A Vietoris-Begle Theorem for Submersions

DONALD R. SHORT* & J. WOLFGANG SMITH**

Communicated by S. S. CHERN

1. Introduction. Among existing versions of the Vietoris-Begle Mapping Theorem the sheaf theoretic version, as given in G. E. Bredon's *Sheaf Theory* [1], is particularly germane to the considerations of the present paper. By a slight restriction of generality this result may be stated in the following form.

Vietoris-Begle Theorem. *Let X be completely paracompact, $f: X \rightarrow Y$ a closed surjective map and $\mathcal{A}$ a sheaf of modules on Y . For all $y \in Y$ let $f^{-1}(y)$ be connected and let $H^i(f^{-1}(y); \mathcal{A}_y) = 0$ for $0 < i \leq p$. The induced map*

$$f^!: H^i(Y; \mathcal{A}) \rightarrow H^i(X; f^* \mathcal{A})$$

is then an isomorphism for $0 \leq i \leq p$ and a monomorphism on dimension $p + 1$.

Here $\mathcal{A}_y$ denotes the stalk of $\mathcal{A}$ at y (considered as a sheaf on X), $f^* \mathcal{A}$ denotes the inverse image of $\mathcal{A}$ under f and H^i denotes sheaf cohomology. For basic facts, terminology and notations relating to sheaf theory we refer to Bredon's text.

The condition that f be closed is quite indispensable, unless it is assumed that f admits some other nice properties. In the present paper we shall be concerned with the case where X and Y are differentiable manifolds (of class C^∞ , let us say) and $f: X \rightarrow Y$ is a *submersion*, i.e. a C^∞ -map whose differential is everywhere surjective. More precisely, we shall take X to be open and paracompact but will allow Y to be an arbitrary (non-Hausdorff) C^∞ -manifold. One observes that a surjective submersion $f: X \rightarrow Y$ whose fibres $f^{-1}(y)$ are connected is now precisely equivalent to the natural projection of X onto the quotient space induced by a regular foliation on X (for basic facts pertaining to foliations we refer to Palais [3], Chapter I). The question is whether one obtains a Vietoris-Begle Theorem for this class of maps without the requirement that f be closed. The main result of this paper (Theorem 1) provides an affirmative answer. However, the resulting Vietoris-Begle Theorem is slightly weaker than its classical counterpart in two respects. Firstly, the acyclicity condition on the

* Research supported by an NDEA Fellowship.

** Supported by the National Science Foundation, Grant GP-11843.

fibres must be somewhat strengthened, *i.e.* we require *strong p -acyclicity*, defined as follows: A space W will be called strongly p -acyclic if for every compact $C \subset W$ there exists a compact locally contractible $\tilde{C} \subset W$ whose interior contains C and whose reduced singular integral homology groups vanish in dimensions $\leq p$. Secondly, unless the coefficient sheaf $\mathfrak{A}$ is subject to a special condition we lose one dimension in the conclusion.

Theorem 1. *Let $f: X \rightarrow Y$ be a surjective submersion, where X is an open paracompact differentiable manifold, and let the fibres $f^{-1}(y)$ be strongly p -acyclic. Let $\mathfrak{A}$ be a locally constant sheaf of modules on Y . The induced map*

$$f^!: H^i(Y; \mathfrak{A}) \rightarrow H^i(X; f^*\mathfrak{A})$$

is then an isomorphism for $0 \leq i < p$ and a monomorphism on dimension p . If $\mathfrak{A}$ is a locally constant sheaf of free modules over a commutative integral domain R of characteristic zero, then $f^!$ is an isomorphism for $0 \leq i \leq p$ and a monomorphism on dimension $p + 1$.

Our conditions on the ring R , it may be observed, serve to insure that a free module over R is torsion-free as an abelian group.

It would be nice to replace the requirement of strong p -acyclicity which we have imposed on the fibres by a simpler condition. Of course when the fibre dimension is 1 or 2 there is no problem, for in these dimensions strong p -acyclicity reduces to acyclicity in the usual sense. Beyond this we have the following result, whose proof depends on the Engulfing Theorem [7].

