Corrigendum on “Two Conditions Preventing Secondary Bifurcation of Eigenfunctions of Sublinear Hammerstein Operators” *

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In a previous paper [1] the author made the following statement, calling it Lemma 2: “The quantity

$$\Lambda_p = \sup_{\varphi \in S_1^+} \frac{\mu_n(K, \varphi)}{\mu_{p-1}(K, \varphi)} > 0, \quad p = 1, 2, \dots$$

is less than unity, and the maximum is actually assumed for a function $\varphi^* \in S_1^+$, $\varphi^*(s) \neq 0$. Here, $S_1^+ = [\varphi \mid \varphi \in C, \|\varphi\|_c = 1, \varphi \geq 0]$.” In this statement, $\mu_n(K, \varphi)$ is the necessarily simple n ’th eigenvalue of the linear operator $K\varphi h = \int_0^1 K(s, t) \cdot \varphi(t)h(t) dt$, with the assumption that $K(s, t)$ is a continuous symmetric oscillation kernel of Gantmacher and Krein, ([1], [2], p. 53). $C = C(0, 1)$ is of course the space of continuous functions $\varphi(s)$, $0 \leq s \leq 1$, with $\|\varphi\|_c = \sup_{0 \leq s \leq 1} |\varphi(s)|$.

Recently the author found a mistake in the proof of this lemma. We must either eliminate the possibility that a maximizing sequence might converge weakly to zero, or show that such a sequence yet keeps the values of the functional $F_p(\varphi) = [\mu_n(K, \varphi)/\mu_{p-1}(K, \varphi)]$ bounded below unity. The author took the former course, but in so doing he passed from the basic functional $F_p(Kg)$ which is weakly continuous, to a functional $\Phi_p(g) = (g, g) + F_p(Kg)$ which lacks the needed property of weak continuity.

Subsequently, Dr. Richard G. Kellner of the Los Alamos Scientific Laboratory found, as a counter-example, a sequence of functions $\{\varphi_n\}$, $\|\varphi\|_c = 1$, weakly convergent to zero, and such that $F_1(\varphi_n)$ actually approaches unity for a certain oscillation kernel. (Kellner’s sequence $\{\varphi_n\}$ corresponded to beads on a string, and his oscillation kernel was the Green’s function for $-\nabla^2$ with homogeneous boundary conditions.) Hence the statement of the lemma given in [1] is not true as it stands.

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Virtually the same statement appears in another publication by the author, namely Lemma 6.4 on p. 66 of [2]. Hence this Corrigendum will serve for the latter work also.

We substitute a more restricted statement for the lemma in question:

Lemma. *The quantity*

$$\Lambda_p^{ab} = \sup_{\varphi \in S_{ab}} \frac{\mu_p(K, \varphi)}{\mu_{p-1}(K, \varphi)} > 0, \quad p = 1, 2, \dots$$

is less than unity, and the maximum is assumed for a function $\varphi^* \in S_{ab}$. Here, $S_{ab} = [\varphi \mid \varphi \text{ measurable, } a(s) \leq \varphi \leq b(s)]$, where $a(s) \geq 0$, $a(s) \not\equiv 0$, and $b(s)$ are two elements of $C(0, 1)$.

Let us note that $K(s, t)$ is an oscillation kernel, and that $\mu_n(K, \varphi)$, $n = p - 1$, p , has the same definition as before.

Proof. Let us select a maximizing sequence $\{\varphi_n\}$ in S_{ab} . Thus $\lim_{n \rightarrow \infty} F_p(\varphi_n) = \sup_{\varphi \in S_{ab}} F_p(\varphi)$ where $F_p(\varphi) = [\mu_p(K, \varphi) / \mu_{p-1}(K, \varphi)]$. Since $S_{ab} \subset L_2(0, 1)$, $\{\varphi_n\}$ contains a subsequence L_2 -weakly convergent to some element $\varphi^* \in S_{ab}$. We consider the subsequence already extracted.

Then we have the following sequence of eigenvalue problems:

$$(1) \quad \mu^{(n)} h^{(n)}(s) = \int_0^1 K(s, t) \varphi_n(t) h^{(n)}(t) dt, \quad n = 1, 2, \dots,$$

where we remember that $K(s, t)$ is a continuous symmetric oscillation kernel. Accordingly the q 'th eigenvalue $\mu_q^{(n)}(K, \varphi_n)$ of the n 'th problem, $q = 0, 1, 2, \dots$, is simple and the corresponding q 'th normalized eigenfunction $h_q^{(n)}(s)$ is continuous and has q nodal zeros (and no other zeros) in the interval $0 < s < 1$.

