Existence of Solutions of the Equilibrium Equations for Nonlinearly Elastic Rings and Arches

STUART ANTMAN

Communicated by IAN SNEDDON

Introduction. In this paper we study the nonlinear sixth order system of ordinary differential equations for the equilibrium of extensible nonlinearly elastic rings and arches under hydrostatic pressure. The material of these rods is characterized by a strain energy function that depends upon a measure of stretching and a measure of bending. By the direct methods of the calculus of variations, we prove the existence and regularity of at least one solution of these equations for each value of the change in area "enclosed" by the ring or arch. In a like manner, we furnish a sufficient condition on the response of the material to tension that ensures the existence of at least one classical solution for each value of the pressure. We also show that the initially circular uniform ring has at least one noncircular solution for certain ranges of pressure.

This work represents a marked improvement over that of [2] because it treats in detail three questions not discussed there, namely, the satisfaction of the continuity condition, the regularity of solutions, and the existence of buckled states of circular rings. (The continuity condition, expressing the impossibility of compressing a segment of the rod of nonzero length into one of zero length, appears as a strict inequality for one of the dependent variables of the problem. The presence of a strict inequality among the conditions defining the admissibility class of functions for a variational problem makes it questionable whether the function that renders the variational functional an infimum actually belongs to the admissibility class. In an abstract sense, the functions that satisfy such a unilateral constraint do not form a subset of a linear space that is convenient for the purposes of variational methods.) In this paper we show that the introduction of a mild and physically reasonable hypothesis on the material response enables us to reformulate our problem so that both the continuity condition and the regularity theory can be handled in a natural way. Moreover we employ a somewhat more convenient function space than that of [2] and we refine and simplify some proofs that have analogs in [2].

There have appeared several studies [5], [6], [7], [9], [10], [12] of the global behavior of solutions of the equilibrium equations for certain one and two dimensional models for problems of nonlinear elasticity. References [5], [6], [9] deal with partial differential equations, the others with ordinary differential equations. Variational methods were employed in [5], [6], [7], [10]. None of these authors treated general classes of materials, general reference configurations, or the continuity condition, as we do here. On the other hand, they were frequently able to exploit special features of their problems to obtain sharp results of a type that we do not get. In particular, the authors of [5], [6], [12] were able to count the number of bifurcating solutions that subsist for large norms. Our work here has been most strongly influenced by [10]; we shall indicate the connection between their work and ours in the course of this exposition.

1. Formulation of the boundary value problems. We regard a ring or an arch as a plane curve that can deform by bending and stretching in its plane under the action of a force system. Let τ denote the arclength of this curve in a given reference configuration, $s(\tau)$, the arclength in an arbitrary configuration, $\Phi(\tau)$, the given continuously differentiable tangent angle to the rod in the reference configuration, and $\phi(\tau)$, the tangent angle to the rod in an arbitrary configuration. These angles are measured in a counter-clockwise sense with respect to the x -axis of a fixed Cartesian (x, y) -coordinate system. We require that the continuity condition

$$(1.1) \quad s'(\tau) > 0,$$

be satisfied. We set

$$(1.2) \quad \psi(\tau) = \phi(\tau) - \Phi(\tau), \quad \delta(\tau) = s'(\tau) - 1.$$

$\psi'(\tau)$ measures flexural strain and $\delta(\tau)$ measures extensional strain.

We assume that the ring or arch is subjected to a hydrostatic pressure q that is taken positive when q acts from left to right with respect to the tangent to the curve in the direction of increasing τ .

The coordinates $(x(\tau), y(\tau))$ of a material point satisfy the equations

$$(1.3) \quad x'(\tau) = [1 + \delta(\tau)] \cos \phi(\tau), \quad y'(\tau) = [1 + \delta(\tau)] \sin \phi(\tau).$$

The material behavior is characterized by the strain energy function $W(\psi', \delta, \tau)$ in terms of which the bending moment M and the axial stress resultant N are specified by

$$(1.4) \quad M(\tau) = W_{\psi'}(\psi'(\tau), \delta(\tau), \tau), \quad N(\tau) = W_{\delta}(\psi'(\tau), \delta(\tau), \tau).$$

We assume that $W(\psi', \delta, \tau)$ is thrice continuously differentiable in its arguments for $-\infty < \psi' < \infty$, $-1 < \delta < \infty$, $0 < \tau < L$ where L is the arc length of the rod in the reference configuration. We also require that the Hessian matrix

$$(1.5) \quad \begin{bmatrix} W_{\psi'\psi'} & W_{\psi'\delta} \\ W_{\delta\psi'} & W_{\delta\delta} \end{bmatrix},$$

be positive definite for all values of the arguments in the above ranges. This ensures that (1.4) has physically reasonable monotonicity and invertibility properties.

