A Nonlinear Version of the Principle of Limiting Amplitude

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1. Introduction. If a dynamical system is governed by a linear equation and driven by an externally imposed harmonic force and if there is some kind of dissipation, then for large times the system is expected to tend to an harmonic motion, with the period of the external force, regardless of its initial configuration. This is called the Principle of Limiting Amplitude. A classical example of this behaviour is furnished by the motion of a damped spring subject to an harmonically varying external force.

Suppose now, that the equation of motion in the above system is replaced by a nonlinear equation. Unlike linear differential equations, nonlinear equations often do not admit solutions which can be continuously extended whenever the equations themselves remain regular. In addition, although the principle of superposition holds for linear waves, it is not valid for nonlinear waves. Thus, in general, one can only expect this system to tend in some generalized sense to a periodic motion of the same period as the external force and then only when an additional condition; such as that the amplitude of the forcing term be sufficiently small, is imposed on it.

It is our purpose here to prove such a result and to obtain an estimate on the rate at which the periodic motion is approached for a system governed by the nonlinear equation of evolution

$$(1-1) \quad u_{,t}(t) + A(t)u(t) + du_{,t}(t) = f(t),$$

in the interior of a finite reflecting body Ω where the forcing term $f(t)$ is periodic in t of period τ , d is a real constant called the damping coefficient, $u(t)$ is a vector-valued function defined on a subinterval of the real line into a space of complex-valued functions $u(x)$ defined on a bounded domain Ω in the real Euclidean

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space E^N of N dimensions, $u_t(t) \equiv du(t)/dt$ and $A(t) = A_1(t) + A_2(t)$ is the family of partial differential operators of order $2m$ ($m = 1, 2, \dots$) on Ω given in generalized divergence form by

$$A_1(t)u(x) = \sum_{|\alpha|, |\beta| \leq m} D^\alpha [a_{\alpha\beta}(x, t) D^\beta u(x)],$$

$$A_2(t)u(x) = F(x, t, u(x), \dots, D^m u(x)),$$

with

$$D_j = \frac{1}{i} \frac{\partial}{\partial x_j} \quad \text{for } 1 \leq j \leq N,$$

and for each N -tuple $\alpha = (\alpha_1, \dots, \alpha_N)$ of nonnegative integers, $D^\alpha = D_1^{\alpha_1} \dots D_N^{\alpha_N}$ and $|\alpha| = \sum_{i=1}^N \alpha_i$. Reflection is associated with the Dirichlet boundary conditions

$$(1-2) \quad D^\alpha u(t) |_{\partial\Omega} = 0, \quad |\alpha| \leq m - 1.$$

The amplitude of the forcing term is just $\|f\|_{L^\infty(\tau; L^2(\Omega))}$ and shall be denoted simply by $\|f\|$. The dissipative mechanism for the system considered here is positive damping, that is, the coefficient d is required to be positive. It is established in Section 4 that if $A(t)$, $f(t)$ and d satisfy conditions (I)–(V) formulated in Section 3 and if $\|f\|$ is sufficiently small, then the system defined by (1-1), (1-2) and the given initial conditions

$$(1-3) \quad v(0) = v_0 \in W_0^{m,2}(\Omega),$$

$$v_t(0) = v_1 \in L^2(\Omega),$$

has a unique solution $v(t)$ and for large time this solution tends exponentially in $C([0, \infty), L^2(\Omega))$ to periodic motion in t of period τ . If, in addition, $N \leq 2m$, then $v(t) = V(x, t)$ tends exponentially to the periodic motion pointwise a.e. on Ω as t tends to infinity where $V(x, t)$ denotes the unique representation of $v(t)$ on $\Omega \times [0, \infty)$. Section 2 consists of the mathematical preliminaries necessary to give a precise treatment of the problem. An example of the family $A(t)$ is considered in Section 5.

2. Notation, function spaces. Ω is a fixed bounded domain (an open, connected set) in E^N with boundary and closure written $\partial\Omega$ and $\bar{\Omega}$ respectively. A point $(x_1, \dots, x_N)$ of E^N is denoted simply by x and the volume element $dx_1 \dots dx_N$ by dx . Integration over subsets of E^N is taken with respect to the N -dimensional Lebesgue measure.

The Lebesgue spaces $L^p(\Omega)$ are, for $1 \leq p \leq \infty$, the collection of equivalence classes of (real or complex-valued) almost everywhere equal functions $u(x)$ defined on Ω and such that

$$\|u\|_{0,p}^p \equiv \int_{\Omega} |u(x)|^p dx < \infty \quad \text{if } 1 \leq p < \infty,$$

$$\|u\|_{0,p} \equiv \operatorname{esssup}_{\Omega} |u(x)| < \infty \quad \text{if } p = \infty.$$

$L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$(u, v) = \int_{\Omega} u(x) \overline{v(x)} dx.$$

The Sobolev space $W^{m,2}(\Omega)$ is, for $m = 1, 2, \dots$, the collection of functions in $L^2(\Omega)$ all derivatives in the distribution sense of order not exceeding m of which also belong to $L^2(\Omega)$. $W^{m,2}(\Omega)$ is a Hilbert space with norm

$$||u||_{m,2} = \left(\sum_{|\alpha| \leq m} ||D^{\alpha}u||_{0,2}^2 \right)^{1/2}.$$

$C_0^{\infty}(\Omega)$ is the set of functions which are infinitely often continuously differentiable and have compact support in Ω . $W_0^{m,2}(\Omega)$ is the closure of this linear manifold in the norm of $W^{m,2}(\Omega)$. For brevity in notation, we shall always denote the $L^2(\Omega)$ norm $||\cdot||_{0,2}$ by $||\cdot||$.

