The Bénard Problem for General Fluid Dynamical Equations and Remarks on the Boussinesq Approximation

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1. Introduction. The object of this paper is to give a rigorous mathematical treatment of Bénard convection, using very general fluid dynamical equations. Bénard convection is the steady periodic motion of a slightly compressible viscous fluid between two horizontal planes held at constant temperatures, the temperature on the lower plane being greater than that on the upper. This situation has been the object of extensive experimental investigation, as well as of analytical study within the framework of the “Boussinesq approximation” to the compressible Navier–Stokes Equations. Those analytical investigations closest in spirit to the present one are found in papers by Velte [8], Judovich [3], Rabinowitz [5] and Fife and Joseph [2].

The present paper proceeds from the general dynamical equations of a compressible fluid with an equation of state of the form

$$\rho = \rho_0(1 + \alpha(T - T_0) + h(T)).$$

Here ρ is the density and T the temperature (ρ_0 and T_0 are a reference density and temperature, respectively), and h is a function of T which is arbitrary except that it is assumed “small enough” in a certain sense. The requirement that ρ depend only on T is the most restrictive assumption on our treatment. The stress tensor, heat conduction coefficient, and internal energy are allowed to depend on the temperature and pressure; furthermore the stress tensor is allowed to be nonlinear as a function of the deformation tensor (this goes beyond the Navier–Stokes Equations, in which linearity is assumed). Restrictions on the temperature and pressure dependence of these quantities are given in Section 2, but are typified by the fact that the heat conduction coefficient is supposed to

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equal a constant plus a relatively small arbitrary nonlinear term. It might be remarked that allowing ρ to depend on pressure causes no difficulty in the formal part of the procedure used here, but leads to serious problems in our convergence proof in Section 4.

Our main result is an existence proof for small convective motions, together with an iteration process for constructing such motions. We also determine under which circumstances the Boussinesq approximation will be valid for the motions considered, and in fact prove its validity when these circumstances are fulfilled. The essential feature of the Boussinesq approximation is that variations in density are neglected throughout the equations except in the term involving the force of gravity (the buoyant force term), in which it is assumed to be a linear function of the temperature.

Finally, we exhibit the phenomenon of bifurcation (as in [2, 3, 5, 8]) in the solutions of the general equations. In this connection the Rayleigh number R is taken to be the relevant eigenvalue parameter, and convective solutions are found for R near eigenvalues of the linearized Boussinesq equations.

Our treatment proceeds as follows. The problem is set up in such a way that the relevant equations governing the flow differ from the linearized Boussinesq equations by nonlinear perturbation terms. Successive approximations are then employed to construct an exact solution. As in [5] this does not involve the usual type of successive approximation procedure, in which an operator is inverted at each step. Rather, the operator which would ordinarily be inverted has no inverse, and an orthogonality condition must be applied at each step in the iteration, in order that the equations for the corresponding approximation will have a solution. The unperturbed problem is a linear eigenvalue problem and accordingly the problem we solve can be thought of as a nonlinear eigenvalue problem.

The convergence of the successive approximations is proved with the aid of a prior Schauder estimates for solutions of the linearized Boussinesq equations. These powerful estimates enable us to treat nonlinearities of general type in the functions spoken of above.

Fluid motions are sought and found which are periodic (with prescribed periods) in each of the two horizontal coordinates. This program could be varied in several ways. First, one could seek only the special kinds of periodic motion which form hexagonal convection cells. Or one could treat a fluid confined in a bounded smooth but arbitrarily shaped vessel, with temperature prescribed in an appropriate way on the boundary. In this case the proof of our basic Lemma 3 would be based on Odkvist's Green's tensor [4] rather than the Fourier expansion techniques used here. Finally, one could take different boundary conditions, corresponding to different physical pictures. For details on these modifications see [2]. In the case of the last two modifications described, the simplicity of the eigenvalues of the linearized problem has not been proved. The existence of at least one such simple eigenvalue is needed in the present method.

Some important aspects of our treatment (especially the use of an iterative

procedure in constructing solutions) are found also in the paper of Rabinowitz [5]. Our approach is a type of generalization of the now classical methods of Lyapunov and Schmidt ([7] is a good survey article). In [2] the present author and Joseph went part of the way to a full generalization of the Boussinesq equations by allowing, for example, the equation of state to have quadratic terms in the dependence of ρ on T . In that paper the solution was constructed as a power series in its own norm, thus directly proving analytic dependence on that norm. If the arbitrary functions used in the present paper (specifically, those in (2.6)) are in fact analytic, then such a construction in power series could no doubt be performed in the present framework as well.

2. Formulation of the Problem. We proceed from the following standard equations of steady compressible viscous flow under the influence of gravity [6] (vectors and tensors are in bold face):

$$(2.1) \quad \rho(\mathbf{v} \cdot \text{grad})\mathbf{v} = -\rho g \mathbf{k} + \text{div } \mathbf{T},$$

$$(2.2) \quad \rho(\mathbf{v} \cdot \text{grad})E = \mathbf{T} : \mathbf{D} + \text{div } (k \text{ grad } T),$$

$$(2.3) \quad \text{div } (\rho \mathbf{v}) = 0,$$

$$(2.4) \quad E = E(T, p),$$

$$(2.5) \quad \rho = \rho(T, p),$$

where v , ρ , E , p , k , T , and $\mathbf{T}$ are velocity, density, internal energy, pressure, heat conduction coefficient, temperature, and stress tensor. Also $\mathbf{D}$ is the deformation tensor, $D_{ij} = 1/2(\partial v_i / \partial x_j + \partial v_j / \partial x_i)$. Finally $\mathbf{k}$ is the unit vector in the vertical direction.

The following assumptions will be made concerning the form of the functions E , ρ , and $\mathbf{T}$. The first (and most crucial) one states that the fluid is not too far from being incompressible, and the density depends only on T .

$$(2.6a) \quad \rho = \rho_0(1 + \gamma T(1 + \epsilon f_1(T))),$$

where ρ_0 is some reference density, γ is a constant, and f_1 is an arbitrary three times differentiable function. The artificial dimensionless parameter ϵ is introduced to emphasize that we shall be assuming the nonlinear part of the above dependence to be small in some sense. Similarly

$$(2.6b) \quad E = cT(1 + \epsilon f_2(T, p))$$

$$(2.6c) \quad \mathbf{T} = (-p + \lambda \text{div } \mathbf{v})\mathbf{I} + 2\mu \mathbf{D} : (\mathbf{I} + \mathbf{N}(T, p, \mathbf{D})),$$

$$(2.6d) \quad \lambda = \lambda_0(1 + \epsilon f_3(T, p)),$$

$$(2.6e) \quad \mu = \mu_0(1 + \epsilon f_4(T, p)),$$

$$(2.6f) \quad k = k_0(1 + \epsilon f_5(T, p)).$$

Again, the functions f_i and the components of $\mathbf{N}$ are three times continuously differentiable in all their arguments.

The Navier–Stokes Equations result when $\mathbf{T}$ is assumed to be linear in $\mathbf{D}$. It then follows that $\mathbf{T}$ is of the form (2.6c) with $\mathbf{N} = 0$; λ and μ are the viscosity coefficients. In our case we allow the general form (2.6c), where $\mathbf{N}$ is “higher order” in the sense that

$$(2.7) \quad \mathbf{N}(T, p, 0) = 0.$$

It should be explained that $\mathbf{I}$ and $\mathbf{N}$ are tensors of rank 4. Specifically $I_{ijkl} = \delta_{ik}\delta_{jl}$, so that $(\mathbf{D}:\mathbf{I})_{ij} = D_{kl}I_{klji} = D_{ij}$ (the summation convention will be used throughout).

We seek solutions of (2.1)–(2.3) in the strip $0 \leq x_3 \leq d$ which are periodic with given periods l_1 and l_2 in x_1 and x_2 respectively, and which satisfy

$$(2.8a) \quad T = \mathbf{v} = 0 \quad \text{for } x_3 = 0; \quad T = T_1 < 0, \quad \mathbf{v} = 0 \quad \text{for } x_3 = d.$$

It will prove convenient, and in fact necessary, to impose a condition on the pressure as well. It takes the form

$$(2.8b) \quad \frac{1}{l_1 l_2} \int_0^{l_1} \int_0^{l_2} p(x_1, x_2, 0) dx_1 dx_2 = p_0,$$

where p_0 is some reference pressure.

A “conduction solution” of (2.1)–(2.3) is defined as one for which $\mathbf{v} \equiv 0$.

Lemma 1. *For small enough ϵ , there exists a conduction solution $\mathbf{v} \equiv 0$, $T = T_0(x_3)$, $p = p_0(x_3)$ of (2.1)–(2.3), (2.8). It is such that $T_0(x_3)$ is monotone.*

Proof. Setting $\mathbf{v} \equiv 0$, $T = T_0(x_3)$, $p = p_0(x_3)$ into the equations we obtain

$$\begin{aligned} \frac{dp_0}{dx_3} &= -\rho(T_0)g, \\ \frac{d}{dx_3} \left(k(T_0, p_0) \frac{dT_0}{dx_3} \right) &= 0, \\ p_0(0) &= p_0, \quad T_0(0) = 0, \quad T_0(d) = T_1. \end{aligned}$$

On the basis of (2.6f) the second equation may be written

$$(2.9) \quad \frac{dT_0}{dx_3} = \frac{C}{1 + \epsilon f_5(T_0, p_0)},$$

where C is a constant of integration.