Theorem 2. *Let M be a piecewise-linear p -connected n -manifold with $p \leq n - 3$. Then M is strongly p -acyclic.*

By way of application we consider the extension problem for regular foliations [5]. A p -dimensional foliation F on X is said to *extend* on a subset $A \subset X$ provided there exists a $(p + 1)$ -dimensional foliation $\hat{F}$ on A such that $F \subset \hat{F}$. Theorems 1 and 2, combined with the methods of [5], yield

Theorem 3. *Let X be an orientable paracompact differentiable n -manifold and F an orientable p -dimensional regular foliation on X with $n \leq 2p - 3$. If X has trivial integral cohomology on dimension $n - p$ and the leaves of F are $(n - p)$ -connected, then F extends over relatively compact subsets of X .*

We remark that the condition $n \leq 2p - 3$ is superfluous when $p = 1$ or 2.

2. Proof of Theorem 1. It will be convenient to introduce the following definition. Given a sheaf $\mathfrak{B}$ and an augmentation $\epsilon: \mathfrak{B} \rightarrow \mathfrak{L}^0$, where $\mathfrak{L}^*$ is a differential sheaf, we shall say that $\mathfrak{L}^*$ (together with ϵ) constitutes an *n -truncated resolution* of $\mathfrak{B}$ provided

$$0 \rightarrow \mathfrak{B} \xrightarrow{\epsilon} \mathfrak{L}^0 \rightarrow \mathfrak{L}^1 \rightarrow \cdots \rightarrow \mathfrak{L}^n \rightarrow \mathfrak{L}^{n+1},$$

is exact.

Lemma 1. *If $\mathfrak{B}$ is a sheaf of modules over X and $\mathfrak{L}^*$ is an n -truncated resolution of $\mathfrak{B}$ there exist natural homomorphisms*

$$\gamma^i: H^i(\Gamma(\mathfrak{L}^*)) \rightarrow H^i(X; \mathfrak{B}),$$

for $0 \leq i \leq n$ (where Γ denotes the section functor). Moreover, if $\mathfrak{L}^*$ is acyclic then the γ^i are isomorphisms and there exists a natural monomorphism

$$\psi: H^{n+1}(X; \mathfrak{B}) \rightarrow H^{n+1}(\Gamma(\mathfrak{L}^*)).$$

A proof of Lemma 1 is achieved by a slight refinement of the argument given in [1, p. 33]. Specifically one defines $\mathfrak{z}^i = \text{Ker}(\mathfrak{L}^i \rightarrow \mathfrak{L}^{i+1})$ and $\mathfrak{B}^i = \text{Im}(\mathfrak{L}^{i-1} \rightarrow \mathfrak{L}^i)$ with the identifications $\mathfrak{z}^0 = \mathfrak{L}^{-1} = \mathfrak{B}$. By exactness one has $\mathfrak{z}^i = \mathfrak{B}^i$ for $0 \leq i \leq n$, and further $\mathfrak{B}^{n+1} \subset \mathfrak{z}^{n+1}$. From the short exact sequences

$$0 \rightarrow \mathfrak{z}^{i-1} \rightarrow \mathfrak{L}^{i-1} \rightarrow \mathfrak{z}^i \rightarrow 0,$$

for $0 < i \leq n$ one obtains the exact sequences

$$(2.1) \quad 0 \rightarrow \Gamma(\mathfrak{z}^{i-1}) \rightarrow \Gamma(\mathfrak{L}^{i-1}) \rightarrow \Gamma(\mathfrak{z}^i) \rightarrow H^1(X; \mathfrak{z}^{i-1}) \rightarrow \dots,$$

and this gives monomorphisms

$$\alpha^i: \frac{\Gamma(\mathfrak{z}^i)}{\text{Im}(\Gamma(\mathfrak{L}^{i-1}) \rightarrow \Gamma(\mathfrak{z}^i))} \rightarrow H^1(X; \mathfrak{z}^{i-1}),$$

for $0 < i \leq n$. By left exactness of Γ one has exact sequences

$$0 \rightarrow \Gamma(\mathfrak{z}^i) \rightarrow \Gamma(\mathfrak{L}^i) \rightarrow \Gamma(\mathfrak{L}^{i+1}),$$

and consequently

$$H^i(\Gamma(\mathfrak{L}^*)) \equiv \frac{\text{Ker}(\Gamma(\mathfrak{L}^i) \rightarrow \Gamma(\mathfrak{L}^{i+1}))}{\text{Im}(\Gamma(\mathfrak{L}^{i-1}) \rightarrow \Gamma(\mathfrak{L}^i))} \approx \frac{\Gamma(\mathfrak{z}^i)}{\text{Im}(\Gamma(\mathfrak{L}^{i-1}) \rightarrow \Gamma(\mathfrak{L}^i))}.$$