If X is a Banach space, then X^* shall denote its dual or conjugate space, that is, the space of all bounded, conjugate linear functionals on X , and $\langle w, u \rangle$ the natural pairing of an element $u \in X$ and $w \in X^*$. Strong continuity and differentiability of vector-valued functions on a subinterval S of E^1 into X are defined in the usual way (see, Hille and Phillips [3]).

$C^k(\tau; X)$, $k = 1, 2, \dots$, is the collection containing every vector-valued function which, together with all its strong derivations of order not exceeding k , is defined on E^1 into X , is strongly continuous and is periodic in t of period τ . $C^*(S; X)$, $k = 1, 2, \dots$, is the set of k -times strongly continuously differentiable vector-valued functions defined on S into X .

Let $||\cdot||_X$ be the norm of X . We denote by $L^p(\tau; X)$ for $1 \leq p \leq \infty$, the collection of equivalence classes of almost everywhere equal functions $u(t)$ defined on E^1 into X and periodic in t of period τ such that

$$||u||_{L^p(\tau; X)}^p \equiv \int ||u(t)||_X^p dt < \infty \quad \text{if} \quad 1 \leq p < \infty,$$

$$||u||_{L^{\infty}(\tau; X)} \equiv \text{esssup} ||u(t)||_X < \infty \quad \text{if} \quad p = \infty,$$

where the integration with respect to t is taken over any interval of periodicity. If X is separable and reflexive, $L^p(\tau; X)$ for $1 < p < \infty$ is a reflexive Banach space with conjugate space isomorphic to $L^q(\tau; X^*)$ where $1/p + 1/q = 1$. In addition, the dual space $[L^1(\tau; X)]^*$ is isomorphic to the Banach space $L^{\infty}(\tau; X^*)$. The spaces $L^p(S; X)$ for $1 \leq p \leq \infty$ and the linear spaces of complex-valued functions $u(x, t)$ over $\Omega \times E^1$ are defined in the usual way.

When considering the dependence of the function F on its several variables without considering the latter variables to be functions, we adopt the notation

$$F(x, t, z) = F(x, t, z_1, \dots, z_M),$$

where the complex variables z_k , $k = 1, 2, \dots, M$ are indexed to correspond to $D^{\beta}u(x)$ for all fixed $\beta = (\beta_1, \dots, \beta_N)$, $|\beta| \leq m$.

3. The conditions on the system. The following conditions on $A(t)$, d and $f(t)$ in (1-1) are assumed to be satisfied.

(I) $F(x, t, z_1, \dots, z_M)$ is a complex-valued function measurable in x on Ω for fixed $t, z_1, \dots, z_M$ and jointly continuous in the real variable t and the complex variables $z_1, \dots, z_M$ for almost every fixed x in Ω . F is once continuously differentiable in t on E^1 and periodic in t of period τ for almost every fixed x in Ω and fixed $z_1, \dots, z_M$. There exists a continuous function $g(s)$ of the real variable s such that for all $u(x)$ in $W_0^{m,2}(\Omega)$ and almost all x in Ω

$$|F(x, t, u(x), \dots, D^m u(x))| \leq g(\|u\|_{m,2}) \left\{ \sum_{|\beta| \leq m} |D^\beta u(x)|^{1+C_\beta} + 1 \right\},$$

for all t in E^1 where the C_β are real constants such that

$$0 \leq C_\beta \leq \left(\frac{1}{2} - (m - |\beta|)/N\right)^{-1} [(2m - |\beta|)/N],$$

with equality only if $\frac{1}{2} - (m - |\beta|)/N \geq 0$. $a_{\alpha\beta}(x, t) = \overline{a_{\alpha\beta}(x, t)}$ for every fixed α and β and each $a_{\alpha\beta}(x, t)$ is measurable in x for fixed t , once continuously differentiable in t on E^1 and periodic in t of period τ for almost every fixed x in Ω and is in $L^\infty(\tau; L^\infty(\Omega)) \cap L^\infty(\Omega \times E^1)$.

Before imposing additional conditions, we require the following definition.