The solution proceeds by iteration. We begin it by defining $T_0^0 \equiv p_0^0 \equiv 0$. Once $T_0^n(x_3)$, $p_0^n(x_3)$ have been found, we define T_0^{n+1} , p_0^{n+1} by the equations

$$\begin{aligned} \frac{dT_0^{n+1}}{dx_3} &= \frac{C^{(n+1)}}{1 + \epsilon f_5(T_0^n, p_0^n)}, \\ \frac{dp_0^{n+1}}{dx_3} &= -g\rho(T_0^n), \end{aligned}$$

with initial conditions

$$T_0^{n+1}(0) = 0, \quad p_0^{n+1}(0) = p_0,$$

where $C^{(n+1)}$ is chosen so that $T_0^{n+1}(d) = T_1 < 0$. Assuming for the moment that $\epsilon |f_5(T_0^n, p_0^n)| \leq \frac{1}{2}$, it is clear that

$$\frac{dT_0^{n+1}}{dx_3} < 0, \quad C^{(n+1)} < 0, \quad \frac{dT_0^{n+1}}{dx_3} \leq \frac{2}{3} C^{(n+1)},$$

and

$$T_1 = \int_0^d \frac{dT_0^{n+1}}{dx_3} dx_3 \leq \frac{2d}{3} C^{(n+1)},$$

so

$$\frac{3T_1}{2d} \leq C^{(n+1)} < 0,$$

and

$$\left| \frac{dT_0^{n+1}}{dx_3} \right| \leq 2 \left| C^{(n+1)} \right| \leq \frac{3T_1}{d}.$$

Also since $0 \geq T_0^n \geq T_1$, we have

$$\left| \frac{dp_0^{n+1}}{dx_3} \right| \leq g \max_{[T_1, 0]} \rho(T).$$

Thus we have bounds on p_0^{n+1} , T_0^{n+1} , and their derivatives independently of n . By choosing ϵ small enough we may therefore guarantee that $\epsilon f_5(T_0^{n+1}, p_0^{n+1}) \leq \frac{1}{2}$. An easy induction argument then establishes the solvability of all the iteration equations, together with uniform bounds on T_0 , p_0 , and their first derivatives. It is seen from the differential equations that these first derivatives are equicontinuous. Arzela's Theorem then shows that a subsequence of these iterates converges to a solution, proving the lemma.

We "subtract off" the conduction solution, setting

$$T = T_0 + T', \quad p = p_0 + p', \quad \rho = \rho^* + \rho', \quad k = k^* + k',$$

where $\rho^* = \rho(T_0(x_3))$, $k^* = k(T_0(x_3), p_0(x_3))$. Equations (2.1-2.3) then become

$$(2.10a) \quad \rho(\mathbf{v} \cdot \text{grad}) \mathbf{v} = -\rho' g \mathbf{k} - \text{grad } \rho' + \text{div} [\lambda(\text{div } \mathbf{v}) \mathbf{I} + 2\mu \mathbf{D}: (\mathbf{I} + \mathbf{N})],$$

$$(2.10b) \quad \rho(\mathbf{v} \cdot \text{grad}) E = \mathbf{T}: \mathbf{D} + \text{div} [(k - k^*) \text{grad } T + k^* \text{grad } T'],$$

$$(2.10c) \quad \text{div} (\rho \mathbf{v}) = 0,$$

$$(2.10d) \quad \mathbf{v} = T' = 0 \quad \text{for } x_3 = 0 \quad \text{and } x_3 = d;$$

$$(2.10e) \quad \mathbf{v}, T', p' \text{ are periodic with periods } l_1 \text{ and } l_2 \text{ in } x_1 \text{ and } x_2,$$

$$(2.10f) \quad \int_0^{l_1} \int_0^{l_2} p'(x_1, x_2, 0) dx_1 dx_2 = 0.$$

At this point the object will be to rewrite these equations in more convenient form by introducing dimensionless variables and special notation. In terms of the parameters already introduced in the problem, we define the further positive parameters

$$\nu = \frac{\mu_0}{\rho_0}, \quad \kappa = \frac{k_0}{\rho_0 c}, \quad \beta = -T_1/d,$$

and the following dimensionless combinations of the above:

$$R = \frac{\gamma \beta d^4 g}{\kappa \nu}, \quad \pi = \frac{\nu}{\kappa}, \quad \tau = \gamma \beta d, \quad e = \frac{g}{\beta c}.$$

Here R is called the Rayleigh number and π the Prandtl number. In Section 5 it will be shown that the Boussinesq approximation is valid for small solutions (in the case $\mathbf{N} = 0$) provided ϵ and τ are small. In fact throughout our treatment, τ as well as ϵ will be considered a perturbation parameter.

A characteristic velocity V , temperature Θ , and pressure P are defined as

$$V = \kappa/d, \quad \Theta = \frac{\beta d}{R}, \quad P = \frac{\rho_0 \kappa^2 \pi}{d^2}.$$

For purposes of our later estimates, we assume that there is a number M serving as an upper bound for the parameters

$$(2.11) \quad R, \pi, \tau, e, \frac{1}{R}, \frac{1}{\pi}, \frac{\tau^2 \lambda_0}{\mu_0}, V, \Theta, P, \text{ and } \epsilon.$$

Our new dimensionless variables will be defined in terms of the above, and also an extra small dimensionless parameter a which for the moment is arbitrary, but which will later be given a meaning:

$$(2.12a) \quad u_i = \frac{\rho v_i}{a \rho_0 V}, \quad i = 1, 2, 3,$$

$$(2.12b) \quad \theta = \frac{T'}{a \Theta},$$

$$(2.12c) \quad q = \frac{p'}{a P};$$

and the independent variables are $z_i = x_i/d$, $i = 1, 2, 3$. Also we define

$$(2.12d) \quad \theta_0(z_3) = \frac{T_0(x_3)}{\Theta}, \quad q_0(z_3) = \frac{p_0(x_3)}{P}.$$

Derivatives will be denoted by the symbol $\partial_i = \partial/\partial z_i$; also $\partial^2 = \sum_1^3 \partial_i^2 = \Delta$.

Lemma 2. *Equations (2.10a, b) may be written in the form*

$$(2.13a) \quad \partial^2 u_i + \theta \delta_{i3} - \partial_i q = a s_{1i} + \tau s_{2i} + \epsilon s_{3i}, \quad i = 1, 2, 3,$$

$$(2.13b) \quad \partial^2 \theta + R u_3 = a s_4 + \epsilon s_5,$$

where δ_{ij} is the Kronecker symbol, and the s_{ij} and s_i are continuously differentiable functions of

- (1) u_i and θ and their first and second order derivatives,
- (2) q and its first derivatives,
- (3) the parameters listed in (2.11), and a as well, and
- (4) the coordinates z_i .

The proof of this lemma will be given in Section 7.

Of course (2.10c) takes the form

$$(2.13c) \quad \mathfrak{a} \cdot \mathbf{u} = 0,$$

and the auxiliary conditions are

$$(2.13d) \quad \theta = \mathbf{u} = 0 \quad \text{for } z_3 = 0, 1,$$

$\mathbf{u}$, θ , and q are periodic with periods

$$(2.13e) \quad l_1/d \text{ and } l_2/d \text{ in } z_1 \text{ and } z_2 \text{ respectively,}$$

$$(2.13f) \quad \int_0^{l_1/d} \int_0^{l_2/d} q(z_1, z_2, 0) dz_1 dz_2 = 0.$$

The system (2.10), hence the above system (2.13), always has the trivial solution $\mathbf{u} \equiv \theta \equiv q = 0$. We seek nonzero solutions. They can be constructed for values of R near simple eigenvalues of the same problem with the right hand side set equal to zero and R considered an eigenvalue parameter. In other words, bifurcation takes place at such eigenvalues.

We further simplify the notation of (2.13) in the following manner. Let $\mathbf{U}$ be the 4-vector (u_1, u_2, u_3, θ) , and $\mathbf{W}$ the 5-vector $(u_1, u_2, u_3, \theta, q)$. Let $\mathfrak{a} = (\partial_1, \partial_2, \partial_3, 0)$ and let $\mathbf{A}$ be the matrix $a_{ij} = \delta_{i3}\delta_{j4} + R\delta_{i4}\delta_{j3}$.

Then the system (2.13a-c) can be written

$$(2.14a) \quad \Delta \mathbf{U} + \mathbf{A} \mathbf{U} - \mathfrak{a} q = a \mathbf{S}_1(\mathbf{W}) + \epsilon \mathbf{S}_2(\mathbf{W}) + \tau \mathbf{S}_3(\mathbf{W}),$$

$$(2.14b) \quad \mathfrak{a} \cdot \mathbf{U} = 0,$$

for some vector functions $\mathbf{S}_i$ with the same properties as the s_{ij} of (2.13).

In the following R_0 will denote a fixed value of R for which there exists exactly one (linearly independent) nontrivial solution $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)$, ω , of

$$(2.15a) \quad \Delta \Phi + \mathbf{A}_0 \Phi - \mathfrak{a} \omega = 0,$$

$$(2.15b) \quad \mathfrak{a} \cdot \Phi = 0$$

together with conditions (2.13d-f). Here we have denoted by $\mathbf{A}_0$ the matrix $\mathbf{A}$ with R set equal to R_0 .

The question as to whether there exist such values R_0 has been answered by Judovich [3] and Rabinowitz [5]. This is of course a linear eigenvalue problem with eigenvalue parameter R , and the requirement is that R_0 be a real simple eigenvalue. There exists an infinite discrete set of real eigenvalues for (2.15a, b)

characterized variationally; this was shown, for example, in [3] and [5]. Furthermore Judovich and Rabinowitz have shown that for almost all pairs (l_1, l_2) , every eigenvalue is simple. In fact even in the nonsimple case Rabinowitz was able to reformulate the problem so that any eigenvalue becomes simple.