By symmetrizing the kernels $K(s, t) \varphi_n(t)$ in eq. (1) after the fashion of Lemma 1 of [1], and using variational principles, we can become convinced that $\mu_q(K, a(s)) \leq \mu_q(K, \varphi_n) \leq \mu_q(K, b(s))$, $q = 0, 1, 2, \dots$. Thus $\{\mu_q(K, \varphi_n)\}$, $n = 1, 2, \dots$, is a numerical sequence, bounded and bounded away from zero. We extract a subsequence $\{\varphi_{n_1}\}$ such that $\mu_q(K, \varphi_{n_1}) \rightarrow \mu_{k_q}^* > 0$, where $\mu_{k_q}^*$ is some number.

Next we note that the sequence $\{h_q^{(n_1)}(s)\}$ of q 'th eigenfunctions of problems (1), $n_1 = 1, 2, \dots$, with $\sup_{0 \leq s \leq 1} |h_q^{(n_1)}(s)| = 1$, forms a bounded equicontinuous set:

$$\begin{aligned} |h_q^{(n_1)}(s + \Delta) - h_q^{(n_1)}(s)|^2 & \leq \left(\frac{1}{\mu_q(K, a)} \int_0^1 |K(s + \Delta, t) - K(s, t)| \cdot |\varphi_{n_1}(t)| \cdot |h_q^{(n_1)}(t)| dt \right)^2 \\ & \leq \frac{1}{|\mu_q(K, a)|^2} \int_0^1 |K(s + \Delta, t) - K(s, t)|^2 dt \int_0^1 |b(t)|^2 dt, \end{aligned}$$

where the latter estimate vanishes as $\Delta \rightarrow 0$. Again, the functions $\sigma_{n_1}(s) = \int_0^s \varphi_{n_1}(t) dt \leq \int_0^1 b(t) dt$ are bounded and equicontinuous. Hence we can extract a subsequence $\{\varphi_{n_s}\}$ such that, for fixed q , $h_q^{(n_s)}(s) \rightarrow h_{k_q}^*(s)$, where $h_{k_q}^*$ is some

function with $\sup_{0 \leq s \leq 1} |h_{k_0}^*(s)| = 1$, and $\sigma_{n_3}(s) \rightarrow \sigma^*(s) \uparrow$ with $\sigma(1) \geq \int_0^1 a(s) ds > 0$, the convergence being uniform in both cases. We note that, because of weak convergence of $\{\varphi_{n_3}\}$, $\sigma^*(s) = \int_0^s \varphi^*(t) dt$.

The number $\mu_{k_q}^*$ and the function $h_{k_q}^*(s)$ are eigenvalue and eigenfunction respectively for a limit problem, $q = 0, 1, 2, \dots$, but we cannot yet say that $\mu_{k_q}^* = \mu_q(K, \varphi^*)$ and $h_{k_q}^*(s) = h_q^*(s)$ except in the case $q = 0$. For $q = 0$ it is clear that $\lim_{n_3 \rightarrow \infty} h_0^{(n_3)}(s) = h_0^*(s) \geq 0$ so that, in view of the oscillatoriness of $K(s, t)$, $\mu_{k_0}^* = \mu_0(K, \varphi^*)$. We use this later.

Thus it is possible to arrive at a maximizing sequence $\{\varphi_{n_3}\}$ with $\varphi_{n_3} \xrightarrow{w^*} \varphi^* \in S_{ab}$ such that $\mu_{p-1}(K, \varphi_{n_3}) \rightarrow \mu_{k_{p-1}}^*$, $h_{p-1}^{(n_3)}(s) \rightarrow h_{k_{p-1}}^*(s)$ with $\sup_{0 \leq s \leq 1} |h_{k_{p-1}}^*(s)| = 1$, and $\mu_p(K, \varphi_{n_3}) \rightarrow \mu_{k_p}^*$, $h_p^{(n_3)}(s) \rightarrow h_{k_p}^*(s)$ with $\sup_{0 \leq s \leq 1} |h_{k_p}^*(s)| = 1$. Moreover we have $\sigma_{n_3}(s) \rightarrow \sigma^*(s) \uparrow$, and the convergence is uniform in each case. Thus as $n_3 \rightarrow \infty$ we have the following relationships in the limit:

$$(2) \quad \mu_{k_{p-1}}^* h_{k_{p-1}}^*(s) = \int_0^1 K(s, t) h_{k_{p-1}}^*(t) d\sigma^*(t),$$

$$\mu_{k_p}^* h_{k_p}^*(s) = \int_0^1 K(s, t) h_{k_p}^*(t) d\sigma^*(t), \quad \sigma^*(s) = \int_0^s \varphi^*(t) dt.$$

Because of the oscillatoriness of $K(s, t)$, we have $\mu_p(K, \varphi_{n_3}) < \mu_{p-1}(K, \varphi_{n_3})$, $n_3 = 1, 2, 3, \dots$; in the limit however we can say only that $\mu_{k_p}^* = \mu_{k_p}(K, \varphi^*) \leq \mu_{k_{p-1}}(K, \varphi^*) = \mu_{k_{p-1}}^*$. For $n_3 = 1, 2, 3, \dots$, though, we also know that $(h_p^{(n_3)}, h_{p-1}^{(n_3)})_{\sigma_{n_3}} = 0$, where $(\cdot, \cdot)_{\sigma_{n_3}}$ is the inner product formed with the measure σ_{n_3} . Thus in the limit we have $(h_{k_p}^*, h_{k_{p-1}}^*)_{\sigma^*} = 0$. Since $\mu_{k_p}(K, \varphi^*)$ and $\mu_{k_{p-1}}(K, \varphi^*)$ are necessarily simple eigenvalues, we must therefore have $\mu_{k_p}(K, \varphi^*) < \mu_{k_{p-1}}(K, \varphi^*)$.