Combining these isomorphisms with α^i we obtain monomorphisms

$$\bar{\alpha}^i: H^i(\Gamma(\mathfrak{L}^*)) \rightarrow H^1(X; \mathfrak{z}^{i-1}),$$

for $0 < i \leq n$. Moreover, the short exact sequences

$$0 \rightarrow \mathfrak{z}^{i-r} \rightarrow \mathfrak{L}^{i-r} \rightarrow \mathfrak{z}^{i-r+1} \rightarrow 0,$$

for $0 < i \leq n$ and $r > 0$ induce exact sequences

$$(2.2) \quad \rightarrow H^{r-1}(X; \mathfrak{L}^{i-r}) \rightarrow H^{r-1}(X; \mathfrak{z}^{i-r+1}) \xrightarrow{\delta_{i-r+1}^{r-1}} H^r(X; \mathfrak{z}^{i-r}) \rightarrow \dots,$$

and we define the desired homomorphisms γ^i by setting

$$\gamma^i = \begin{cases} \text{identity} & \text{for } i = 0, \\ \delta_1^{i-1} \circ \delta_2^{i-2} \circ \dots \circ \delta_{i-1}^1 \circ \bar{\alpha}^i & \text{for } 0 < i \leq n. \end{cases}$$

If $\mathfrak{L}^*$ is acyclic it follows by exactness of (2.1) and (2.2) that $\bar{\alpha}^i$ and δ_{i-r+1}^{r-1} are isomorphisms for $0 < i \leq n$, and hence that γ^i is an isomorphism for $0 \leq i \leq n$.

It remains to construct the monomorphism ψ assuming that $\mathfrak{L}^*$ is acyclic. Clearly we get an isomorphism

$$\beta: H^{n+1}(X; \mathfrak{B}) \rightarrow H^1(X; \mathfrak{B}^n),$$

by setting

$$\beta = (\delta_1^n \circ \delta_2^{n-1} \circ \cdots \circ \delta_n^1)^{-1}.$$

Again we note that the short exact sequence

$$0 \rightarrow \mathfrak{B}^n \rightarrow \mathfrak{L}^n \rightarrow \mathfrak{B}^{n+1} \rightarrow 0,$$

induces a long exact sequence similar to (2.1), and using acyclicity of $\mathfrak{L}^*$ this gives an isomorphism

$$\alpha^{n+1}: \frac{\Gamma(\mathfrak{B}^{n+1})}{\text{Im } (\Gamma(\mathfrak{L}^n) \rightarrow \Gamma(\mathfrak{B}^{n+1}))} \rightarrow H^1(X; \mathfrak{B}^n).$$

By left exactness of Γ one has a monomorphism $\Gamma(\mathfrak{B}^{n+1}) \rightarrow \Gamma(\mathfrak{B}^{n+1})$ and hence a monomorphism

$$\kappa: \frac{\Gamma(\mathfrak{B}^{n+1})}{\text{Im } (\Gamma(\mathfrak{L}^n) \rightarrow \Gamma(\mathfrak{B}^{n+1}))} \rightarrow \frac{\Gamma(\mathfrak{B}^{n+1})}{\text{Im } (\Gamma(\mathfrak{L}^n) \rightarrow \Gamma(\mathfrak{L}^{n+1}))}.$$

Now define $\psi = \kappa \circ (\alpha^{n+1})^{-1} \circ \beta$. Clearly ψ is a natural monomorphism.