Definition 1. $B(t)$, $t \in E^1$, is said to be a family of admissible lower order operators if

$$B(t)u(x) = \sum_{|\beta| \leq m-1} D^\beta B_\beta(x, t, u(x), \dots, D^{m-1}u(x)),$$

for all t in E^1 , each $B_\beta(x, t, z_1, \dots, z_M)$ satisfies the same measurability, continuity differentiability and periodicity requirements as the function F and there exists a continuous function $g_1(s)$ such that for all $u(x)$ in $W_0^{m,2}(\Omega)$, all $\beta = (\beta_1, \dots, \beta_N)$ with $|\beta| \leq m-1$ and almost all x in Ω ,

$$|B_\beta(x, t, u(x), \dots, D^{m-1}u(x))| \leq g_1(\|u\|_{m,2}) \left\{ \sum_{|\gamma| \leq m-1} |D^\gamma u(x)|^{1+C_{\gamma\beta}} + 1 \right\},$$

for all t in E^1 where the $C_{\gamma\beta}$ satisfy

$$0 \leq C_{\gamma\beta} < \left(\frac{1}{2} - \frac{m - |\gamma|}{N}\right)^{-1} \{[2m - (|\gamma| + |\beta|)]/N\}.$$

(II) The family of forms

$$a(t; u, v) = a_1(t; u, v) + a_2(t; u, v),$$

with

$$a_1(t; u, v) = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\alpha|} (a_{\alpha\beta}(x, t) D^\beta u, D^\alpha v),$$

and

$$a_2(t; u, v) = \langle F(x, t, u, \dots, D^m u), v \rangle,$$

associated with the family of operators $A(t)$, $t \in E^1$, $A(t) = A_1(t) + A_2(t)$, satisfies:

(i) There exists a positive constant k_1 such that for all $u(x)$ in $W_0^{m,2}(\Omega)$ and all t in E^1

$$\operatorname{Re} a(t; u, u) \geq k_1 \|u\|_{m,2}^2,$$

and

$$a_1(t; u, u) \geq k_1 \|u\|_{m,2}^2.$$

(ii) There exists a family of admissible lower order operators $B(t)$ such that

$$\operatorname{Re} \{a_2(t; u, u - v) - a_2(t; v, u - v) + b(t; u, u - v) - b(t; v, u - v)\} \geq 0,$$

for all $u(x)$ and $v(x)$ in $W_0^{m,2}(\Omega)$ and all t in E^1 where $b(t; u, v)$ is the family of forms

$$b(t; u, v) = \sum_{|\beta| \leq m-1} (-1)^{|\beta|} \langle B_\beta(x, t, u, \dots, D^{m-1}u), D^\beta v \rangle.$$

(iii) For all $u(x)$ in $W_0^{m,2}(\Omega)$ and all t in E^1

$$\frac{d}{dt} a_1(t; u, u) = a'_1(t; u, u) \leq 0.$$

(iv) There exists a continuous real-valued function $h(s)$ of the real variable s which is monotone increasing in s with $h(0) = 0$ such that for all $\rho \geq 0$

$$|a_2(t; e^{-\rho t}u, e^{\rho t}w) - a_2(t; e^{-\rho t}v, e^{\rho t}w)| \\ \leq h(e^{-\rho t}[\|u\|_{m,2} + \|v\|_{m,2}]) \cdot \{\|u - v\|_{m,2}^2 + \|w\|^2\},$$

for all $u(x)$, $v(x)$ and $w(x)$ in $W_0^{m,2}(\Omega)$ and all $t \geq 0$.

(III). There exists a family $r(t; \cdot)$, $t \in E^1$, of real-valued functions periodic in t of period τ such that for all $t \in E^1$, all $\psi(t)$ in $C^1(E^1; W_0^{m,2}(\Omega))$ and all $\rho \geq 0$, $r(t; \psi(t))$ is once continuously differentiable in t ,

$$\operatorname{Re} a_2(t; e^{-\rho t}\psi(t), e^{\rho t}\psi_t(t)) \geq \frac{d}{dt} r(t; \psi(t)),$$

and for some constant k_2 , all $u(x)$ in $W_0^{m,2}(\Omega)$ and all t in E^1

$$0 \leq r(t; u) \leq k_2 \operatorname{Re} a(t; u, u).$$

(IV) $d > 0$.

(V) $f(t) \in L^\infty(\tau; L^2(\Omega))$.

Weak continuity and semi-boundedness (growth) requirements similar to those in condition (I) were employed by this author [2] to obtain existence theorems for boundary value problems involving nonlinear evolution equations of the form (1-1) with

$$A(t)u(x) = \sum_{|\alpha| \leq m} D^\alpha A_\alpha(x, t, u(x), \dots, D^m u(x)).$$

Condition (II)-(i) asserts in effect that the family $A(t)$ represents forces acting to restore the system to equilibrium. Since $B(t) \equiv 0$, $t \in E^1$, is clearly a family

of admissible lower order operators, condition (II)–(ii) is weaker than a monotonicity condition. In fact, it only requires that those expressions in F explicitly involving $D^m u$ be monotone. Although (II)–(iii) is an annoying restriction, it is also required by J. L. Lions and W. A. Strauss [4] in order to establish the existence of weak solutions of initial-boundary value problems for evolution equations of the form