If we define $R_{k_p} = [\mu_{k_p}(K, \varphi^*) / \mu_{k_{p-1}}(K, \varphi^*)] < 1$, we then have

$$\lim_{n_3 \rightarrow \infty} \frac{\mu_p(K, \varphi_{n_3})}{\mu_{p-1}(K, \varphi_{n_3})} = \lim_{n_3 \rightarrow \infty} F_p(\varphi_{n_3}) = R_{k_p} = \sup_{\varphi \in S_{ab}} F_p(\varphi).$$

As for whether or not this supremum R_{k_p} of $F_p(\varphi)$ on S_{ab} is assumed, consider the case $p = 1$. As we have noted, $k_0 = 0$; i.e. $\mu_{k_0}^* = \mu_0(K, \varphi^*)$ and $h_{k_0}^*(s) = h_0^*(s) \geq 0$. Thus $R_{k_1} = [\mu_{k_1}(K, \varphi^*) / \mu_0(K, \varphi^*)] \leq [\mu_1(K, \varphi^*) / \mu_0(K, \varphi^*)] = F_1(\varphi^*)$, which is impossible unless $k_1 = 1$ since R_{k_1} is the supremum of $F_1(\varphi)$ on S_{ab} . Hence $\mu_{k_1}^* = \mu_1(K, \varphi^*)$ and $h_{k_1}^*(s) = h_1^*(s)$ with just one nodal zero (and no other zeros) in the interval $0 < s < 1$. In general suppose we know that $k_{p-1} = p - 1$, so that $\mu_{k_{p-1}}^* = \mu_{p-1}(K, \varphi^*)$ and $h_{k_{p-1}}^*(s) = h_{p-1}^*(s)$ with just $p - 1$ nodal zeros (and no other zeros) in $0 < s < 1$. Then $R_{k_p} = [\mu_{k_p}(K, \varphi^*) / \mu_{p-1}(K, \varphi^*)] \leq [\mu_p(K, \varphi^*) / \mu_{p-1}(K, \varphi^*)] = F_p(\varphi^*)$, which again is impossible unless $k_p = p$ since R_{k_p} is the supremum of $F_p(\varphi)$ on S_{ab} . Thus $\mu_{k_p}^* = \mu_p(K, \varphi^*)$ and $h_{k_p}^*(s) = h_p^*(s)$ with just p nodal zeros (and no other zeros) in $0 < s < 1$. The induction is complete, and we see that

$$F_p(\varphi^*) = \frac{\mu_p(K, \varphi^*)}{\mu_{p-1}(K, \varphi^*)} = R_{k_p} = \sup_{\varphi \in S_{ab}} F_p(\varphi) < 1, \quad p = 1, 2, \dots$$

which was to be proven.

In the work of the author on conditions preventing secondary bifurcation (cf. [1]), the lemma proven above can be used in place of faulty Lemma 2 (and the resulting function φ^* used in Theorem 6), if we make the following identifications: $a(s) = A(s) \geq 0$, $A(s) \not\equiv 0$, $b(s) = f'_x(s, 0)$ in the sublinear case, and $a(s) = f'_x(s, 0)$, $b(s) = A(s)$ in the superlinear case. Likewise the new lemma can be substituted for faulty Lemma 6.4 on pp. 66 of [2].

Unfortunately we are thereby restricted to asymptotically linear cases of the Hammerstein operator, *i.e.* cases where $A(s) = \lim_{|x| \rightarrow \infty} [f(s, x)/x]$ is defined, bounded and nontrivial (see [1], p. 792, H-3, or [2], p. 49). Other cases are approximated by truncating and linearly extending the function $f(s, x)$ to be differentiable. Attention should be called here to the suggestion of Prof. S. V. Parter, of the University of Wisconsin, that possession of an *a priori* bound for all the solutions of the eigenvalue problem for the Hammerstein operator would enable truncation of $f(s, x)$ with no loss of information in a non-asymptotically-linear case.

In the previous paper [1] there is a gap in the proof of Theorem 9, which however is filled in essentially the same result in [2], namely Theorem 7.3 on p. 83.

REFERENCES

- [1] GEORGE H. PIMBLEY JR., Two Conditions Preventing Secondary Bifurcation for Eigenfunctions of Sublinear Hammerstein Operators, *J. Math. Mech.*, **8** (1969) 787-804.
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