The well-known equilibrium equations for such a rod (Cf. [3], [10]) can be put into the form

$$(1.6) \quad \frac{d}{d\tau} \left[\frac{M'(\tau)}{1 + \delta(\tau)} \right] - \phi'(\tau)N(\tau) + q[1 + \delta(\tau)] = 0,$$

$$(1.7) \quad \phi'(\tau)M'(\tau) + [1 + \delta(\tau)]N'(\tau) = 0.$$

The system of differential equations (1.3), (1.4), (1.6), (1.7) is defined for

$$(1.8) \quad 0 < \tau < L.$$

We now turn to boundary conditions. Let a denote either the end 0 or the end L of the τ -interval. An arch that is clamped at a satisfies the boundary condition

$$(1.9) \quad \psi(a) = 0;$$

an arch that is hinged at a satisfies the boundary condition

$$(1.10) \quad M(a) = 0.$$

If a is hinged and if

$$(1.11) \quad W_{\psi'}(0, \delta, a) = 0,$$

then our hypothesis that (1.5) is positive definite implies that (1.10) is equivalent to

$$(1.12) \quad \psi'(a) = 0.$$

We restrict our consideration to arches with fixed ends. Thus

$$(1.13) \quad x(L) - x(0) \quad \text{and} \quad y(L) - y(0) \text{ are prescribed.}$$

We do not bother to specify $x(0)$ and $y(0)$ since these just determine the integration constants for (1.3) and merely fix the rigid body translation of the arch.

For the ring we assume that the functions $\Phi'(\tau)$ and $W(\psi', \delta, \tau)$ can be extended by periodicity to the whole τ -axis while retaining their smoothness. Moreover, for the reference configuration to satisfy the boundary conditions (1.16) below, $\Phi(\tau)$ must satisfy

$$(1.14) \quad \int_0^L \exp i\Phi(\tau) d\tau = 0.$$

The appropriate boundary conditions are the periodicity conditions

$$(1.15) \quad \psi(0) = \psi(L), \quad \psi'(0) = \psi'(L), \quad \delta(0) = \delta(L),$$

$$(1.16) \quad x(0) = x(L), \quad y(0) = y(L).$$

We fix the rigid body rotation of the ring by

$$(1.17) \quad \int_0^L \psi(\tau) d\tau = 0,$$

but again we do not bother to fix the rigid body translation.

Our boundary value problems for arches consist of the differential equations (1.3), (1.4), (1.6), (1.7) on $0 < \tau < L$, an appropriate set of boundary conditions, and the unilateral constraint

$$(1.18) \quad \delta(\tau) > -1,$$

obtained from (1.1). For the ring the boundary conditions are replaced by the periodicity and orientation conditions (1.14)–(1.17).

In Section 4 we shall show that the equations (1.4), (1.6), (1.7) are equivalent to the Euler equations (a set of integro-differential equations) for the potential energy functional

$$(1.19) \quad V[\psi, \delta] = V_1[\psi, \delta] - qV_2[\psi, \delta],$$

with

$$(1.20) \quad V_1[\psi, \delta] = \int_0^L W(\psi'(\tau), \delta(\tau), \tau) d\tau,$$

$$(1.21) \quad V_2[\psi, \delta] = \frac{1}{2} \int_0^L [1 + \delta(\tau)] \int_0^\tau [1 + \delta(\xi)] \sin [\phi(\tau) - \phi(\xi)] d\xi d\tau.$$

The functional V_2 is just the area expression,

$$\frac{1}{2} \int_0^L [x(\tau)y'(\tau) - y(\tau)x'(\tau)] d\tau.$$

Using the integral expression for x and y obtained from (1.3) we have been able to express V in terms of only ψ and δ , thereby eliminating the differential equations (1.3) as side conditions at the expense of introducing an integral into the integrand of (1.21). In like manner we write the boundary conditions (1.13) and (1.16) as

$$(1.22) \quad F[\psi, \delta] \equiv \int_0^L [1 + \delta(\tau)] \exp i[\Phi(\tau) + \psi(\tau)] d\tau = \int_0^L \exp i\Phi(\tau) d\tau.$$

By (1.14) the right hand side vanishes for rings.