We turn to the proof of Theorem 1. The maps $f^i: H^i(Y; \mathfrak{A}) \rightarrow H^i(X; f^*\mathfrak{A})$ in question are induced by the sheaf cohomomorphism $f^*: \mathfrak{A} \rightarrow f^*\mathfrak{A}$ given by the inverse image functor. On the other hand, the direct image functor induces an f -cohomomorphism $f: ff^*\mathfrak{A} \rightarrow f^*\mathfrak{A}$ which induces maps $f^i: H^i(Y; ff^*\mathfrak{A}) \rightarrow H^i(X; f^*\mathfrak{A})$. As it turns out, the maps f^i are more easily investigated than the maps f^i . Consequently one considers the natural factorization $f^* = f \circ h$, where $h: \mathfrak{A} \rightarrow ff^*\mathfrak{A}$ is the homomorphism induced by the cohomomorphism $\mathfrak{A} \rightarrow f^*\mathfrak{A}$ (see [1, p. 55]). Our hypothesis insures that $f^{-1}(y)$ is taut in X for all $y \in Y$, and this implies (as in [1, p. 55]) that h is an isomorphism. It therefore suffices to show that the f^i are isomorphisms (or a monomorphism) on the dimensions in question. To this end one considers the canonical resolution $\mathfrak{C}^*(Y; ff^*\mathfrak{A})$ of the sheaf $ff^*\mathfrak{A}$ and observes that f induces an f -cohomomorphism $\mathfrak{C}^*(Y; ff^*\mathfrak{A}) \rightarrow \mathfrak{C}^*(X; f^*\mathfrak{A})$ which admits a natural factorization

$$\mathfrak{C}^*(Y; ff^*\mathfrak{A}) \xrightarrow{\theta_1} f\mathfrak{C}^*(X; f^*\mathfrak{A}) \xrightarrow{\theta_2} \mathfrak{C}^*(X; f^*\mathfrak{A}),$$

(see [1, p. 10]). Let us suppose for the moment that $f\mathfrak{C}^*(X; f^*\mathfrak{A})$ constitutes an n -truncated resolution of $ff^*\mathfrak{A}$. By Lemma 1 we then obtain commutative diagrams

$$\begin{array}{ccc} H^i(\Gamma\mathfrak{C}^*(Y; ff^*\mathfrak{A})) & \xrightarrow{\theta_1^i} & H^i(\Gamma f\mathfrak{C}^*(X; f^*\mathfrak{A})) \\ \gamma_1^i \searrow & & \swarrow \gamma_2^i \\ & H^i(Y; ff^*\mathfrak{A}) & \end{array}$$

for $0 \leq i \leq n$. Clearly γ_i^i is an isomorphism (in fact, it is the identity map). Moreover, since $\mathcal{C}^*(X; f^*\mathcal{A})$ is flabby, $f\mathcal{C}^*(X; f^*\mathcal{A})$ is also flabby and hence acyclic. Hence γ_i^i is an isomorphism by Lemma 1 and consequently θ_i^i is an isomorphism for $0 \leq i \leq n$. On dimension $n + 1$ one obtains the commutative diagram

$$\begin{array}{ccc} H^{n+1}(\Gamma\mathcal{C}^*(Y; ff^*\mathcal{A})) & \xrightarrow{\theta_1^{n+1}} & H^{n+1}(\Gamma f\mathcal{C}^*(X; f^*\mathcal{A})) \\ \gamma_1^{n+1} \searrow & & \nearrow \psi \\ & H^{n+1}(Y; ff^*\mathcal{A}) & \end{array}$$

(again by Lemma 1), where ψ is a monomorphism. Since γ_1^{n+1} is a monomorphism, θ_1^{n+1} is a monomorphism. On the other hand, θ_2 clearly induces an isomorphism of global sections and consequently an isomorphism on cohomology.

Thus our problem is reduced to showing that $f\mathcal{C}^*(X; f^*\mathcal{A})$ is an n -truncated resolution of $ff^*\mathcal{A}$ with $n = p - 1$ (or $n = p$ when $\mathcal{A}$ is a locally constant sheaf of free modules over R). Thus we must show that

$$\mathcal{H}^i(f\mathcal{C}^*(X; f^*\mathcal{A})) = 0,$$

for $0 < i \leq p - 1$ (or p) where $\mathcal{H}$ denotes the derived sheaf. For $y \in Y$ we have

$$\begin{aligned} \mathcal{H}^i(f\mathcal{C}^*(X; f^*\mathcal{A}))_y &= \lim_{U \ni y} H^i(f\mathcal{C}^*(X; f^*\mathcal{A})(U)) \\ &= \lim_{U \ni y} H^i(\mathcal{C}^*(X; f^*\mathcal{A})(f^{-1}U)) \\ &= \lim_{U \ni y} H^i(f^{-1}(U); f^*\mathcal{A}) \\ &= \lim_{U \ni y} H^i(f^{-1}(U); M), \end{aligned}$$