For two 4-vectors Φ and Ψ we shall employ the special scalar product

$$(2.16) \quad (\Phi, \Psi) = \int_{\mathfrak{D}} \left(R_0 \sum_{i=1}^3 \Phi_i \Psi_i + \Phi_4 \Psi_4 \right) dz_1 dz_2 dz_3$$

where $\mathfrak{D}$ is the period parallelepiped $0 \leq z_1 \leq l_1/d$, $0 \leq z_2 \leq l_2/d$, $0 \leq z_3 \leq 1$.

We suppose Φ is normalized so that $(\Phi, \Phi) = 1$ (this of course determines Φ only up to change in sign). Then we impose the following additional condition on the solution $\mathbf{U}$ of (2.14):

$$(2.17) \quad (\mathbf{U}, \Phi) = 1.$$

Although this may seem like too many conditions to be imposed on the solution of our problem, one should remember that we essentially have a nonlinear eigenvalue problem, and in such problems one may in some sense specify the magnitude of the solution. (2.17) is such a specification. On the other hand, once this specification is made, the eigenvalue parameter R is no longer arbitrary. In fact, part of our problem will be to determine the discrepancy

$$\eta = R - R_0,$$

along with the solution $\mathbf{W}$.

Incidentally, this is the point at which the physical relevance of the parameter a (introduced artificially in (2.12)) appears. The definitions (2.12a, b) together with (2.17) yield

$$a\mathbf{U} = \left(\frac{\rho v_1}{\rho_0 V}, \frac{\rho v_2}{\rho_0 V}, \frac{\rho v_3}{\rho_0 V}, \frac{T'}{\theta} \right).$$

hence

$$a = \int_{\mathfrak{D}} \left(\frac{\rho R_0}{\rho_0 V} \sum_i v_i \Phi_i + \frac{1}{\theta} T' \Phi_4 \right) dz_1 dz_2 dz_3.$$

Thus a is a weighted scalar product of the physical solution (v_1, v_2, v_3, T') with Φ . It turns out not to differ too much from the actual weighted norm of the solution (compare [2]).

We define a matrix $\mathbf{B}$ with elements $-\delta_{i4}\delta_{j3}$, and observe that $\mathbf{A} - \mathbf{A}_0 = \eta\mathbf{B}$. We may now write our problem in its final form

$$(2.18a) \quad \Delta \mathbf{U} + \mathbf{A}_0 \mathbf{U} - \mathfrak{a}q = \eta \mathbf{B} \mathbf{U} + a \mathbf{S}_1(\mathbf{W}, \eta) + \epsilon \mathbf{S}_2(\mathbf{W}, \eta) + \tau \mathbf{S}_3(\mathbf{W}, \eta)$$

$$(2.18b) \quad \mathfrak{a} \cdot \mathbf{U} = 0,$$

$$(2.18c) \quad \mathbf{U} = 0 \quad \text{for } z_3 = 0, 1,$$

$$(2.18d) \quad \mathbf{W} \text{ is periodic with periods } l_1/d \text{ and } l_2/d \text{ in } z_1 \text{ and } z_2,$$

$$(2.18e) \quad \int_0^{1_1/d} \int_0^{1_2/d} q(z_1, z_2, 0) dz_1 dz_2 = 0,$$

$$(2.18f) \quad (\mathbf{U}, \Phi) = 1.$$

Solving the problem will consist in finding a vector $\mathbf{W}$ and a scalar η which satisfy (2.18). This will be possible provided a , ϵ , and τ are small enough.

In (2.18a) we have emphasized the fact that the S_i depends on R by writing them explicitly as functions of η as well as of W . This fact was not considered important before; but now the determination of η is part of the problem itself.

3. The Solution Procedure. An iteration process will be used to prove our existence theorem. The steps are based on the following basic definitions and results.

Definition. Let $0 < \alpha < 1$. Then we take $\mathcal{C}_\alpha^4$ to be the set of all 4-vector functions $\Phi(z) = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)$ defined in $\mathcal{S} = \{z: 0 \leq z_3 \leq 1\}$, periodic in z_1 and z_2 with periods l_1/d , l_2/d and such that the norm

$$|\Phi|_\alpha^4 \equiv \sup_{\mathcal{S}} |\Phi(z)| + \sup_{z_1, z_2 \in \mathcal{S}} \frac{|\Phi(z_1) - \Phi(z_2)|}{|z_1 - z_2|^\alpha},$$

is finite. Similarly for any integer l we take $\mathcal{C}_{l+\alpha}^4$ to be the set of such functions with finite norm

$$|\Phi|_{l+\alpha} \equiv \sum |\partial^i \Phi|_\alpha^4,$$

where the sum is over all partial derivatives ∂^i of order $i \leq l$.

Again, $\mathcal{C}_{l+\alpha}^1$ and $\mathcal{C}_{l+\alpha}^3$ are defined the same way except that their elements are scalar functions and 3-vector functions.

Lemma 3. Let $\mathbf{f} \in \mathcal{C}_4^\alpha$ and

$$(3.1) \quad (\mathbf{f}, \Phi) = 0,$$

where Φ is the solution of (2.15). Then there exists one solution $\mathbf{U}$, q of

$$(3.2a) \quad \Delta \mathbf{U} + A_0 \mathbf{U} - \partial q = \mathbf{f},$$

$$(3.2b) \quad \partial \cdot \mathbf{U} = 0,$$

together with (2.18c–f). This solution is such that $\mathbf{U} \in \mathcal{C}_{2+\alpha}^4$, $q \in \mathcal{C}_{1+\alpha}^1$, and there exists a constant C depending only on l_1 , l_2 , d , and R_0 , such that

$$(3.3) \quad |\mathbf{U}|_{2+\alpha}^4 + |q|_{1+\alpha}^1 \leq C(1 + |\mathbf{f}|_\alpha^4).$$

If the condition (2.18f) is replaced by

$$(3.4) \quad (\mathbf{U}, \Phi) = 0,$$

then the same result holds with the right hand side of (3.3) replaced by $C |\mathbf{f}|_\alpha^4$.

The proof will be given in Section 6. We now describe the steps of the iteration process.

(1) We define $\mathbf{W}^0 = (\mathbf{U}^0, q^0)$ as the solution of (2.18) with $\eta = a = \epsilon = \tau = 0$. Thus we must have (by uniqueness of the solution of (2.15)) $\mathbf{U}^0 = \pm \Phi$, since Φ was originally defined as normalized in the appropriate way. At this point we decide on one of the signs (for definiteness, take it to be “+”) which will be kept through the latter discussion. Applying the iteration process to the second solution $(-\Phi)$ would give us a second solution of the nonlinear problem.

(2) For all $n > 0$ we define successively $\mathbf{W}^n = (\mathbf{U}^n, q^n)$, η^n as the solution of

$$(3.5a) \quad \Delta \mathbf{U}^n + \mathbf{A}_0 \mathbf{U}^n - \partial q^n = \eta^n \mathbf{B} \mathbf{U}^{n-1} + a \mathbf{S}_1(\mathbf{W}^{n-1}, \eta^{n-1}) \\ + \epsilon \mathbf{S}_2(\mathbf{W}^{n-1}, \eta^{n-1}) + \tau \mathbf{S}_3(\mathbf{W}^{n-1}, \eta^{n-1}),$$

$$(3.5b) \quad \mathfrak{a} \cdot \mathbf{U}^n = 0,$$

$$(3.5c) \quad (\mathbf{U}^n, \Phi) = 1,$$

together with the usual boundary and periodicity conditions.

Implicit in this is that η^n must be chosen in accordance with (3.1):

$$(3.6) \quad (\mathbf{f}^{n-1}, \Phi) = 0,$$

where $\mathbf{f}^{n-1}$ is the right side of (3.5a). We investigate the possibility of doing this. First observe that

$$(\mathbf{B} \mathbf{U}^{n-1}, \Phi) = \int_{\mathfrak{D}} U_3^{n-1} \delta_{i4} \Phi_i = \int_{\mathfrak{D}} U_3^{n-1} \Phi_4 = \int \Phi_3 \Phi_4 + \int (U_3^{n-1} - \Phi_3) \Phi_4.$$

But $\Delta \Phi_4 = -R_0 \Phi_3$, so $\int \Phi_3 \Phi_4 = -1/R_0 \int \Phi_4 \Delta \Phi_4 = 1/R_0 \int |\Delta \Phi_4|^2$. This cannot vanish, since if it did, then (using the boundary conditions) $\Phi_4 \equiv 0$ and we would have $\Delta \Phi_i - \partial_i \omega = 0$, $i = 1, 2, 3$. Again multiplying by Φ_i , summing over i and integrating over $\mathfrak{D}$, we would obtain $\Phi \equiv 0$, contradicting the definition of Φ as being nontrivial. Thus we may denote

$$(3.7) \quad b \equiv \int \Phi_3 \Phi_4 > 0.$$

The condition

$$(3.8) \quad \left| \int_{\mathfrak{D}} (U_3^{n-1} - \Phi_3) \Phi_4 \right| < b/2,$$

would then guarantee that

$$(3.9) \quad |(\mathbf{B} \mathbf{U}^{n-1}, \Phi)| \geq b/2.$$

Lemma 4. *If $\mathbf{U}^{n-1} \in \mathcal{C}_{2+\alpha}^4$ and $q^{n-1} \in \mathcal{C}_{1+\alpha}^1$ are known, $|\eta^{n-1}| \leq 1$, and (3.8) is satisfied, then (3.5), (3.6) may be solved uniquely for $\mathbf{U}^n, q^n, \eta^n$.*

Furthermore $\mathbf{U}^n \in \mathcal{C}_{2+\alpha}^4$, $q^n \in \mathcal{C}_{1+\alpha}^1$, and setting $P^n = |\mathbf{U}^n|_{2+\alpha}^4 + |q^n|_{1+\alpha}^1$, we have

$$(3.10) \quad P^n \leq C + (\epsilon + a + \tau) h_1(P^{n-1}),$$

$$(3.11) \quad |\eta^n| \leq (\epsilon + a + \tau)h_2(P^{n-1}),$$

where $h_i(s)$ are certain positive nondecreasing functions of s .