2. The extremum problem. We wish to demonstrate that the boundary value problems defined in the last section have classical solutions by using the direct methods of the calculus of variations. We first show that the potential energy functional V has a minimum on some suitable set of functions that satisfy the boundary conditions and then show that the minimizing functions are classical solutions. The essential tool for the first part of this program is the

Fundamental Minimization Theorem. *A weakly lower semicontinuous functional on a bounded, weakly-closed, nonempty subset of a reflexive Banach space assumes its minimum there. (See [11] for proof.)*

Unfortunately, any set of function pairs ψ, δ for which (1.18) holds cannot be weakly closed in any of the common Banach spaces. (E.g., the sequence of constant functions $\delta_n(\tau) = -1 + 1/n$ converges weakly to $\delta(\tau) = -1$ in the Lebesgue and Sobolev spaces.) We must reformulate our problem to avoid this difficulty (which is symptomatic of all problems of continuum mechanics).

We define a one-to-one mapping Γ of $(-1, \infty)$ onto $(-\infty, \infty)$ and its inverse $\Delta = \Gamma^{-1}$ by

$$(2.1) \quad \gamma = \Gamma(\delta), \quad -1 < \delta < \infty,$$

$$(2.2) \quad \delta = \Delta(\gamma), \quad -\infty < \gamma < \infty,$$

where Γ is twice continuously differentiable on its domain, $\Gamma(0) = 0$, $\Gamma'(0) = 1$, $\Gamma'(\delta) > 0$, $\delta\Gamma''(\delta) \geq 0$, $\Gamma(\delta) \rightarrow -\infty$ as $\delta \rightarrow -1$. (See Fig. 1.) The function $\gamma(\tau) = \Gamma(\delta(\tau))$ is just another measure of extensional strain with the desirable property that if $\gamma(\tau)$ is a continuous function of τ then the corresponding function $\delta(\tau)$ is not only continuous, but also satisfies (1.18). We define

$$(2.3) \quad \Omega(\psi', \gamma) = W(\psi', \delta),$$

$$(2.4) \quad U[\psi, \gamma] = U_1[\psi, \gamma] - qU_2[\psi, \gamma],$$

FIGURE 1.

where

$$(2.5) \quad U_1[\psi, \gamma] = \int_0^L \Omega(\psi'(\tau), \gamma(\tau), \tau) d\tau,$$

$$(2.6) \quad U_2[\psi, \gamma] = \frac{1}{2} \int_0^L [1 + \Delta(\gamma(\tau))] \int_0^\tau [1 + \Delta(\gamma(\xi))] \sin [\phi(\tau) - \phi(\xi)] d\xi d\tau.$$

Imposing some mild restrictions on Ω , we shall show that U is minimized on some set of function pairs ψ, γ that satisfy

$$(2.7) \quad F[\psi, \Delta(\gamma)] \equiv \int_0^L [1 + \Delta(\gamma(\tau))] \exp i[\psi(\tau) + \Phi(\tau)] d\tau = \int_0^L \exp i\Phi(\tau) d\tau,$$

and either the arch conditions:

$$(2.8) \quad \psi(a) = 0 \quad \text{or} \quad M(a) = 0 \quad \text{at} \quad a = 0, L,$$

or the ring conditions:

$$(2.9) \quad \psi(0) = \psi(L), \quad \psi'(0) = \psi'(L), \quad \gamma(0) = \gamma(L),$$

$$(2.10) \quad \int_0^L \psi(\tau) d\tau = 0.$$

The introduction of γ has obviated the need for the inequality (1.18).

3. Existence theory. The following two lemmas are immediate consequences of the properties of Γ .

Lemma 3.1. *If the Hessian matrix of Ω*

$$(3.1) \quad \begin{bmatrix} \Omega_{\psi' \psi'} & \Omega_{\psi' \gamma} \\ \Omega_{\gamma \psi'} & \Omega_{\gamma \gamma} \end{bmatrix}$$

is positive definite then so is (1.5).