for the stalks at y , where $\lim$ denotes the direct limit as U ranges over open neighborhoods of y , $\mathcal{B}(U)$ denotes the module of sections over U in a given sheaf $\mathcal{B}$, H_*^i denotes singular cohomology and M denotes a module such that $\mathcal{A}$ is locally isomorphic to the constant sheaf M . The first of the above equalities holds by definition of the derived sheaf, the second by definition of the direct image sheaf, the third by definition of sheaf cohomology, and the fourth by the fact that for U sufficiently small, the restriction $f^*\mathcal{A} \mid f^{-1}(U)$ reduces to the constant sheaf M and sheaf cohomology may be identified with singular cohomology on paracompact manifolds. Moreover, for U sufficiently small there exists a cross-section $s: U \rightarrow f^{-1}(U)$. It should be observed that U is consequently Hausdorff. It now remains to prove

Lemma 2. *Let $f: W \rightarrow U$ be a submersion admitting a cross-section $s: U \rightarrow W$, where U is an open ball and $W = f^{-1}(U)$ is a paracompact differentiable manifold. If every fibre $f^{-1}(y)$ is strongly p -acyclic, then the reduced singular cohomology groups $\tilde{H}_i^*(W; M)$ vanish for $0 \leq i \leq n$, where $n = p - 1$ in general and $n = p$ when M is a free module over R .*

By the axiom of compact support for singular homology [6, p. 203] and the universal coefficient theorem for cohomology [6, p. 243] it will suffice to show that for every compact $C \subset W$ there exists a compact $\tilde{C} \subset W$ with $C \subset \tilde{C}$ such that the reduced singular integral homology groups $\tilde{H}_i(\tilde{C})$ vanish for $0 \leq i < p$ and $H_p(\tilde{C})$ is a torsion group. To construct $\tilde{C}$ one will require the following tubular neighborhood theorem.

Lemma 3. *Let $f: W \rightarrow U$ be a submersion, where W denotes a paracompact differentiable manifold. Given $y \in U$ and a compact subset $\Sigma \subset f^{-1}(y)$ there exists a neighborhood D of y and a homeomorphism α of $\Sigma \times D$ into W such that*

- (i) $f \circ \alpha = \pi$ (where $\pi: \Sigma \times D \rightarrow D$ is the natural projection)
- (ii) $\alpha(x, y) = x$ for all $x \in \Sigma$.

To construct a tubular neighborhood of Σ we will use a Riemannian metric on W to obtain a splitting of the tangent bundle τ of W . Thus $\tau = \xi \oplus \eta$, where ξ denotes the subbundle tangential to the fibres of f and η denotes the orthogonal complement to ξ . We may without loss of generality take U to be an affine space, so that for every point $y' \in U$ one has a corresponding ray $yy': I \rightarrow U$, where I denotes the unit interval. For Δ a sufficiently small neighborhood of $\Sigma \times y$ in $\Sigma \times U$ and $(x, y') \in \Delta$ the ray yy' lifts to a unique curve $\gamma: I \rightarrow W$ tangential to η with $\gamma(0) = x$. Setting $\alpha(x, y') = \gamma(1)$ we obtain a continuous map $\alpha: \Delta \rightarrow W$ satisfying conditions (i) and (ii) above. It remains to be shown that there exists a neighborhood D of y such that $\Sigma \times D \subset \Delta$ and $\alpha|_{\Sigma \times D}$ is a homeomorphism onto its image. One way to see this is to observe that our definition of α reduces to the usual construction of tubular neighborhoods using geodesic sprays. More precisely, a Euclidean metric on U induces an inner product on η which, together with an arbitrary inner product on ξ , gives a bundle-like metric on W in the sense of Reinhart [4]. It follows by [4, Propositions 1 and 2] that our lifted rays γ are precisely the normal geodesics to Σ with respect to this bundle-like metric. The existence of D follows now by the usual argument [2, p. 73].

A slight refinement of our construction yields

Lemma 4. *Let $f: W \rightarrow U$ be a submersion, where again W is paracompact. Given a compact subset $C \subset W$ and a compact neighborhood $\Sigma \subset f^{-1}(y)$ of $C \cap f^{-1}(y)$ there exists a tubular neighborhood $\alpha: \Sigma \times D \rightarrow W$ (as in Lemma 3) which satisfies the additional condition*

- (iii) $C \cap f^{-1}(D) \subset \alpha(\Sigma \times D)$.