Proof. According to Lemma 3, a solution exists if (3.6) is satisfied; i.e.,

$$(3.12) \quad \eta^n(\mathbf{B}\mathbf{U}^{n-1}, \Phi) = - (a\mathbf{S}_1(\mathbf{W}^{n-1}, \eta^{n-1}) - \epsilon\mathbf{S}_2(\mathbf{W}^{n-1}, \eta^{n-1}) \\ - \tau\mathbf{S}_3(\mathbf{W}^{n-1}, \eta^{n-1}), \Phi).$$

Taking note of (3.9), we may solve for η^n , obtaining an estimate of the form (3.11). For example, (3.11) will be satisfied for any h_2 such that

$$h_2(s) \geq \frac{2}{b} \sum_{i=1}^3 \sup |(S_i(\mathbf{W}, \eta), \Phi)|,$$

the suprema being taken over all $\mathbf{W} = (\mathbf{U}, q)$, η , such that $|\eta| \leq 1$ and

$$|\mathbf{U}|_{2+\alpha}^4 + |q|_{1+\alpha}^1 \leq s.$$

These suprema are finite since the $\mathbf{S}_i(\mathbf{W}, \eta)$ may be estimated pointwise in terms of $|\mathbf{U}|_{2+\alpha}^4$, $|q|_{1+\alpha}^1$, and M (see (2.11)), and Φ is a fixed bounded function.

With η^n so chosen, $\mathbf{W}^n$ is determined and is subject to the estimate (3.3); i.e.,

$$P^n \leq C(1 + |\mathbf{f}^{n-1}|_\alpha^4).$$

It only remains to be observed that $|\mathbf{f}^{n-1}|_\alpha^4$ is bounded in terms of P^{n-1} alone. In fact, the $\mathbf{S}_i(\mathbf{W}, \eta)$ are functions of the components of $\mathbf{W}$ and certain of their derivatives (also the parameters (2.11), which are bounded by the constant M). Thus the Hölder difference quotients of the $\mathbf{S}_i$ may also be expressed as functions of the components of $\mathbf{W}$, certain of their derivatives, and Hölder difference quotients of these components and derivatives. In all, we find that (3.10) holds for any h_1 satisfying

$$\frac{1}{C} (\epsilon + a + \tau)h_1(s) \geq |\eta^n| s + a \sup |\mathbf{S}_1(\mathbf{W}, \eta)|_\alpha^4 \\ + \epsilon \sup |\mathbf{S}_2(\mathbf{W}, \eta)|_\alpha^4 + \tau \sup |\mathbf{S}_3(\mathbf{W}, \eta)|_\alpha^4,$$

the suprema being taken over all vectors $\mathbf{W} = (\mathbf{U}, q)$ with $|\mathbf{U}|_{2+\alpha}^4 + |q|_{1+\alpha}^1 \leq s$. By combining this with (3.11) one may easily construct an appropriate $h_1(s)$. This completes the proof of Lemma 4.

It now remains to be shown that for small enough ϵ , a , and τ , (3.8) and $|\eta| \leq 1$ will be satisfied at each step. This will establish that the complete iteration process may be carried out. Then, of course, we must show that the process converges to a solution. These things will be done in the next section.

4. Convergence of the Process. First we establish a uniform bound on the norms P^n of the iterates.

Lemma 5. *Suppose, for some integer $N_0 \leq \infty$, (3.8) and $|\eta^{n-1}| \leq 1$ are satisfied for all $n = 1, 2, \dots, N_0$. Then there exist positive numbers ϵ_0 , M_1 , and M_2 such*

that

$$(4.1) \quad P^n \leq M_1,$$

$$(4.2) \quad |\eta^n| \leq M_2(\epsilon + a + \tau),$$

for all $n \leq N_0$, provided $\epsilon + a + \tau \leq \epsilon_0$.

Proof. There is no loss of generality in assuming the function $h_1(s)$ in (3.10) to be continuously differentiable for $s > 0$. Then the implicit function theorem implies the existence, for some ϵ_0 , of a continuous solution $P(\epsilon) \geq C$ of

$$P = C + \epsilon h_1(P), \quad P(0) = C,$$

in the range $0 \leq \epsilon \leq \epsilon_0$ for some $\epsilon_0 > 0$. Setting $M_1 = P(\epsilon_0)$, so that

$$M_1 = C + \epsilon_0 h_1(M_1),$$

we verify (4.1) by induction on n . For $n = 0$, we have

$$P^0 = |\Phi|_{2+\alpha}^4 + |\omega|_{1+\alpha}^4 \leq C$$

by (3.3). Since $M_1 \geq C$, (4.1) follows for this case.

Assume (4.1) to be true for $n \leq N$. Then by (3.10), and the nondecreasing nature of h_1 ,

$$P^{N+1} \leq C + (\epsilon + a + \tau)h_1(P^n) \leq C + \epsilon_0 h_1(M_1) = M_1$$

for $\epsilon + a + \tau \leq \epsilon_0$. This completes the induction argument and the proof of (4.1). Finally (4.2) follows from (3.11) and (4.1) after setting $M_2 = h_2(M_1)$.

Now we estimate the increments $\delta \mathbf{U}^n \equiv \mathbf{U}^n - \mathbf{U}^{n-1}$, $\delta q^n = q^n - q^{n-1}$, and $\delta \eta^n = \eta^n - \eta^{n-1}$ associated with the iterations. We define

$$(4.3) \quad \delta^n = |\delta \mathbf{U}^n|_{2+\alpha}^4 + |\delta q^n|_{1+\alpha}^4.$$

Lemma 6. Suppose for some integer $N_0 \leq \infty$ that (3.8) and $|\eta^{N_0-1}| \leq 1$ are satisfied, for all integers $n \leq N_0$. Then there is an ϵ_1 and a constant C_7 such that

$$(4.4) \quad \delta^n \leq C_7 \epsilon_1 2^{-n},$$

for $\epsilon + a + \tau \leq \epsilon_1$, $n \leq N_0$.

Proof. Suppose $n \geq 2$. Write (3.5) for indices n and $n-1$ and subtract, obtaining

$$(4.5a) \quad \Delta \delta \mathbf{U}^n + \mathbf{A}_0 \delta \mathbf{U}^n - \partial \delta q^n = \delta \eta^n \mathbf{B} \mathbf{U}^{n-1} + \eta^{n-1} \mathbf{B} \delta \mathbf{U}^{n-1} + a \mathbf{R}_1 + \epsilon \mathbf{R}_2 + \tau \mathbf{R}_3,$$

$$(4.5b) \quad \partial \cdot (\delta \mathbf{U}^n) = 0,$$

$$(4.5c) \quad (\delta \mathbf{U}^n, \Phi) = 0,$$

where $\mathbf{R}_i = \mathbf{S}_i(\mathbf{W}^{n-1}, \eta^{n-1}) - \mathbf{S}_i(\mathbf{W}^{n-2}, \eta^{n-2})$. By the Mean Value Theorem, the $\mathbf{R}_i$ are expressible as linear combinations of $\delta \eta^{n-1}$, $\delta \mathbf{U}^{n-1}$, δq^{n-1} , and the derivatives of the latter two to orders two and one respectively. The coefficients

in these linear combinations are known functions of $\mathbf{W}^{n-1}$ and its derivatives. By Lemma 5 we know that these coefficients therefore must be bounded in the Hölder norm $|\cdot|_\alpha$ independently of n if $\epsilon_1 \leq \epsilon_0$. Thus we obtain

$$(4.6) \quad |\mathbf{R}_i|_\alpha^4 \leq M_3(|\delta \mathbf{U}^{n-1}|_{2+\alpha}^4 + |\delta W^{n-1}|_{1+\alpha}^4 + \delta \eta^{n-1}) \equiv M_3(\delta^{n-1} + \delta \eta^{n-1}).$$

Denoting by $\delta \mathbf{f}^n$ the right hand side of (4.5a), we obtain the following from (4.6), (4.1), (4.2):

$$(4.7) \quad |\delta \mathbf{f}^n|_\alpha^4 \leq |\delta \eta^n| M_1 + (\epsilon + a + \tau)(M_2 + M_3)\delta^{n-1}.$$

Next, write (3.12) for indices n and $n-1$ and take the difference:

$$\delta \eta^n (\mathbf{B} \mathbf{U}^{n-1}, \Phi) + \eta^{n-1} (\mathbf{B} \delta \mathbf{U}^{n-1}, \Phi) = -(a \mathbf{R}_1 + \epsilon \mathbf{R}_2 + \tau \mathbf{R}_3, \Phi).$$

Again divide by $(\mathbf{B} \mathbf{U}^{n-1}, \Phi)$ and use (4.6), (4.2), and (3.9). Thus for some M_4 ,

$$(4.8) \quad |\delta \eta^n| \leq (\epsilon + a + \tau) M_4 \delta^{n-1}.$$

Combining this with (4.7), we obtain

$$(4.9) \quad |\delta \mathbf{f}^n|_\alpha^4 \leq M_5(\epsilon + a + \tau)\delta^{n-1}.$$

We are now ready to apply Lemma 3 to the system (4.5). That lemma and (4.9) yield

$$\delta^n \equiv |\delta \mathbf{U}^n|_{2+\alpha}^4 + |\delta q^n|_{1+\alpha}^4 \leq C |\delta \mathbf{f}^n|_\alpha^4 \leq C M_5(\epsilon + a + \tau)\delta^{n-1}$$

for $1 \leq n \leq N_0$. We now need only choose $\epsilon_1 = \min \{(2CM_5)^{-1}, \epsilon_0\}$ to obtain $\delta^n \leq \frac{1}{2}\delta^{n-1}$, hence

$$(4.10) \quad \delta^n \leq 2^{1-n}\delta^1, \quad (n \geq 2),$$

for $\epsilon + a + \tau \leq \epsilon_1$.