$$u_{it}(t) + L(t)u(t) + \beta(t; u(t), u_t(t)) = f(t),$$

where each $L(t)$ is an unbounded self-adjoint linear operator and β is a nonlinear function of u and u_t and is, in some sense, close to a “dissipative” operator. Conditions (II)–(iv) and (III) are the most restrictive requirements and indicate that the family of forms $a_2(t; u, v)$ must be close to multilinear. Condition (III) might be best termed an “energy condition” since it enables us to obtain the required *a priori* estimates on the solutions of the Dirichlet boundary value problems for (1-1). An example of a family $A_2(t)$ for which (III) is satisfied is given by

$$A_2(t)u(x) = |u(x)|^\gamma u(x), \quad (\gamma > 0),$$

where Ω is a bounded domain in E^2 . For all $\psi(t)$ in $C^1(E^1; W_0^{m,2}(\Omega))$ and all $t \in E^1$,

$$\begin{aligned} \operatorname{Re} a_2(t; e^{-\rho t} \psi(t), e^{\rho t} \psi_t(t)) &= \frac{d}{dt} (e^{-\gamma \rho t} \|\psi(t)\|_{0,\gamma+2}^{\gamma+2} / (\gamma + 2)) \\ &\quad + \gamma \rho e^{-\gamma \rho t} \|\psi(t)\|_{0,\gamma+2}^{\gamma+2} / (\gamma + 2), \end{aligned}$$

and thus, for all $\rho \geq 0$, $r(t; u) = e^{-\gamma \rho t} \|u\|_{0,\gamma+2}^{\gamma+2} / (\gamma + 2)$ satisfies (III).

The null Dirichlet boundary conditions (1-2) are replaced by the condition that $u(t)$ be defined on a subinterval of E^1 into $W_0^{m,2}(\Omega)$. This is justified by the fact that a function $u(x)$ in $C^m(\bar{\Omega})$ will satisfy

$$D^\alpha u(x)|_{\partial\Omega} = 0, \quad |\alpha| \leq m - 1,$$

if and only if $u(x)$ is in $W_0^{m,2}(\Omega)$.

4. The principle of limiting amplitude. Let $\{w_j\}$, $j = 1, 2, \dots$, be a countable set of distinct basis elements which generate $W_0^{m,2}(\Omega)$. Let P_n be the orthogonal projection in $L^2(\Omega)$ onto the subspace generated by $w_1, \dots, w_n$.

It can easily be verified that conditions (I)–(V) together with (1-3) are stronger than conditions (II.1)–(II.6) on problem II in section 4 of [2]. Hence, there exists [2, section 7] a sequence $v^n(t) = \sum_{k=1}^n c_{nk}(t)w_k$ with each $c_{nk}(t)$ once continuously differentiable in t for $t \geq 0$ such that for every $n = 1, 2, \dots$

$$\begin{aligned} (v_{it}^n(t), w_i) + a(t; v^n(t), w_i) + d(v_i^n(t), w_i) &= (f(t), w_i), \quad t \geq 0, \\ j &= 1, \dots, n, \\ v^n(0) &= P_n v_0, \\ v_i^n(0) &= P_n v_1, \end{aligned} \tag{4-1}$$

and a subsequence $v^\nu(t)$ of $v^n(t)$ with

$$v^\nu(t) \rightarrow v(t) \text{ in the weak* topology of } [L^1([0, T]; W_0^{m,2}(\Omega))]^*$$

$$v_i^\nu(t) \rightarrow v_i(t) \text{ in the weak* topology of } [L^1([0, T]; L^2(\Omega))]^*$$

for all finite T as $\nu \rightarrow \infty$ such that $v(t)$ satisfies the initial-boundary value problem defined by (1-1) and (1-2) for $t \geq 0$ and the initial conditions (1-3) (Problem II in [2]) in the following weak sense:

$$\begin{aligned} \int_0^\infty \{-(v_i(t), \phi_i(t)) + a(t; v(t), \phi(t)) + d(v_i(t), \phi(t))\} dt \\ = (v_1, \phi(0)) + \int_0^\infty (f(t), \phi(t)) dt, \end{aligned}$$

for all $\phi(t)$ in $\mathcal{G}_0$ and $v(0) = v_0$ a.e. in Ω . Here $\mathcal{G}_0$ is the set

$$\mathcal{G}_0 = \{\phi(t) \mid \Phi(t) \in L^1([0, \infty); W_0^{m,2}(\Omega)), \phi_i(t) \in L^1([0, \infty); L^2(\Omega))\}$$

and $\phi(t) \equiv 0$ for all $t \geq T$ for some finite T .

It is proved in Lemma 2 that there is only one such $v(t)$. Derivatives in the above definition of a weak solution $v(t)$ are, of course, considered in the sense of distributions.