To show this we assume that a tubular neighborhood $\alpha: \Sigma \times D' \rightarrow W$ has been constructed as in Lemma 3, and we consider

$$A = C \cap [W - \text{Int } \alpha(\Sigma \times D')],$$

where Int denotes the interior. Clearly A is compact, so that $f(A)$ is compact. But we may here assume without loss of generality that U is Hausdorff, so that $f(A)$ is also closed. Since A does not meet $f^{-1}(y)$ there exists an open ball $D \subset D'$

such that $y \in D$ and $D \cap f(A)$ is empty. One easily verifies now that the tubular neighborhood given by the restriction of α to $\Sigma \times D$ satisfies condition (iii).

We return to the proof of Lemma 2. Given a compact subset $C \subset W$ we wish to construct a compact set $\tilde{C}$ containing C which satisfies the acyclicity condition stated above, i.e. $\tilde{H}_i(\tilde{C}) = 0$ for $0 \leq i < p$ and $H_p(\tilde{C})$ is a torsion group. This will be accomplished in a finite number of steps, each step involving the same basic construction. It will suffice to describe this basic construction as it applies to the first step. To simplify our notation we will write $E \mid F = E \cap f^{-1}(F)$ and $E \mid y = E \cap f^{-1}(y)$ whenever $E \subset W$, $F \subset U$ and $y \in U$.

Basic construction (1st application). Let $B \subset U$ denote a closed ball such that $f(C) \subset B$. Since the fibres $f^{-1}(y)$ are strongly p -acyclic one can associate with every $y \in f(C)$ a compact locally contractible $\Sigma \subset f^{-1}(y)$ such that $C \mid y \subset \text{Int } \Sigma$ and $\tilde{H}_i(\Sigma) = 0$ for $0 \leq i \leq p$. By Lemma 4 we now associate with every $y \in f(C)$ a tubular neighborhood $\alpha: \Sigma \times D \rightarrow W$, and we will write $V = \alpha(\Sigma \times D)$. By compactness of $f(C)$ one obtains a finite family $V_1, \dots, V_r$ of these neighborhoods which covers C . It should be observed that we are here utilizing condition (iii) of Lemma 4. With each neighborhood V_i there is thus associated a fibre Σ_i and a base space D_i . Setting $D_0 = U - f(C)$ one obtains an open covering $D_0, \dots, D_r$ of B . Let K denote a triangulation of B subordinate to this covering. Thus there exists a function j which associates to every m -simplex $\sigma \in K$ (where $m = \text{dimension of } K$) an index $j(\sigma)$ such that $\sigma \subset D_{j(\sigma)}$. With every m -simplex $\sigma \in K$ we will now associate a subset $C_\sigma \subset W$ by the rule

$$C_\sigma = \begin{cases} V_{j(\sigma)} \mid \sigma & \text{for } j(\sigma) > 0, \\ s(U) \mid \sigma & \text{for } j(\sigma) = 0. \end{cases}$$

Moreover, we let C_0 denote the union of the C_σ as σ ranges over all m -simplexes of K . This describes our basic construction.

Clearly $C \subset C_0$ and C_0 is compact. Moreover, for every m -simplex σ and $y \in \text{Int } \sigma$, $C_0 \mid y$ is locally contractible and acyclic through dimension p . On the other hand $C_0 \mid y$ may fail to be acyclic through dimension p for points y belonging to the $(m-1)$ -skeleton K^{m-1} . To remedy this situation we shall build up $\tilde{C}_0$ over K^{m-1} by repeating our basic construction with C_0 replacing C and K^{m-1} (regarded as a compact subset of B) replacing B . In place of the triangulation K of B the construction will now involve a suitable barycentric subdivision K_1 of K^{m-1} , and it will furnish a compact set C_1 such that $C_0 \mid K^{m-1} \subset C_1$ and $C_1 \mid y$ is locally contractible and p -acyclic for all $y \in \text{Int } \sigma$ as σ ranges over the $(m-1)$ -simplexes of K_1 . Continuing in this manner one obtains a finite sequence $C_0, C_1, \dots, C_m$ of compact subsets and we let $\tilde{C}$ denote their union. Thus $\tilde{C}$ is compact, $C \subset \tilde{C}$ and $\tilde{C} \mid y$ is locally contractible and acyclic through dimension p for all $y \in B$.