We must now estimate δ^1 . We know that $\delta \mathbf{U}^1$ and δq^1 satisfy (4.5) with the right hand side of (4.5a) replaced by

$$\mathbf{f}^0 \equiv \eta^1 \mathbf{B} \Phi + a \mathbf{S}_1(\mathbf{W}^0, 0) + \epsilon \mathbf{S}_2(\mathbf{W}^0, 0) + \tau \mathbf{S}_3(\mathbf{W}^0, 0).$$

Using (4.2) with $n=1$, we obtain

$$|\mathbf{f}^0|_\alpha^4 \leq M_6(\epsilon + a + \tau),$$

and from Lemma 3,

$$\delta^1 \leq C M_6(\epsilon + a + \tau) \leq C M_6 \epsilon_1,$$

and (4.10) becomes

$$(4.11) \quad \delta^n \leq C M_6 \epsilon_1 2^{1-n} = C_7 \epsilon_1 2^{-n},$$

$1 \leq n \leq N_0$, completing the proof of Lemma 6.

The following is the main result of the paper.

Theorem. *There exists an $\epsilon_2 > 0$ such that for $\epsilon + a + \tau \leq \epsilon_2$, the iterative solutions of (3.5), (3.6) exist for all n , and converge to a solution of (2.18) uniformly*

as $n \rightarrow \infty$. In fact the iterates U^n and q^n converge in the spaces $\mathcal{C}_{2+\alpha}^4$ and $\mathcal{C}_{1+\alpha}^1$ respectively.

Proof. We must show that conditions (3.8) and $|\eta^{n-1}| \leq 1$ remain valid at each stage of the iteration; i.e., that $N_0 = \infty$, in the terminology of Lemmas 5 and 6. Clearly (3.8) is trivially satisfied for $n = 1$, so $N_0 \geq 1$. Suppose $N_0 < \infty$. Then set $n = N_0 + 1$:

$$\begin{aligned} |U_3^{n-1} - \Phi_3|_0 &= |U_3^{N_0} - \Phi_3|_0 \leq |\mathbf{U}^{N_0} - \Phi|_{2+\alpha}^4 \\ &\leq \sum_{i=1}^{N_0} |\delta \mathbf{U}^i|_{2+\alpha}^4 \leq \sum_{i=1}^{N_0} \delta^i \leq C_7 \epsilon_1 \sum_{i=1}^{\infty} 2^{-i} = C_7 \epsilon_1. \end{aligned}$$

Hence

$$\left| \int (U_3^{n-1} - \Phi_3) \Phi_4 \right| \leq C_8 |U_3^{n-1} - \Phi_3|_0 \leq C_8 C_7 \epsilon_1 \leq b/2,$$

for ϵ_1 small enough, say $\epsilon_1 \leq \epsilon_2$. This shows that (3.8) holds for $n = N_0 + 1$ as well. Also (4.2) shows that for ϵ_2 small enough, $|\eta^{n-1}| \leq 1$ for $n = N_0 + 1$ as well. This contradicts the finiteness of N_0 . Thus $N_0 = \infty$ when $\epsilon + a + \tau \leq \epsilon_2$; and all the iterates may be constructed.

As for the convergence of the process, this follows again from (4.11), which implies the sequence $\mathbf{W}^n$ is a Cauchy sequence. In fact, for $m > n$,

$$|\mathbf{U}^n - \mathbf{U}^m|_{2+\alpha}^4 + |q^n - q^m|_{1+\alpha}^4 \leq \sum_{i=n}^m \delta^i \leq C_7 \epsilon_1 \sum_{i=n}^m 2^{-i} \leq C_7 \epsilon_1 2^{-n},$$

which approaches 0 as n and $m \rightarrow \infty$. Therefore the $\mathbf{U}^n$ and q^n converge uniformly, together with their derivatives of orders two and one respectively. It is seen from (4.8) that the η^n also converge to some limit η . Finally passing to the limit in (3.5), we see that the limit is a solution of the problem (2.18).

5. The Boussinesq Approximation. A more detailed form of the equations (2.13) is given in (7.6) and in the equations preceding that in Section 7. If one assumes that $N = 0$ (so that the stress tensor is linear in D), then it is clear that in (2.13),

$$S_{1,i} = \frac{1}{\pi} u_i \partial_i u_i,$$

and

$$S_4 = Ru_i \partial_i \Theta.$$

Therefore if one formally sets $\epsilon = \tau = 0$ in (2.13), the standard Boussinesq equations result.

All the preceding analysis (iterative procedure and convergence proof) carries through as easily in the case $\epsilon = \tau = 0$ (in fact this was done by another method in [5]). In this way, a convective solution of the Boussinesq equations is found

for each (small enough) value of the parameter a . It may be shown for small ϵ and τ , this solution is an approximation to the solution we have here constructed (for the case $\mathbf{N} = 0$).

In fact, for given a, ϵ, τ with $a + \epsilon + \tau \leq \epsilon_2$, let $\mathbf{W} = (\mathbf{U}, q)$, η be the solution of (2.18) here constructed and let $\mathbf{W}^* = (\mathbf{U}^*, q^*)$, η^* be the solution of the Bousinesq equations, *i.e.* (2.18) with $\mathbf{S}_2 = \mathbf{S}_3 = 0$, for the same value of a . Then the difference $\mathbf{W}' = (\mathbf{U}', q') = (\mathbf{U} - \mathbf{U}^*, q - q^*)$, $\eta' = \eta - \eta^*$ satisfies

$$(5.1) \quad \Delta \mathbf{U}' + \mathbf{A}_0 \mathbf{U}' - \partial q' = \eta' \mathbf{B} \mathbf{U} + \eta^* \mathbf{B} \mathbf{U}' + a(\mathbf{S}_1(\mathbf{W}) - \mathbf{S}_1(\mathbf{W}')) + \epsilon \mathbf{S}_2(\mathbf{W}) + \tau \mathbf{S}_3(\mathbf{W})$$

$$(5.2) \quad \partial \cdot \mathbf{U}' = 0$$

$$(5.3) \quad (\mathbf{U}', \Phi) = 0,$$

and the boundary and periodicity conditions (2.18c-e).

Through the Mean Value Theorem and the *a priori* bounds given by Lemma 5, we may estimate

$$(5.4) \quad |\mathbf{S}_1(\mathbf{W}) - \mathbf{S}_1(\mathbf{W}')| \leq M_s P',$$

where we use the notation

$$P' = |\mathbf{U}'|_{2+\alpha}^4 + |q'|_{1+\alpha}^1.$$

An estimate for η' may be made by noting that the right side of (5.1) is orthogonal to Φ . We apply that orthogonality condition; the resulting expression for η' yields the following estimate based on (5.4), (3.9) with $\mathbf{U}^{n-1}$ replaced by $\mathbf{U}$, and (4.2) with η^n replaced by η^* :

$$|\eta'| \leq M_9(aP' + \epsilon + \tau).$$

With this result, the last statement in Lemma 2 now yields

$$P' \leq M_{10}(aP' + \epsilon + \tau).$$

If it is supposed that a is so small that $M_{10}a \leq \frac{1}{2}$, then we obtain

$$(5.5) \quad P' \leq 2M_{10}(\epsilon + \tau).$$

Thus we have shown that $\mathbf{U} - \mathbf{U}^*$ is small when ϵ and τ are small. This is not the final result, however, since $\mathbf{U}$ and $\mathbf{U}^*$ have different interpretations. In Bousinesq approximation, $\mathbf{U}^*$ is interpreted as the normalized velocity and temperature:

$$U_i^* = \frac{v_i^*}{av}, \quad i = 1, 2, 3,$$

$$U_4^* = \frac{T'^*}{a\Theta}.$$

However we have defined

$$U_i = \frac{\rho v_i}{\rho_0 a V},$$

$$U_4 = \frac{T'}{a\Theta}.$$

Thus from (7.1)

$$\begin{aligned} \frac{v_i}{aV} - \frac{v_i^*}{aV} &= \frac{\rho_0}{\rho} U_i - U_i^* = (U_i - U_i^*) + \frac{\rho_0 - \rho}{\rho} U_i \\ &= (U_i - U_i^*) - \frac{\tau}{R} (\theta_0 + a\theta)(1 + \epsilon g_1) U_i. \end{aligned}$$

The final term is bounded in absolute value by τM_{11} . Combining this with (5.5), we find

$$\left| \frac{v}{aV} - \frac{v^*}{aV} \right| \leq M_{12}(\epsilon + \tau).$$

The same type of result holds for the normalized temperature and pressure.