In addition, conditions (I)–(V) are stronger than conditions (I.1)–(I.5) on problem I in Section 3 of [2]. Thus, we can conclude that there exists [2, section 6] a sequence $u^n(t) = \sum_{k=1}^n \eta_{nk}(t) w_k$ with each $\eta_{nk}(t)$ once continuously differentiable and periodic in t of period τ for all $t \in E^1$ such that for every positive integer n

$$\begin{aligned} (4-2) \quad (u_i^n(t), w_i) + a(t; u^n(t), w_i) + d(u_i^n(t), w_i) \\ = (f(t), w_i), \quad t \in E^1, \quad j = 1, \dots, n, \end{aligned}$$

and a subsequence $u^\sigma(t)$ of any bounded infinite subsequence $u^\nu(t)$ of $u^n(t)$ with

$$u^\sigma(t) \rightarrow u(t) \text{ in the weak* topology of } [L^1(\tau; W_0^{m,2}(\Omega))]^*$$

$$u_i^\sigma(t) \rightarrow u_i(t) \text{ in the weak* topology of } [L^1(\tau; L^2(\Omega))]^*$$

as $\sigma \rightarrow \infty$ such that $u(t)$ satisfies the boundary value problem defined by (1-1), (1-2) and the requirement that $u(t)$ be periodic in t of period τ (Problem I in [2]) in the following sense:

$$\int \{-(u_i(t), \Phi_i(t)) + a(t; u(t), \Phi(t)) + d(u_i(t), \Phi(t))\} dt = \int (f(t), \Phi(t)) dt,$$

for all $\Phi(t)$ in $\mathcal{F}_0$ where the integration is taken over any interval of periodicity and $\mathcal{F}_0$ is given by

$$\mathcal{F}_0 \equiv \{\Phi(t) \mid \Phi(t) \in L^1(\tau; W_0^{m,2}(\Omega)) \text{ and } \Phi_i(t) \in L^1(\tau; L^2(\Omega))\}.$$

It is our purpose to prove that if conditions (I)–(V) are satisfied then for

any initial conditions v_0 and v_1 in (1-3) there exists a finite number $t_0 \geq 0$ and a constant $\lambda > 0$ such that

$$(4-3) \quad \|u(t) - v(t)\| \leq K_1 e^{-\lambda t},$$

for all $t \geq t_0$ and, if $N \leq 2m$,

$$(4-4) \quad |u(t) - v(t)| = |U(x, t) - V(x, t)| \leq K_2 e^{-\lambda t},$$

for almost all x in Ω and all $t \geq t_0$, provided $\|f\|$ is sufficiently small. K_1 and K_2 are nonnegative fixed constants. Since $u(t)$ is periodic in t of the same period as the external force $f(t)$, this establishes a nonlinear version of the Principle of Limiting Amplitude for the system under consideration and the rate of exponential decay in (4-3) and (4-4) depends on $\|f\|$. We shall assume throughout this section that either $\|f\| \neq 0$ or if $\|f\| = 0$, that we do not have $v_0 = v_1 = 0$, since the latter case yields the trivial estimates $K_1 = K_2 = 0$ in (4-3) and (4-4) for all $t \geq 0$. The following lemmas are required.

Lemma 1. Let $u^n(t)$ and $v^n(t)$, $n = 1, 2, \dots$, be the appropriate solutions as above. Then there exists fixed constants K_3 , K_4 and K_5 such that

$$\|u^n(t)\|_{m,2} \leq K_3 \|f\| \quad \text{for all } t \in E^1,$$

and

$$\|v^n(t)\|_{m,2} \leq K_4 \|f\| + e^{-d t/l} K_5 \quad \text{for all } t \geq 0,$$

independent of n where l is the positive constant

$$(4-5) \quad l = \max(2, 2d^2/k_1).$$

Proof. Let $u^n(t)$, $n = 1, 2, \dots$, be as above. Multiplying (4-2) by $\overline{\eta'_{ni}(t)} \equiv d(\overline{\eta_{ni}(t)})/dt$, summing over $j = 1, \dots, n$, taking the real part of both sides and integrating over any interval of periodicity gives by (I), (II)–(iii), (III), (IV) and the Schwarz Inequality

$$(4-6) \quad \int \|u'_i(t)\|^2 dt \leq \tau d^{-2} \|f\|^2,$$

independent of n . Multiplying (4-2) by $\overline{\eta_{ni}(t)}$, summing over $j = 1, \dots, n$, taking the real part of both sides, integrating over any interval of periodicity and integrating the first term by parts gives by (II)–(i) and (4-6)

$$(4-7) \quad \int \operatorname{Re} a(t; u^n(t), u^n(t)) dt \leq \left(\int \|f(t)\|^2 dt \right)^{1/2} \cdot \left(\int \|u^n(t)\|^2 dt \right)^{1/2} + \int \|u'_i(t)\|^2 dt,$$

and

$$(4-8) \quad \int \|u^n(t)\|^2 dt \leq (k_1^{-1} + d^{-1} k_1^{-1/2}) \tau \|f\|^2,$$

independent of n . Substituting (4-8) into (4-7) gives

$$(4-9) \quad \int \operatorname{Re} a(t; u^n(t), u^n(t)) dt \leq (k_1^{-1} + d^{-1}k_1^{-1/2} + d^{-2}) \cdot \tau \|f\|^2,$$

independent of n . Again multiplying (4-2) by $\overline{\eta_{n_i}(t)}$, summing over $j = 1, \dots, n$, taking the real part of both sides, integrating over $[t_0, t_1]$, $t_0 < t_1$ and then integrating the result with respect to t_0 over the interval of periodicity $[t_1 - \tau, t_1]$ gives by (III)