Lemma 3.2. *For $\beta \geq 0$,*

$$(3.2a) \quad \int_0^L |\delta(\tau)|^\beta d\tau < \int_0^L |\Gamma(\delta(\tau))|^\beta d\tau,$$

or

$$(3.2b) \quad \int_0^L |\Delta(\gamma(\tau))|^\beta d\tau < \int_0^L |\gamma(\tau)|^\beta d\tau.$$

We study the functional U on certain subspaces of the reflexive Banach space $B = W_\alpha^{(1)}(0, L) \times L_\beta(0, L)$, $\alpha > 1$, $\beta > 1$ of function pairs $\{\psi, \gamma\}$ having the norm

$$(3.3) \quad ||\{\psi, \gamma\}|| = \left\{ \int_0^L [|\psi(\tau)|^\alpha + |\psi'(\tau)|^\alpha] d\tau \right\}^{1/\alpha} + \left\{ \int_0^L |\gamma(\tau)|^\beta d\tau \right\}^{1/\beta}.$$

(The Sobolev space $W_\alpha^{(1)}$ consists of L_α functions having distributional derivatives that also lie in L_α). The values of α and β will be fixed in (3.4).

We study the ring problem on the subspace of B for which

$$(3.3a) \quad \psi(0) = \psi(L), \quad \int_0^L \psi(\tau) d\tau = 0,$$

the problem for the arch clamped at both ends on the subspace of B for which

$$(3.3b) \quad \psi(0) = 0, \quad \psi(L) = 0,$$

the problem for the arch clamped at $\tau = 0$ and hinged at $\tau = L$ on the subspace of B for which

$$(3.3c) \quad \psi(0) = 0,$$

and the problem for the arch hinged at both ends on B itself. These subspaces are well defined because $W_\alpha^{(1)}(0, L)$ can be identified with a subset of the space of continuous functions $C[0, L]$. Those boundary conditions from (2.8)–(2.10) that are absent from (3.3a)–(3.3c) are implicit in our variational formulation and will be shown to hold for their corresponding problems. (E.g., the condition $M(a) = 0$ is a natural boundary condition.)

The domain S of the functional U consists of those elements of the appropriate subspace of B that satisfy (2.7).

We now prove a sequence of results needed to show that the hypotheses of the Fundamental Minimization Theorem are met.

Lemma 3.3. *If there are constants $K > 0$, $\alpha > 1$, $\beta > 1$ such that*

$$(3.4) \quad \Omega(\psi', \gamma, \tau) \geq K(|\psi'|^\alpha + |\gamma|^\beta) + f(\tau),$$

where $f(\tau)$ is an integrable function on $[0, L]$, then $U_1[\psi, \gamma] \rightarrow \infty$ as $\|\{\psi, \gamma\}\| \rightarrow \infty$. If (3.4) holds with $\beta > 2$, then $U[\psi, \gamma] \rightarrow \infty$ as $\|\{\psi, \gamma\}\| \rightarrow \infty$.

Proof. If $g(\tau)$ is a function in $W_\alpha^{(1)}(0, L)$ with $g(\tau_0) = 0$ for $\tau_0 \in [0, L]$, then

$$(3.5) \quad \int_0^L |g(\tau)|^\alpha d\tau < C \int_0^L |g'(\tau)|^\alpha d\tau,$$

where C is a positive constant. (This is a Sobolev Inequality readily obtained by the use of the Hölder Inequality.) If $\psi(\tau)$ in $W_\alpha^{(1)}(0, L)$ satisfies either (2.10) or else the boundary condition $\psi(a) = 0$, then $\psi(\tau)$ satisfies (3.5). If the arch is hinged at both ends, (3.5) need not hold for $\psi(\tau)$. We note, however, that the position $x(\tau)$, $y(\tau)$ of the arch is unaffected by the addition of an integral multiple of 2π to ψ . Thus we can define ψ so that it either vanishes on $[0, L]$ or has a mean value less than 2π . In the first case ψ satisfies (3.5); in the second case $L\psi - \int_0^L \psi(\tau) d\tau$ satisfies (3.5). It therefore follows from (3.5) that the functions $\psi(\tau)$ for our ring and arch problems have the property that

$$(3.6) \quad \int_0^L |\psi'|^\alpha d\tau \rightarrow \infty \quad \text{when} \quad \int_0^L |\psi|^\alpha d\tau \rightarrow \infty.$$

Relation (3.4) implies that

$$(3.7) \quad U_1[\psi, \gamma] \geq K \int_0^L [|\psi'(\tau)|^\alpha + |\gamma(\tau)|^\beta] d\tau + \text{const.}$$

Our preceding remarks then show that $U_1 \rightarrow \infty$ as $\|\{\psi, \gamma\}\| \rightarrow \infty$. This completes the proof of the first part of the lemma.