This result may be stated as follows: *for small enough a , and for $\mathbf{N} = 0$, the difference between the Boussinesq solution and the exact solution is uniformly less than $M_{11}(\epsilon + \tau)$, where M_{11} depends only on the upper bound M for the parameters in (2.11), the form of the functions $f_i(T, p)$, and l_1/d , l_2/d .*

6. Proof of Lemma 3. The proof will be based on the following three lemmas.

Lemma 7. *Let $\mathbf{f} \in \mathcal{C}_\infty^3$ (that is to say, f is an infinitely differentiable periodic 3-vector). Then there exists exactly one solution $\mathbf{U} = (U, V, W)$, q of*

$$(6.1a) \quad \Delta \mathbf{U} - \partial q = \mathbf{f},$$

$$(6.1b) \quad \partial \cdot \mathbf{U} = 0,$$

with conditions (2.18c-e). This solution is infinitely differentiable.

Proof. The uniqueness is proved by showing that $\mathbf{U} = 0$ when $\mathbf{f} = 0$. In fact in that case,

$$0 = \int_{\mathfrak{D}} \mathbf{U} \cdot \Delta \mathbf{U} - \int_{\mathfrak{D}} \mathbf{U} \cdot \partial q = \int_{\mathfrak{D}} \mathbf{U} \cdot \Delta \mathbf{U} = - \int_{\mathfrak{D}} |\text{grad } \mathbf{U}|^2.$$

Thus $\text{grad } \mathbf{U} \equiv 0$, hence from the boundary condition $\mathbf{U} \equiv 0$, and from (6.1) (2.18e), $q \equiv 0$.

The existence is easily proved by Fourier series construction. Thus we set (where $\mathbf{k} = (k_1 d/l_1, k_2 d/l_2)$, k_i integers, $\mathbf{x} = (z_1, z_2)$, $k = |\mathbf{k}|$), $\mathbf{f}(z_1, z_2, z_3) = \sum \mathbf{f}_{\mathbf{k}}(z_3) \exp(i\mathbf{k} \cdot \mathbf{x})$ and seek a solution of the form

$$\mathbf{U} = \sum \mathbf{U}_{\mathbf{k}}(z_3) \exp(i\mathbf{k} \cdot \mathbf{x}),$$

$$q = \sum Q_{\mathbf{k}}(z_3) \exp(i\mathbf{k} \cdot \mathbf{x}).$$

We denote the components of $\mathbf{U}_k$ by (U_k, V_k, W_k) . Substituting these expansions into the equations (6.1), and rearranging the results somewhat, we obtain (suppressing the subscript $\mathbf{k}$):

$$\begin{aligned}
 (6.2) \quad & W^{IV} - 2k^2 W'' + k^4 W = -i(k_1 f'_1 + k_2 f'_1) - k^2 f_3, \\
 & W = W' = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad 1; \\
 & Q' = -k^2 W + W'' - f_3, \quad Q'(0) = 0, \\
 & U'' - k^2 U = f_1 + ik_1 Q, \quad U(0) = U(1) = 0, \\
 & V'' - k^2 V = f_2 + ik_2 Q, \quad V(0) = V(1) = 0.
 \end{aligned}$$

These equations may be solved for all the coefficients $\mathbf{U}_k(z_3)$, $\mathbf{Q}_k(z_3)$. Furthermore the solution may be estimated a priori in terms of $\mathbf{k}$, $\mathbf{f}_k$, and the latter's derivatives. Namely, we multiply (6.2) by W and integrate to obtain (still suppressing subscripts $\mathbf{k}$)

$$\int_0^1 (W'')^2 dz_3 + 2k^2 \int_0^1 (W')^2 dz_3 + k^4 \int_0^1 W^2 dz_3 = \int_0^1 W(z) F(z) dz.$$

where $F(z) = i(k_1 f'_1 + k_2 f'_2) - k^2 f_3$. Hence (using Schwartz' Inequality)

$$\begin{aligned}
 \left(\int_0^1 W^2 dz \right)^{1/2} &\leq k^{-4} \left(\int_0^1 F(z)^2 dz \right)^{1/2}, \quad (k \neq 0), \\
 \left(\int_0^1 (W')^2 dz \right)^{1/2} &\leq k^{-3} \left(\int_0^1 F^2 dz \right)^{1/2},
 \end{aligned}$$

and

$$\left(\int_0^1 (W'')^2 dz \right)^{1/2} \leq k^{-2} \left(\int_0^1 F^2 dz \right)^{1/2}.$$

Integral estimates for the higher order derivatives may similarly be obtained by differentiating (6.2). The fact that $\mathbf{f} \in \mathcal{C}_\infty$ implies that $F(z_3) = O(k^{-N})$ as $k \rightarrow \infty$ for any N . The same holds for its z -derivatives. These estimates imply the same of $W_k(z_3)$, from which we therefore conclude that $W(z)$ is in $\mathcal{C}_\infty$. Similar arguments apply to U , V , and Q .

Lemma 8. *Let $\mathbf{f} \in \mathcal{C}_\alpha^3$. Then there exists exactly one solution of (6.1), (2.18c-e) with $\mathbf{U} \in \mathcal{C}_{2+\alpha}^3$, $\mathbf{q} \in \mathcal{C}_{1+\alpha}^1$. Furthermore there exists a C_1 depending only on l_1/d and l_2/d such that*

$$(6.3) \quad |\mathbf{U}|_{2+\alpha}^3 + |\mathbf{q}|_{1+\alpha}^1 \leq C_1 |\mathbf{f}|_\alpha^3.$$

Proof. First we show that (6.3) holds when $\mathbf{f} \in \mathcal{C}_\alpha^3$ for the solutions constructed in Lemma 7. It is an immediate consequence of the Schauder estimates for elliptic systems [1] once we show that our system (6.1) together with boundary conditions (2.18c) falls into the category of the problems treated there. For

this we must check (1) that the system (6.1) is elliptic, and (2) that the boundary condition $\mathbf{U} = 0$ satisfies the "complementing condition" with respect to the given system.

(1) We write the system (6.1) in the form

$$\sum_{j=1}^4 l_{ij}(\partial) u_j(z) = F_i(z), \quad i = 1, \dots, 4,$$

where $u_i = U_i$ for $i = 1, 2, 3$, $u_4 = q$, and the l_{ij} are differential operators satisfying

$$\det l_{ij} \leq s_i + t_i,$$

where $s_1 = s_2 = s_3 = 0$, $s_4 = -1$, $t_1 = t_2 = t_3 = 2$, $t_4 = 1$. Ellipticity is guaranteed if $L(\xi) \equiv \det l_{ij}(\xi) \neq 0$ for real $\xi \neq 0$. In our case it turns out that $L(\xi) = |\xi|^6$, so the system is elliptic.

(2) Our boundary condition takes the form $u_1 = u_2 = u_3 = 0$ at the boundary. In the terminology of [1] this may be expressed

$$\sum_{j=1}^4 B_{hj}(\partial) u_j(z) = 0, \quad h = 1, 2, 3,$$

where $B_{hj} = 1$ for $h = j \leq 3$, and $B_{hj} = 0$ otherwise. Still in the terminology of [1] we denote by $\Xi = (\xi_1, \xi_2)$ any vector in the (z_1, z_2) -plane, and by $\mathbf{n}$ the unit vector parallel to the z_3 axis. Then

$$M^+(\Xi, \tau) = (\tau - i\rho)^3,$$

where $\rho^2 = \xi_1^2 + \xi_2^2$. Also, letting $L^{ik}(\Xi + \tau\mathbf{n})$ be the matrix adjoint to $L_{ij}(\Xi + \tau\mathbf{n})$, and considering the elements of the matrix

$$\sum_{j=1}^4 B_{hj}(\Xi + \tau\mathbf{n}) L^{jk}(\Xi + \tau\mathbf{n})$$

as polynomials in τ , we find that this matrix is proportional (mod M^+) to

$$\begin{bmatrix} \xi_2^3 + \tau^2 & -\xi_1\xi_2 & -\xi_1\tau & -\xi_1(\rho^2 + \tau^2) \\ -\xi_1\xi_2 & \xi_1^2 + \tau^2 & -\xi_2\tau & -\xi_2(\rho^2 + \tau^2) \\ -\xi_1\tau & -\xi_2\tau & \rho^2 & 3i\rho\tau^2 + 2\rho\tau - i\rho^3 \end{bmatrix}.$$

It may be determined that for any ξ_1, ξ_2 not both zero, this matrix (elements considered as polynomials in τ) has rank 3. For example when $\tau = 0$, the determinant of the 2nd, 3rd, and 4th columns is $-\rho^6\xi_1$, whereas that of the 1st, 3rd, and 4th columns is $-\xi_2\rho^6$. They cannot both be zero unless $\xi_1 = \xi_2 = 0$.

This proves (6.3) for the case $\mathbf{f} \in \mathcal{C}_\infty$. However if $\mathbf{f} \in \mathcal{C}_\alpha^3$, we need only construct a sequence $\mathbf{f}_{(n)}$ of $\mathcal{C}_\infty^3$ functions such that $\mathbf{f}_{(n)} \rightarrow \mathbf{f}$ uniformly and $|\mathbf{f}_{(n)}|_\alpha^3$ are all bounded uniformly in n . This is possible by mollifiers. Let $\mathbf{U}_{(n)}, q_{(n)}$ be the solutions constructed as in Lemma 7. Then (6.3) holds with subscripts (n) attached. Since the right hand side is bounded independently of n , $\mathbf{U}_{(n)}$ and their 2nd derivatives and $q_{(n)}$ and their first derivatives are equicontinuous. By Arzela's

Theorem a subsequence of $(\mathbf{U}_{(n)}, q_{(n)})$ exists approaching a limit whose appropriate derivatives satisfy the same Hölder condition. This limit will clearly be a solution of (6.1). Finally, this solution will by [1] again be subject to Schauder estimates (6.3). This proves Lemma 8.

