak{R}} < \epsilon.$$

As a result of Theorem 6.1 and Corollary 6.2, $f_n(z)/||f_n||_{I_n}^2$ converges uniformly on compact subsets of $\mathfrak{R}$ to $f(z)/||f||_{\mathfrak{R}}^2 = K(z, a)$ where $K(z, t)$ is the Bergman kernel function of $\mathfrak{R}$. If $\mathfrak{R}$ is simply-connected, then the discrete analytic function F_n defined on I_n by

$$F_n(z) = \frac{\pi^{1/2}}{||f_n||_{I_n}} \int_a^z f_n(t) \delta t,$$

converges uniformly on compact subsets of $\mathfrak{R}$ to the function F defined on $\mathfrak{R}$ by

$$F(z) = \frac{\pi^{1/2}}{||f||_{\mathfrak{R}}} \int_a^z f(t) dt.$$

As was noted in Section 3, F maps $\mathfrak{R}$ conformally onto the unit disk with $F(a) = 0$ and $F'(a) > 0$.

When $\mathfrak{R}$ is simply-connected, $D_n = \mathfrak{D}_n \cap \Pi_n$ is a simple discrete region. Thus, f_n can be obtained by solving the system of equations developed in Section 3.

Suppose the region $\mathfrak{R}$ is the interior of the union of a finite number of net squares of Π_N for some N . Now for n_0 large enough so that $n_0 \geq N$ and $a \in \Pi_{n_0+3} \cap \mathfrak{R}$, we have $\mathfrak{G}_n = \mathfrak{R}$ for all $n \geq n_0$. Also, for $n \geq n_0 + 3$, $I_n = \mathfrak{R} \cap \Pi_n$. That is, for all sufficiently large n , the domain of the solution of the discrete minimum problem is the set of all nodes of Π_n in $\mathfrak{R}$.

In Section 3 we considered a minimum problem in the Banach space $B(I)$. A similar minimum problem can be stated for the Hilbert space $H(I)$. However, it can be shown that the solution of the minimum problem for $H(I)$ does not converge to the solution of the ordinary minimum problem in the sense of Theorem 6.1.

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Date communicated: NOVEMBER 24, 1969