By Lemma 3.2 and the Hölder Inequality, we have

$$(3.8) \quad |2U_2[\psi, \gamma]| \leq \left\{ \int_0^L |1 + \Delta(\gamma(\tau))| d\tau \right\}^2 \leq \left\{ L + \int_0^L |\Delta(\gamma(\tau))| d\tau \right\}^2 \leq \left\{ L + \int_0^L |\gamma(\tau)| d\tau \right\}^2 \leq \left\{ L + \text{const} \left[\int_0^L |\gamma(\tau)|^\beta d\tau \right]^{1/\beta} \right\}^2.$$

The inequalities (3.7) and (3.8) imply that for fixed q , U_1 dominates qU_2 for large norm, if $\beta > 2$. $\square$

The role of the critical value of $\beta = 2$ will be discussed in the next section.

We define the set of functions

$$(3.9) \quad \Delta(L_\beta) = \{\delta : \delta(\tau) = \Delta(\gamma(\tau)), \gamma \in L_\beta\}.$$

The properties of Δ ensure that $\Delta(L_\beta) \subset L_\beta$.

We wish to show that the functionals $F[\psi, \delta]$ and $V_2[\psi, \delta]$ are weakly continuous on $W_\alpha^{(1)} \times \Delta(L_\beta)$. Thus we must show that

$$(3.10) \quad F[\psi_i, \delta_i] - F[\psi^*, \delta^*] \rightarrow 0,$$

$$(3.11) \quad V_2[\psi_i, \delta_i] - V_2[\psi^*, \delta^*] \rightarrow 0,$$

as $\{\psi_i, \delta_i\} \rightarrow \{\psi^*, \delta^*\}$ weakly. By the Riesz Representation Theorem for our function spaces, the weak convergence of $\{\psi_i, \delta_i\}$ to $\{\psi^*, \delta^*\}$ means that

$$(3.12) \quad \int_0^L \{[\psi_i(\tau) - \psi^*(\tau)]\eta(\tau) + [\psi'_i(\tau) - \psi^{*'}(\tau)]\eta'(\tau)\} d\tau \rightarrow 0,$$

$$(3.13) \quad \int_0^L [\delta_i(\tau) - \delta^*(\tau)]\zeta(\tau) d\tau \rightarrow 0$$

for arbitrary $\eta \in W_\alpha^{(1)}(0, L)$ and for arbitrary $\zeta \in L_{\beta'}(0, L)$ where

$$(3.14) \quad \frac{1}{\alpha} + \frac{1}{\alpha'} = 1, \quad \frac{1}{\beta} + \frac{1}{\beta'} = 1.$$

Lemma 3.4. $F[\psi, \delta]$ is weakly continuous on $W_\alpha^{(1)} \times \Delta(L_\beta)$.

Proof. From (1.22) we have

$$(3.15) \quad F[\psi_i, \delta_i] - F[\psi^*, \delta^*]$$

$$\begin{aligned}
&= \int_0^L [\delta_i(\tau) - \delta^*(\tau)] \exp i[\psi^*(\tau) + \Phi(\tau)] d\tau \\
&+ \int_0^L [1 + \delta_i(\tau)] \{ \exp i[\psi_i(\tau) + \Phi(\tau)] - \exp i[\psi^*(\tau) + \Phi(\tau)] \} d\tau.
\end{aligned}$$

Since $W_\alpha^{(1)}(0, L)$ can be compactly embedded in the space $C[0, L]$ of continuous functions with the maximum norm, the elements of the sequence $\{\psi_i\}$ are (equivalence classes containing) continuous functions converging strongly to ψ^* in $C[0, L]$. Thus

$$(3.16) \quad \max_{\tau} |\psi_i(\tau) - \psi^*(\tau)| \rightarrow 0,$$

and $\psi^*(\tau)$ is continuous. Therefore the function $\exp i[\psi^*(\tau) + \Phi(\tau)]$ is continuous and can be identified with $\zeta(\tau)$ of (3.13) so that the first integral of (3.15) approaches 0 as $j \rightarrow \infty$.