We claim that $\tilde{C}$ is locally contractible. To show this, let $x \in \tilde{C}$ and $y = f(x)$. Let L denote a barycentric subdivision of K such that $K_m \subset L^0$ (i.e. L refines all the simplicial complexes involved in the construction of $\tilde{C}$). Then $y \in \text{Int } \sigma$ for some $(m-s)$ -simplex $\sigma \in L$. Let U_i denote the open star of σ in L^{m-i}

for $0 \leq j \leq s$. There exist then strong deformation retractions $r_j : U_j \rightarrow U_{j+1}$ for $0 \leq j < s$. Using the local product structure of C_j (as given by our construction) one can lift each r_j to a strong deformation retraction $\bar{r}_j : \bar{C} | U_j \rightarrow \bar{C} | U_{j+1}$. Moreover there exists a strong deformation retraction $r_s : U_s \rightarrow \sigma$ which similarly lifts to $\bar{r}_s : \bar{C} | U_s \rightarrow \bar{C} | \sigma$. Composing these maps one obtains a strong deformation retraction $\bar{r} : \bar{C} | U_0 \rightarrow \bar{C} | \sigma$. On the other hand, $x \in C_s | \sigma$ and $C_s | \sigma$ is locally contractible by construction. Now let W be a neighborhood of x in $\bar{C}$. There exists then a contractible neighborhood V of x in $C_s | \sigma$ such that $V \subset W$. Clearly V is also a neighborhood of x in $\bar{C} | \sigma$. Then $\bar{r}^{-1}(V)$ is a neighborhood of x in $\bar{C} | U_0$, and we may assume without loss of generality that $\bar{r}^{-1}(V) \subset W$. It remains to be shown that $\bar{r}^{-1}(V)$ is contractible. To see this we consider

$$V_j = (\bar{r}_s \circ \bar{r}_{s-1} \circ \cdots \circ \bar{r}_j)^{-1}(V),$$

for $0 \leq j \leq s$. Thus $V_0 = \bar{r}^{-1}(V)$, and we may set $V = V_{s+1}$. It now follows by our construction of $\bar{r}_j$ that $\bar{r}_j | V_j$ is a strong deformation retraction of V_j onto V_{j+1} , which implies that V_0 is contractible.

One now observes that the restriction of f to $\bar{C}$, regarded as a map $g : \bar{C} \rightarrow B$, satisfies the conditions of the Vietoris-Begle Theorem, as given in Section 1. Since B is contractible this implies that $\bar{H}_i^s(\bar{C}) = 0$ for $0 \leq i \leq p$, where $\bar{H}_i^s$ denotes the reduced singular cohomology group. Since $\bar{C}$ is a compact Hausdorff HLC space, it follows that its homology groups are finitely generated [6, p. 342]. One may therefore conclude by a strong form of the universal coefficient theorem [6, p. 244] that $\bar{H}_i(\bar{C}) = 0$ for $0 \leq i < p$ and $H_p(\bar{C})$ is a torsion group, as was to be shown.

3. Proof of Theorem 2. Let M be a piecewise-linear p -connected n -manifold with $p \leq n - 3$. To prove that M is strongly p -acyclic it will suffice to show that every compact polyhedron A in M is contained in a compact polyhedron C whose reduced integral homology groups vanish on dimensions $\leq p$.

Let K be a triangulation of M belonging to the given piecewise-linear structure. Then $A = |K_A|$, where K_A is a finite subcomplex of some rectilinear subdivision K' of K . Consider the p -skeleton K_A^p and let $A^p = |K_A^p|$. By the Engulfing Theorem [7] there exists a piecewise-linear ball B in M such that $A^p \subset B$, and we let $C = A \cup B$. Now B corresponds to a subcomplex K_B of some rectilinear subdivision K'' of K , and we let L denote a rectangular subdivision of K which refines K' and K'' . The compact polyhedra A , A^p , B and C correspond now to finite subcomplexes L_A , L_{A^p} , L_B and L_C of L . Let z denote a (simplicial integral) q -cycle in L_C with $0 < q \leq p$. Then we can write $z = z_1 + z_2$ with $z_1 \subset L_A$ and $z_2 \subset L_B$. Clearly z_1 is homologous in L_A to a chain $z^* \subset L_{A^p}$, and z is therefore homologous in L_C to $z^* + z_2 \subset L_B$. Since L_B is acyclic on dimension q , z is homologous to zero in L_C , as was to be shown.