Lemma 9. *Let $\mathbf{f} \in \mathcal{C}_\alpha^4$. There exists exactly one solution of*

$$(6.4) \quad \Delta \mathbf{U} - \partial q = \mathbf{f},$$

with conditions (2.18c-e). Furthermore $\mathbf{U} \in \mathcal{C}_{2+\alpha}^4$, $q \in \mathcal{C}_{1+\alpha}^1$, and

$$(6.5) \quad |\mathbf{U}|_{2+\alpha}^4 + |q|_{1+\alpha}^1 \leq C_1 |\mathbf{f}|_\alpha^4.$$

Proof. The system (6.4) consists of (6.1) plus an extra equation, the Poisson equation, for the fourth component U_4 . This fourth equation is not coupled to the first 3, so a solution of it satisfying $U_4 = 0$ on the boundary, and the periodicity conditions, may be found separately by the same method as in Lemmas 7 and 8. Combining this solution with that in Lemma 8 yields Lemma 9.

At this point we reformulate the basic problem in terms of the solution operator $\mathbf{S}$ of the problem (6.4), (2.18c-e). Namely, let $\mathbf{S}$ be the linear operator taking $\mathcal{C}_\alpha^4$ into $\mathcal{C}_\alpha^4$ defined by $\mathbf{U} = \mathbf{S}\mathbf{f}$, where $\mathbf{U}$ is the solution whose existence and uniqueness is guaranteed in Lemma 9. Then shifting one term in (3.2) to the right, we have

$$(6.6) \quad \mathbf{U} = \mathbf{S}(-\mathbf{A}_0\mathbf{U} + \mathbf{f}).$$

The operator $\mathbf{S}$ is compact, because it is well known that a set of functions (over a compact set) bounded in the $\mathcal{C}_{2+\alpha}^4$ norm is compact in the $\mathcal{C}_\alpha^4$ norm. Since $\mathbf{A}_0$ is bounded, $\mathbf{S}\mathbf{A}_0$ is also compact. Let $\mathcal{C}^*$ denote the dual space to $\mathcal{C}_\alpha^4$, $\mathbf{S}^*$ and $\mathbf{A}_0^*$ the adjoint operators to $\mathbf{S}$ and $\mathbf{A}_0$, and let $\langle \mathbf{U}, \psi_0 \rangle$ denote the value assigned by a functional $\psi_0 \in \mathcal{C}^*$ to an element $\mathbf{U} \in \mathcal{C}_\alpha^4$. The Fredholm Alternative Theorem states that (6.6) is solvable for every $\mathbf{f}$ satisfying

$$(6.7) \quad \langle \mathbf{S}\mathbf{f}, \psi \rangle = 0$$

for every ψ satisfying

$$(6.8) \quad \psi = -(\mathbf{S}\mathbf{A}_0)^*\psi = -\mathbf{A}_0^*\mathbf{S}^*\psi.$$

Note (2.15) that Φ satisfies

$$(6.9) \quad \Phi = -\mathbf{S}\mathbf{A}_0\Phi.$$

By assumption there exists only one such linearly independent Φ , hence only one such linearly independent ψ satisfying (6.8).

Lemma 10. *For any $\mathbf{f} \in \mathcal{C}_\alpha^4$,*

$$\langle \mathbf{S}\mathbf{f}, \psi \rangle = \mu(\mathbf{f}, \Phi),$$

for some constant μ .

Proof. For any $\mathbf{V} \in \mathcal{C}_\alpha^4$, let $\psi_0 = \mathbf{QV}$ denote the linear functional defined by

$$(\mathbf{U}, \mathbf{V}) = \langle \mathbf{U}, \psi_0 \rangle$$

for all $\mathbf{U} \in \mathcal{C}_\alpha^4$. Thus $\mathbf{Q}$ is a mapping from $\mathcal{C}_\alpha^4$ to $\mathcal{C}^*$.

First we show that

$$(6.10) \quad \mathbf{QSA}_0 = \mathbf{S}^* \mathbf{A}_0^* \mathbf{Q}.$$

It may be verified from the definition (2.16) and that of $\mathbf{A}_0$ that $(\phi_1, \mathbf{A}_0 \phi_2) = (\mathbf{A}_0 \phi_1, \phi_2)$ for any $\phi_1, \phi_2 \in \mathcal{C}_\alpha^4$. Furthermore if they satisfy the conditions (2.18c, d), then Green's formula shows $(\phi_1, \Delta \phi_2) = (\phi_2, \Delta \phi_1)$. Finally if $\partial \cdot \phi = 0$, then $(\phi, \partial \omega) = 0$ for any scalar function ω .

Let $\mathbf{f}$ and ϕ be any two elements in $\mathcal{C}_\alpha^4$. Bearing the above facts in mind, we find

$$\begin{aligned} \langle \mathbf{f}, \mathbf{QSA}_0 \phi \rangle &= (\mathbf{f}, \mathbf{SA}_0 \phi) = (\Delta \mathbf{Sf} - \partial \omega, \mathbf{SA}_0 \phi) = (\Delta \mathbf{Sf}, \mathbf{SA}_0 \phi) = (\mathbf{Sf}, \Delta \mathbf{SA}_0 \phi) \\ &= (\mathbf{Sf}, \mathbf{A}_0 \phi + \partial \omega_1) = (\mathbf{Sf}, \mathbf{A}_0 \phi) = (\mathbf{A}_0 \mathbf{Sf}, \phi) \\ &= \langle \mathbf{A}_0 \mathbf{Sf}, \mathbf{Q} \phi \rangle = \langle \mathbf{f}, \mathbf{S}^* \mathbf{A}_0^* \mathbf{Q} \phi \rangle. \end{aligned}$$

This proves (6.10), since $\mathbf{f}$ and ϕ were arbitrary.

Now from (6.9) and (6.10) we have

$$(\mathbf{Q} \Phi) = -\mathbf{QSA}_0 \Phi = -\mathbf{S}^* \mathbf{A}_0^* (\mathbf{Q} \Phi),$$

and from (6.8),

$$(\mathbf{S}^* \psi) = -\mathbf{S}^* \mathbf{A}_0^* (\mathbf{S}^* \psi).$$

Comparing these two results, we have

$$\mathbf{S}^* \psi = \mu \mathbf{Q} \Phi$$

for some scalar μ . Thus

$$\langle \mathbf{Sf}, \psi \rangle = \langle \mathbf{f}, \mathbf{S}^* \psi \rangle = \mu \langle \mathbf{f}, \mathbf{Q} \Phi \rangle = \mu (\mathbf{f}, \Phi),$$

which establishes Lemma 10.

In view of the Fredholm condition (6.7), we can now state that (6.6), hence (3.2), has a solution if and only if f satisfies (3.1). And in case of existence, the solution is unique only up to an arbitrary multiple of Φ . This multiple can clearly be chosen uniquely to satisfy $(\mathbf{U}, \Phi) = k$ for any real number k . (In Lemma 1, the two cases $k = 1$ and $k = 0$ are singled out). Let $\mathfrak{J}$ be the subspace of elements $(\mathbf{f}, k)$ in $\mathcal{C}_\alpha^4 \times R_1$ (where R_1 is the real line) with the property that $(\mathbf{f}, \Phi) = 0$. Then we proved the existence of mapping $\mathfrak{W}$ from all of $\mathfrak{J}$ into $\mathcal{C}_{2+\alpha}^4$, such that $\mathfrak{W}(\mathbf{f}, k) = \mathbf{U}$, where $\mathbf{U}$ satisfies (3.2) and $(\mathbf{U}, \Phi) = k$. It is clear from (3.2) and this condition that the mapping is closed. By the Closed Graph Theorem it must also be bounded:

$$|\mathbf{U}|_{2+\alpha}^4 \leq C (|k| + |\mathbf{f}|_\alpha^4).$$

Similar arguments show that $|q|_{1+\alpha}^1$ is also so bounded. The estimate (3.3) is the particular case of this when $k = 1$. The last statement of Lemma 3 is the particular case $k = 0$. This completes the proof.

7. Proof of Lemma 2. The following uniform notation will be used for perturbation terms. The symbols r_i will denote twice continuously differentiable functions of $\mathbf{u}$, θ , q , the first and second order derivatives of $\mathbf{u}$ and θ , the first derivatives of q , and the parameters listed in (2.11) (and α as well). If an r_i is written in bold face it is a vector or tensor function of the type described.

Applying the transformation (2.12) to (2.6), we define new functions g_i as $g_i(\theta_0 + a\theta, q_0 + aq) \equiv f_i(T, p) \equiv f_i(\Theta(\theta_0 + a\theta), P(q_0 + aq))$, $i = 1, \dots, 5$.