The absolute value of the second integral of (3.15) is less than

$$(3.17) \quad [L + \int_0^L |\delta_i(\tau)| d\tau] \max_{\tau} |\psi_i(\tau) - \psi^*(\tau)|.$$

Since the sequence $\{\delta_i\}$ is weakly convergent in $L_\beta(0, L)$, it is bounded in norm and since $\beta > 1$, $\int_0^L |\delta_i(\tau)| d\tau$ is also bounded. Relation (3.16) then implies that (3.17) approaches 0. Thus F is weakly continuous. $\square$

Lemma 3.5. $V_2[\psi, \delta]$ is weakly continuous on $W_\alpha^{(1)} \times \Delta(L_\beta)$.

Proof. From (1.21) we have

$$(3.18)$$

$$\begin{aligned}
&2V_2[\psi_i, \delta_i] - V_2[\psi^*, \delta^*] \\
&= \int_0^L [\delta_i(\tau) - \delta^*(\tau)] \int_0^\tau [1 + \delta_i(\xi)] \sin [\psi^*(\tau) + \Phi(\tau) - \psi^*(\xi) - \Phi(\xi)] d\xi d\tau \\
&- \int_0^L [\delta_i(\tau) - \delta^*(\tau)] \int_\tau^L [1 + \delta^*(\xi)] \sin [\psi^*(\tau) + \Phi(\tau) - \psi^*(\xi) - \Phi(\xi)] d\xi d\tau \\
&+ \int_0^L \int_0^\tau [1 + \delta_i(\tau)][1 + \delta_i(\xi)] \{ \sin [\psi_i(\tau) + \Phi(\tau) - \psi_i(\xi) - \Phi(\xi)] \\
&- \sin [\psi^*(\tau) + \Phi(\tau) - \psi^*(\xi) - \Phi(\xi)] \} d\xi d\tau.
\end{aligned}$$

Since the coefficient of $[\delta_i(\tau) - \delta^*(\tau)]$ in the integrand of the second integral of (3.18) is continuous, the second integral approaches 0 by (3.13). The absolute value of the third integral of (3.18) is bounded above by

$$\int_0^L [1 + \delta_i(\tau)] \int_0^\tau [1 + \delta_i(\xi)] [|\psi_i(\tau) - \psi^*(\tau)| + |\psi_i(\xi) - \psi^*(\xi)|] d\xi d\tau$$

$$\leq 2 \left\{ \int_0^L [1 + \delta_i(\tau)] d\tau \right\}^2 \max_{\tau} |\psi_i(\tau) - \psi^*(\tau)|,$$

which, according to the proof of Lemma 3.4, approaches 0 as $j \rightarrow \infty$.

We define the continuous functions

$$\rho(t, \xi) = \begin{cases} \sin [\psi^*(\tau) + \Phi(\tau) - \psi^*(\xi) - \Phi(\xi)], & \xi \leq \tau, \\ 0, & \xi > \tau, \end{cases}$$

$$h_i(\tau) = \int_0^L [\delta_i(\xi) - \delta^*(\xi)] \rho(\tau, \xi) d\xi.$$

Then the first integral of (3.18) becomes

$$(3.19) \quad \int_0^L [\delta_i(\tau) - \delta^*(\tau)] \int_0^{\tau} [1 + \delta^*(\xi)] \rho(\tau, \xi) d\xi d\tau \\ + \int_0^L [\delta_i(\tau) - \delta^*(\tau)] h_i(\tau) d\tau.$$

By (3.13) the first term of (3.19) approaches 0. Also by (3.13), $h_i(\tau) \rightarrow 0$. Since $\{\delta_i\}$ is bounded in norm, we can readily show that $\{h_i\}$ is uniformly bounded and equicontinuous. By Ascoli's Theorem $\{h_i\}$ has a subsequence uniformly convergent to 0. It follows that $\{h_i\}$ itself must converge uniformly to 0. Thus the second term of (3.19) also approaches 0. $\square$

Theorem 3.1. *If (3.1) is positive definite and if (3.4) holds with $\beta > 2$, then for arbitrary fixed q , there exists an element $\{\psi_0, \gamma_0\}$ of S that furnishes $U[\psi, \gamma]$ with a minimum on S .*