$$\begin{aligned} \|u_i^n(t)\|^2 + k_1 \|u^n(t)\|_{m,2}^2 &\leq 2\tau \left(\int \|f(t)\|^2 dt \right)^{1/2} \cdot \left(\int \|u_i^n(t)\|^2 dt \right)^{1/2} \\ &+ \int \|u_i^n(t)\|^2 dt + (k_2 + 1) \cdot \int \operatorname{Re} a(t; u^n(t), u^n(t)) dt, \end{aligned}$$

for all t in E^1 and by (4-6) and (4-9)

$$\|u^n(t)\|_{m,2} \leq K_3 \|f\|,$$

for all t in E^1 independent of n where K_3 is a fixed constant.

Let $v^n(t)$, $n = 1, 2, \dots$, be as above and let $v^n(t) = e^{-dt/l} z^n(t)$ for each n where l is the constant in (4-5). Then $z^n(t) = \sum_{k=1}^n e^{dt/l} c_{nk}(t) w_k$ for every n and (4-1) becomes

$$\begin{aligned} e^{-dt/l} (z_{i,t}^n(t), w_i) + a(t; e^{-dt/l} z_i^n(t), w_i) + d(1 - 2/l) e^{-dt/l} (z_i^n(t), w_i) \\ + (d^2/l^2)(1 - l) e^{-dt/l} (z^n(t), w_i) = (f(t), w_i), \\ t \geq 0, \\ (4-10) \quad j = 1, \dots, n, \\ z^n(0) = P_n v_0, \\ z_i^n(0) = (d/l) P_n v_0 + P_n v_1. \end{aligned}$$

Multiplying the representative differential equation in (4-10) by $e^{2dt/l} ((d/l) \overline{c_{n,i}(t)} + \overline{c'_{n,i}(t)})$, summing over $j = 1, \dots, n$ and taking the real part of both sides gives

$$\begin{aligned} \operatorname{Re} (z_{i,t}^n(t), z_i^n(t)) + \operatorname{Re} a_1(t; z^n(t), z_i^n(t)) + \operatorname{Re} a_2(t; e^{-dt/l} z^n(t), e^{dt/l} z_i^n(t)) \\ + d(1 - 2/l) \|z_i^n(t)\|^2 + (d^2/l^2)(1 - l) \operatorname{Re} (z^n(t), z_i^n(t)) \leq e^{dt/l} \|f\| \|z_i^n(t)\|, \end{aligned}$$

for $t \geq 0$ and for all n or, by (II)-(iii) and (III) and by (4-5)

$$\begin{aligned} \frac{d}{dt} y_0^n(t) &= \frac{d}{dt} \{ \|z_i^n(t)\|^2/2 + (d^2/2l^2)(1 - l) \|z^n(t)\|^2 \\ &+ a_1(t; z^n(t), z^n(t))/2 + r(t; z^n(t)) \} \\ &\leq 2e^{dt/l} \|f\| (y_0^n(t))^{1/2}, \quad t \geq 0, \end{aligned}$$

for every n . That is

$$\frac{d}{dt} (y_0^n(t))^{1/2} \leq e^{dt/l} \|f\|, \quad t \geq 0,$$

and integrating this expression from 0 to s gives

$$(y_0^n(s))^{1/2} \leq \int_0^s e^{dt/l} \|f\| dt + (y_0^n(0))^{1/2}, \quad s \geq 0,$$

and

$$(4-11) \quad (y_0^n(t))^{1/2} \leq (l/d)e^{dt/l} \|f\| + (y_0^n(0))^{1/2}, \quad t \geq 0.$$

By conditions (II)–(i) and (III) and our choice of l

$$(4-12) \quad \begin{aligned} (y_0^n(t))^{1/2} &\geq (k_1^{1/2}/2) \|z^n(t)\|_{m,2} \\ &= (k_1^{1/2}/2)e^{dt/l} \|v^n(t)\|_{m,2}, \end{aligned}$$

for every n and all $t \geq 0$. In addition, by (I) and (III), $(y_0^n(0))^{1/2} \leq K_5$ independent of n where K_5 is a fixed constant. Hence, (4-11) and (4-12) give the second estimate

$$\|v^n(t)\|_{m,2} \leq K_4 \|f\| + e^{-dt/l} K_5,$$

for all $t \geq 0$ independent of n and the lemma is proved.