We now proceed to rewrite the various terms in (2.10), on the basis of (2.6) and (2.12). We begin with new expressions for ρ and E :

$$(7.1) \quad \rho = \rho_0(1 - \gamma T(1 + \epsilon f_1)) = \rho_0 \left(1 - \frac{\tau}{R} (\theta_0 + a\theta)(1 + \epsilon g_1) \right),$$

$$(7.2) \quad E = cT(1 + \epsilon f_2) = \frac{c\tau}{\alpha R} (\theta_0 + a\theta)(1 + \epsilon g_2).$$

For convenience we shall use the symbol

$$(7.3) \quad \sigma \equiv \frac{\rho_0}{\rho} = \frac{1}{1 - \frac{\tau}{R} (\theta_0 + a\theta)(1 + \epsilon g_1)} \equiv 1 + \tau r_1.$$

From (2.10c) we have

$$0 = \operatorname{div}(\rho \mathbf{v}) = (\operatorname{grad} \rho) \cdot \mathbf{v} + \rho \operatorname{div} \mathbf{v};$$

hence

$$\begin{aligned} (7.4) \quad \operatorname{div} \mathbf{v} &= \frac{v_i}{\rho d} \frac{\partial_i \rho}{\partial} = \frac{-\rho_0 \tau}{d R \rho} v_i \partial_i [(\theta_0 + a\theta)(1 + \epsilon g_1)] \\ &= \frac{-a \tau \kappa}{d^2 R} \sigma^2 u_i \partial_i [(\theta_0 + a\theta)(1 + \epsilon g_1)] = \frac{a \tau \kappa}{d^2} r_2; \\ \rho(\mathbf{v} \cdot \operatorname{grad}) v_i &= \frac{1}{d} \rho v_i \partial_i v_i = \frac{1}{d} \left(\frac{a \kappa \rho_0}{d} \right) u_i \partial_i \left(\frac{a \kappa \rho_0}{\rho d} u_i \right) \\ &= \frac{a^2 \kappa^2 \rho_0}{d^3} [\sigma u_i \partial_i u_i + u_i u_i \partial_i \sigma] \\ &= \frac{a^2 \tau \rho_0 g}{\pi R} [u_i \partial_i u_i + \tau r_{3i}]; \\ \rho' &= \rho - \rho(\theta_0) = -\rho_0 \frac{\tau}{R} [(\theta_0 + a\theta)(1 + \epsilon g_1(a\theta))] \end{aligned}$$

$$\begin{aligned}
- \theta_0(1 + \epsilon g_1(\theta_0)) &= \frac{-\rho_0 a \tau}{R} [\theta(1 + \epsilon g_1(\theta))] \\
+ \epsilon \theta_0(g_1(a\theta) - g_1(\theta_0)) &\equiv \frac{-\rho_0 a \tau}{R} [\theta + \epsilon r_4]; \\
(\text{grad } p')_i &= \frac{1}{d} \partial_i p' = \frac{a \rho_0 \kappa^2 \pi}{d^3} \partial_i q = \frac{a \tau \rho_0 g}{R} \partial_i q; \\
\text{div } (\lambda(\text{div } \mathbf{v}) \mathbf{I})_i &= \frac{1}{d} \partial_i (\lambda \text{div } \mathbf{v}) = \frac{a \tau \kappa \lambda_0}{d^3} \partial_i [(1 + \epsilon g_2) r_1] = \frac{a \tau \kappa \lambda_0}{d^3} r_{5i}; \\
D_{ii} &= \frac{1}{2d} [\partial_i v_i + \partial_i v_i] = \frac{a \kappa}{2d^2} [\sigma(\partial_i u_i + \partial_i u_i) \\
&\quad + u_i \partial_i \sigma + u_i \partial_i \sigma] = \frac{a \kappa}{2d^2} (r_6)_{ii}; \\
(\text{div } (2\mu \mathbf{D}))_i &= \frac{1}{d} \partial_i (2\mu D_{ii}) = \frac{\mu}{d^2} \partial_i (\partial_i v_i + \partial_i v_i) + \frac{2}{d} D_{ii} \partial_i \mu \\
&= \frac{\mu}{d} \partial_i (\text{div } \mathbf{v}) + \frac{\mu}{d^2} \partial^2 v_i + \frac{2}{d} (\partial_i \mu) \frac{a \kappa}{2d^2} r_{6ii} \\
&= \frac{\mu}{d^2} \partial^2 v_i + \frac{a \tau \kappa \mu_0}{d^3} r_{7i} + \frac{\epsilon a \kappa \mu_0}{d^3} r_{8i} \\
&= \frac{\mu}{d^2} \frac{a \kappa}{d} [\sigma \partial^2 u_i + 2 \partial_i \sigma \partial_i u_i + u_i \partial^2 \sigma] + \dots \\
&= \frac{a \tau \rho_0 g}{R} (\partial^2 u_i + \tau r_{9i} + \epsilon r_{10i}).
\end{aligned}$$

We now come to the tensor $\mathbf{N}(T, p, \mathbf{D}) = \mathbf{N}(\Theta(\theta_0 + a\theta), P(aq + q_0), (a\kappa/2d^2)\mathbf{r}_6)$. Since we require (2.7) $\mathbf{N} = \mathbf{0}$ when $\mathbf{D} = \mathbf{0}$, $\mathbf{N}$ is of at least first order in $\mathbf{D} = (a\kappa/2d^2)\mathbf{r}_6$. Thus we may factor this last "a" from the entire expansion leaving a smooth function of θ, q , derivatives of u , and the parameters in (2.11). In short, $\mathbf{N} = a\mathbf{r}_{11}$. The same is true of the derivatives of $\mathbf{N}$ with respect to components of z . From all this we determine

$$\text{div } [2\mu \mathbf{D} : \mathbf{N}] = \frac{a^2 \tau \rho_0 g}{R} \mathbf{r}_{12}.$$

Next we rewrite the terms in (2.10b):

$$\begin{aligned}
\rho \mathbf{v} \cdot \text{grad } E &= \frac{1}{d} \rho v_i \partial_i E = \frac{a c \kappa \rho_0 \beta}{R d} u_i \partial_i [(\theta_0 + a\theta)(1 + \epsilon g_2)] \\
&= \frac{a c \kappa \rho_0 \beta}{R d} \{u_3 \partial_3 \theta_0 + a u_i \partial_i \theta + \epsilon r_{13}\}.
\end{aligned}$$

But

$$\partial_3 \theta_0 = \frac{d}{\Theta} \frac{dT_0}{dx_3} = \frac{R}{\beta} \frac{dT_0}{dx_3}.$$

Now from (2.9) we may write

$$(7.5) \quad \frac{dT_0}{dx_3} = C(1 + \epsilon r_{14}).$$

Integrating from 0 to d and using $T_0(d) = T_1$, we find

$$C = -\beta(1 + \epsilon r_{15}).$$

Plugging back into (7.5), we see that

$$\frac{dT_0}{dx_3} = -\beta(1 + \epsilon r_{16}),$$

so in all,

$$\rho \mathbf{v} \cdot \text{grad } E = \frac{-ac\kappa\rho_0\beta}{d} \{u_3 + au_i \partial_i \theta + \epsilon r_{17}\};$$

$$\mathbf{I} : \mathbf{D} = \text{div } \mathbf{v};$$

$$\mathbf{T} : \mathbf{D} = (-p + \lambda \text{div } \mathbf{v}) \text{div } \mathbf{v} + 2\mu \mathbf{D} : (\mathbf{I} + \mathbf{N}) : \mathbf{D}$$

$$\begin{aligned} &= \left(-\frac{\rho_0 \kappa^2 \pi}{d^2} (aq + q_0) + \frac{\lambda a^2 \tau \kappa r_2}{d^2} \right) \frac{a^2 \tau \kappa r_2}{d^2} + \frac{\mu_0 a^2 \kappa^2}{d^4} r_{18} \\ &= \frac{\mu_0 a^2 \kappa^2}{d^4} \left[\frac{\tau \rho_0 \pi \kappa}{\mu_0} r_{19} + \left(\frac{\tau^2 \lambda_0}{\mu_0} \right) r_{20} + r_{21} \right] = \frac{\mu_0 a^2 \kappa^2}{d^4} r_{22}; \end{aligned}$$

$$\text{div} [(k - k^*) \text{grad } T] = \frac{\beta d}{d^2 R} \partial_i [(k - k^*) \partial_i (\theta_0 + a\theta)]$$

$$= \frac{\beta k_0 \epsilon}{Rd} \partial_i [(g_{50} - g_5) \partial_i (\theta_0 + a\theta)] = \frac{a\beta k_0 \epsilon}{Rd} r_{23} = \frac{\epsilon a\beta \kappa \rho_0 c}{Rd} r_{23};$$

$$\text{div} (k^* \text{grad } T') = \frac{a\beta d k_0}{d^2 R} \partial_i [(1 + \epsilon g_{50}) \partial_i \theta] = \frac{a\beta \kappa \rho_0 c}{Rd} \partial^2 \theta + \frac{\epsilon a\beta \kappa \rho_0 c}{dR} r_{24}.$$

If we employ these expressions for the terms in (2.10a) and multiply the equation thus obtained by $R/a\rho_0\tau g$, the result (after rearranging terms) is:

$$\begin{aligned} (7.6a) \quad \partial^2 u_i + \theta \delta_{i3} - \partial_i q &= \frac{a}{\pi} u_i \partial_i u_i + \frac{a\tau}{\pi} (r_3)_i - \epsilon (r_4)_i \\ &\quad - \frac{\tau \lambda_0}{\mu_0} (r_5)_i - \tau r_{9i} - \epsilon r_{10i} - a r_{12i} \\ &= \frac{a}{\pi} u_i \partial_i u_i + a (r_{25})_i + \epsilon (r_{26})_i - \tau (r_9)_i. \end{aligned}$$

Similarly we insert the above expressions into (2.10b) and multiply the result by $dR/a\beta\kappa\rho_0c$ to obtain

$$(7.6b) \quad \partial^2\theta + Ru_3 = aRu_3\partial_1\theta - R\epsilon r_{17} - a\epsilon r_{22} - \epsilon r_{23} - \epsilon r_{24} = ar_{27} + \epsilon r_{28}.$$

The functions r_i on the right have the properties desired of the s_i ; hence Lemma 2 is proved.

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