4. Proof of Theorem 3. Let X be an orientable para-compact differentiable n -manifold and F an orientable p -dimensional regular foliation on X . Let Y denote the quotient space (leaf space) induced by F and let $f : X \rightarrow Y$ denote

the natural projection. Thus f is a surjective submersion. According to [5, Theorem A] the foliation F extends on relatively compact subsets of X provided Y has trivial Euler class. But this Euler class χ is an element of the singular integral cohomology $H_*^{n-p}(Y)$. We note that since X is paracompact there exists a natural isomorphism $H_*(X) \approx H^*(X; Z)$, where Z denotes the sheaf of integers. Under the hypothesis of Theorem 3 one may therefore conclude by Theorems 1 and 2 that $H^{n-p}(Y; Z) = 0$. However, since Y may fail to be paracompact we cannot identify singular and sheaf cohomology of Y . To obviate this difficulty we must consider the proof of Theorem A as given in [5] to see precisely how the vanishing of χ has been employed. One finds (see proof of Theorem 1 in [5, Section 2]) that the desired extension of F comes from a nonzero cross-section in an orientable $(n - p)$ -plane bundle ξ over a compact polyhedron $\bar{E}$. There exists, moreover, a natural bundle map $g: \xi \rightarrow \tau$, where τ denotes the tangent bundle of Y . One now employs the vanishing of χ (plus naturality of the Euler class) to conclude that ξ has trivial Euler class. To establish Theorem 3 it will therefore suffice to prove the following

Lemma. *Let $g: \xi \rightarrow \tau$ be a bundle map (as above) and let $H^{n-p}(Y; Z) = 0$. Then ξ has trivial Euler class.*

Since Y is HLC the sheaf $\mathcal{S}^*$ of singular integral cochains on Y constitutes a resolution of Z (see [1, p. 25]). By Lemma 1 there exists a natural homomorphism $\gamma: H^*(\Gamma(\mathcal{S}^*)) \rightarrow H^*(Y; Z)$. Precomposing γ with the natural map $H_*(Y) \rightarrow H^*(\Gamma(\mathcal{S}^*))$ gives a natural homomorphism $\varphi: H_*(Y) \rightarrow H^*(Y; Z)$. Thus one obtains a commutative diagram

$$\begin{array}{ccc} H_*^{n-p}(Y) & \xrightarrow{\varphi} & H^{n-p}(Y; Z) \\ \psi_1 \downarrow & & \downarrow \psi_2 \\ H_*^{n-p}(\bar{E}) & \xrightarrow{\varphi} & H^{n-p}(\bar{E}; Z), \end{array}$$

where ψ_i is induced by the map $\bar{E} \rightarrow Y$ associated with g . By naturality, the Euler class of ξ is precisely $\psi_1(\chi)$. But since $\bar{E}$ is a paracompact HLC space φ_2 is an isomorphism. Hence $\psi_1(\chi)$ is zero, as was to be shown.

Remark. Our argument actually proves a slightly strengthened version of Theorem 3. For if $n - p$ is odd, χ is known to be of order 2, and we need only to require that $H^{n-p}(Y; Z)$ admits no torsion of order 2. Hence for $n - p$ odd the hypothesis of Theorem 3 can be weakened in regard to $H^{n-p}(X)$. One now requires only that $H^{n-p}(X)$ shall admit no torsion of order 2.

REFERENCES

- [1] G. E. BREDON, *Sheaf Theory*, McGraw-Hill, 1967.
- [2] S. LANG, *Introduction to Differential Manifolds*, Interscience, 1962.
- [3] R. S. PALAIS, A global formation of the Lie theory of transformation groups, *Amer. Math. Soc. Memoir*, 22 (1957).

- [4] B. L. REINHART, Foliated manifolds with bundle-like metrics, *Ann. of Math.*, **69** (1959) 119–131.
- [5] J. W. SMITH, Extending regular foliations, *Ann. Inst. Fourier* (Grenoble), **19** (1970) 155–168.
- [6] E. H. SPANIER, *Algebraic Topology*, McGraw-Hill, 1966.
- [7] J. R. STALLINGS, The piecewise-linear structure of Euclidean space, *Proc. Cambridge Philos. Soc.*, **58** (1962) 481–488.

San Diego State College

Oregon State University

Date communicated: NOVEMBER 18, 1969