Lemma 2. *The solution $v(t)$ of the initial-boundary value problem defined by (1-1) and (1-2) for $t \geq 0$ and by (1-3) is unique.*

Proof. Let $v_1(t)$ and $v_2(t)$ be weak solutions of the given initial-boundary value problem. Then $v_1(t)$ and $v_2(t)$ are in $L^\infty([0, \infty); W_0^{m,2}(\Omega))$ by Lemma 1 and if we set $W(t) = v_1(t) - v_2(t)$, $W(t)$ is a weak solution of the initial-boundary value problem

$$W_{,tt}(t) + A_1(t)W(t) + dW_t(t) = A_2(t)v_1(t) - A_2(t)v_2(t),$$

$$t \geq 0,$$

$$\nabla^\alpha W(t)|_\Omega = 0, \quad |\alpha| \leq m-1, \quad t \geq 0,$$

$$W(0) = 0,$$

$$W_{,t}(0) = 0.$$

In addition, since $A_1(t)$ is linear (see [4, p. 81])

$$\begin{aligned} \|W_{,t}(t)\|^2 + a_1(t; W(t), W(t)) &\leq a_1(0; W(0), W(0)) \\ &+ \|W_{,t}(0)\|^2 + \int_0^t a_1'(s; W(s), W(s)) ds \\ &+ 2 \int_0^t |a_2(s; v_1(s), W(s)) - a_2(s; v_2(s), W(s))| ds, \end{aligned}$$

for almost every $t \geq 0$ and, by conditions (II)–(i), (II)–(iii) and (II)–(iv) and

the initial conditions

$$\begin{aligned} ||W_t(t)||^2 + k_1 ||W(t)||_{m,2}^2 &\leq \int_0^t 2h(||v_1(s)||_{m,2} + ||v_2(s)||_{m,2}) \\ &\quad \cdot \{||W(s)||_{m,2}^2 + ||W_s(s)||^2\} ds, \end{aligned}$$

for almost all $t \geq 0$ where h is as in (II)–(iv). Since $v_1(t)$ and $v_2(t)$ are in $L^\infty([0, \infty))$; $W_0^{m,2}(\Omega)$ and $h(s)$ is continuous and monotone increasing in s , there exists a fixed constant K_6 such that

$$||W_t(t)||^2 + ||W(t)||_{m,2}^2 \leq K_6 \int_0^t \{||W(s)||_{m,2}^2 + ||W_s(s)||^2\} ds,$$

for almost all $t \geq 0$. This immediately gives $W(t) = 0$ almost everywhere and the lemma is proved.

We can now prove the estimates (4-3) and (4-4). Let $u^n(t)$ and $v^n(t)$, $n = 1, 2, \dots$, be as above. We subtract their defining equations, substitute $u^n(t) - v^n(t) = e^{-dt/l} \cdot (w^n(t) - z^n(t))$, where l is given by (4-5), replace w_i by $e^{dt/l}(w_i^n(t) - z_i^n(t))$ and take the real part of both sides, to obtain by (II)–(iii) and (II)–(iv),

$$\begin{aligned} \frac{d}{dt} y^n(t) &= \frac{d}{dt} \{||w_i^n(t) - z_i^n(t)||^2/2 + (d^2/2l^2)(1-l) ||w^n(t) - z^n(t)||^2 \\ &\quad + a_1(t; w^n(t) - z^n(t), w^n(t) - z^n(t))/2\} \leq h(||u^n(t)||_{m,2} + ||v^n(t)||_{m,2}) \\ &\quad \cdot \{||w^n(t) - z^n(t)||_{m,2}^2 + ||w_i^n(t) - z_i^n(t)||^2\}, \end{aligned}$$

for all $t \geq 0$ and all $n = 1, 2, \dots$. By (II)–(i)

$$y^n(t) \geq \{||w_i^n(t) - z_i^n(t)||^2/2 + (k_1/4) ||w^n(t) - z^n(t)||_{m,2}^2\},$$

for all $t \geq 0$ independent of n . Since h is monotone increasing and continuous in s and $h(0) = 0$, we have by Lemma 1

$$h(||u^n(t)||_{m,2} + ||v^n(t)||_{m,2}) \leq h(K_3 ||f|| + K_4 ||f|| + K_5 e^{-dt/l}),$$

for all $t \geq 0$ independent of n , and

$$h((K_3 + K_4) ||f|| + K_5 e^{-dt/l}),$$

is continuous in $||f||$ for fixed $t \geq 0$, continuous in t for fixed $||f||$, monotone increasing in $||f||$ and monotone decreasing in t for all $t \geq 0$. Hence, we can choose $t_0 \geq 0$ sufficiently large and $||f||$ sufficiently small so that

$$h(||u^n(t)||_{m,2} + ||v^n(t)||_{m,2}) \{||w^n(t) - z^n(t)||_{m,2}^2 + ||w_i^n(t) - z_i^n(t)||^2\} \leq \lambda_1 y^n(t),$$

with $\lambda_1 < d/l$ for all $t \geq t_0$ independent of n . Then, just as in (4-11)

$$(4-13) \quad ||u^n(t) - v^n(t)||_{m,2} \leq K_7 e^{-\lambda t} + K_8 e^{-dt/l} \leq K_1 e^{-\lambda t},$$

for all $t \geq t_0$ independent of n where $\lambda = (d/l) - \lambda_1 > 0$ and K_7, K_8 and K_1 are fixed positive constants. Let $v(t)$ be the unique weak solution of (1-1) and (1-2)

for $t \geq 0$ and of (1-3). Then the subsequence $v'(t)$ converges to $v(t)$ in the appropriate topologies as $\nu \rightarrow \infty$. Now for this ν , $u'(t)$ is a bounded infinite sequence of approximate periodic solutions of (1-1) and (1-2). Hence, we can choose a subsequence $u''(t)$ of $u'(t)$ so that $u''(t)$ converges to a function $u(t)$ which is a weak periodic solution of (1-1) and (1-2) in the sense defined. Thus, setting $n = \sigma$ in (4-13) and taking the limit inferior, we have

$$(4-14) \quad \|u(t) - v(t)\|_{m,2} \leq K_1 e^{-\lambda t},$$

for almost all $t \geq t_0$ where $\lambda > 0$ depends on $\|f\|$. Since the weak solutions are equivalent to continuous functions in $L^2(\Omega)$ (see [7]) and, if $N \leq 2m$, are equivalent to continuous functions for almost all fixed x in Ω , we can redefine them on a set of measure zero so that estimates (4-3) and (4-4) obtained from (4-14) hold for all $t \geq t_0$. This proves the desired result.

Remarks. From the remarks concerning related problems in section 3 and 4 of [2], it is clear that identical results would be obtained if we replaced the Dirichlet boundary conditions (1-2) by more general boundary conditions which corresponded to a closed linear subspace V of $W^{m,2}(\Omega)$ which contained $C_0^\infty(\Omega)$. However, for $V \neq W_0^{m,2}(\Omega)$, $W_0^{m,2}(\Omega)$ must be replaced by $W^{m,2}(\Omega)$ in conditions (I)-(III) and Ω is required to satisfy a strong cone condition (see [5]).

As an example, suppose we let $m = 1$ in (1-1) and replace the boundary condition (1-2) by

$$\left[\frac{\partial u(t)}{\partial n} + u(t) \right] \Big|_{\partial\Omega} = 0,$$

where $\partial u(x)/\partial n$ denotes differentiation of $u(x)$ in the direction of the exterior normal to $\partial\Omega$. Then V is the subspace of $W^{1,2}(\Omega)$ obtained by closing the set of all functions $u(x)$ in $C^2(\Omega)$ which satisfy $[\partial u(x)/\partial n + u(x)]|_{\partial\Omega} = 0$ in the strong topology of $W^{1,2}(\Omega)$.

5. An example. A very simple example of the families $A_1(t)$, $t \in E^1$, and $A_2(t)$, $t \in E^1$, in (1-1) for which the nonlinear version of the Principle of Limiting Amplitude is valid is:

$$\begin{aligned} A_1(t)u(x) &= -\Delta_3 u(x), \\ A_2(t)u(x) &= |u(x)|^2 u(x), \end{aligned}$$

where $\Omega \subset E^3$ and $\Delta_3 \equiv -(D_1^2 + D_2^2 + D_3^2)$. The calculations required in order to verify that (I)-(III) are satisfied for this example are left to the reader. Thus, if (IV) and (V) are satisfied, the system defined by

$$\begin{aligned} v_{tt}(t) - \Delta_3(t) + |v(t)|^2 v(t) + dv_t(t) &= f(t), \quad t \geq 0, \\ v(t) \Big|_{\partial\Omega} &= 0, \quad t \geq 0, \end{aligned}$$

whose initial configuration is

$$\begin{aligned} v(0) &= v_0 \in L^2(\Omega), \\ v_t(0) &= v_1 \in W_0^{1,2}(\Omega), \end{aligned}$$

tends exponentially in $C([0, \infty); L^2(\Omega))$ to periodic motion of period τ in t provided the amplitude of the forcing term f is sufficiently small. This includes as a special case some results obtained by D. H. Sattinger [6].

Now, if we take $N = 2$ in the above example and set $f(t) = ge^{2\pi i t/\tau}$, it follows from the results of F. E. Browder [1] and the separation of variables that $u(t) = We^{2\pi i t/\tau}$ is a periodic solution of

$$u_{tt}(t) - \Delta_2 u(t) + |u(t)|^2 u(t) + du_t(t) = f(t), \quad t \in E^1,$$

satisfying $u(t)|_{\partial\Omega} = 0$, $t \in E^1$, where W is the solution of the boundary value problem

$$(5-1) \quad -\Delta_2 W + \left(\frac{2\pi di}{\tau} - \left(\frac{2\pi}{\tau}\right)^2\right)W + |W|^2 W = g,$$

$$W|_{\partial\Omega} = 0.$$

Then

$$W(x) = \lim_{t \rightarrow \infty} v(t)e^{-2\pi i t/\tau} = \lim_{t \rightarrow \infty} V(x, t)e^{-2\pi i t/\tau},$$

for almost all fixed x in Ω where $v(t)$ is the solution of the nonlinear wave equation

$$v_{tt}(t) - \Delta_2 v(t) - |v(t)|^2 v(t) + dv_t(t) = f(t), \quad t \geq 0,$$

which satisfies $v(t)|_{\partial\Omega} = 0$, $t \geq 0$ and the initial data $v_0 = v_1 = 0$ and we say that W is the limiting amplitude of the (transient) periodic vibration building up for large time under the action of a periodic force. Thus, the nonlinear version of the Principle of Limiting Amplitude provides a principle for originating solutions of (5-1).

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