Proof. By Lemma 3.3, we need only seek a minimum for U on the intersection with S of some closed ball K centered at the origin of B . Let

$$K_{\Delta} = \{ \{\psi, \delta\} : \delta = \Delta(\gamma), \{\psi, \gamma\} \in K \},$$

$$S_{\Delta} = \{ \{\psi, \delta\} : \delta = \Delta(\gamma), \{\psi, \gamma\} \in S \}.$$

Thus we need only seek a minimum for V on $K_{\Delta} \cap S_{\Delta}$. The properties of Δ ensure that K_{Δ} is midpoint convex so that its closure $\bar{K}_{\Delta}$ is convex and therefore weakly closed. By Lemma 3.4, $\bar{K}_{\Delta} \cap S_{\Delta}$ is weakly closed. The positive definiteness of (1.5), ensured by Lemma 3.1, implies that V_1 is weakly lower semicontinuous. Lemma 3.5 implies that V_2 is weakly continuous. Thus all the hypotheses of the Fundamental Minimization Theorem are met so that V has a minimum on $\bar{K}_{\Delta} \cap S_{\Delta}$. U accordingly has a minimum on

$$\{ \{\psi, \gamma\} : \gamma = \Gamma(\delta), \{\psi, \delta\} \in \bar{K}_{\Delta} \} \cap S.$$

(This set may include elements with arbitrarily large norm.) But if U is minimized on S , it must in fact be minimized on $K \cap S$. $\square$

If we specify the amount of deformation rather than the pressure, we obtain

Theorem 3.2. *Let*

$$(3.20) \quad E(R) = \{ \{ \psi, \gamma \} \in S : U_2[\Psi, \gamma] = R \}.$$

If (3.1) is positive definite and if (3.4) holds, then for each fixed R , there exists an element $\{ \psi_0, \gamma_0 \}$ of $E(R)$ that furnishes $U_1[\psi, \gamma]$ with a minimum on $E(R)$.

The proof of this is almost identical to that of Theorem 3.1. Note that the fixing of U_2 by (3.20) obviates the need for the requirement that $\beta > 2$. By the multiplier rule, this minimization problem leads to the same Euler equations as that of Theorem 3.1. The hypothesis that (3.4) holds is consistent both with the positive definiteness of (3.1) and with the physical requirement that $N \rightarrow -\infty$ as $\gamma \rightarrow -\infty$ ($\delta \rightarrow -1$).

Theorem 3.3. *If in addition to satisfying the hypotheses of Theorems 3.1 and 3.2, the function $\Omega(\psi', \gamma, \tau)$ satisfies*

$$(3.21) \quad |\Omega_{\psi'}(\psi', \gamma, \tau) + \Omega_{\gamma}(\psi', \gamma, \tau)| \leq \text{const} (|\psi'|^\alpha + |\gamma|^\beta + 1),$$

then the minimizing functions of Theorems 3.1 and 3.2 are classical solutions of the boundary value problem.

Proof. Our approach is a modification of that for the Tonelli problem described in [1]. We just prove the result for the minimizer pair $\{ \psi_0, \gamma_0 \}$ of Theorem 3.2 since the proof for the minimizer of Theorem 3.1 is identical.

Let $\mathbf{n}(\tau) = \{ \eta_k(\tau) \}$, $k = 1, 2, 3, 4$ be a set of fixed, piecewise smooth functions, each member of which satisfies the conditions from (3.3a)–(3.3c) appropriate for the given problem. Let $\zeta(\tau) = \{ \xi_k(\tau) \}$, $k = 1, 2, 3, 4$ be a set of fixed, piecewise continuous functions. Let $\epsilon = \{ \epsilon_k \}$, $k = 1, 2, 3, 4$ be a set of real parameters. We define the following functions of ϵ :

$$(3.22) \quad \begin{aligned} u_1(\epsilon) &= U_1[\psi_0 + \epsilon \cdot \mathbf{n}, \gamma_0 + \epsilon \cdot \zeta], \\ u_2(\epsilon) &= U_2[\psi_0 + \epsilon \cdot \mathbf{n}, \gamma_0 + \epsilon \cdot \zeta], \\ f_1(\epsilon) &= \text{Re } F[\psi_0 + \epsilon \cdot \mathbf{n}, \gamma_0 + \epsilon \cdot \zeta], \\ f_2(\epsilon) &= \text{Im } F[\psi_0 + \epsilon \cdot \mathbf{n}, \gamma_0 + \epsilon \cdot \zeta]. \end{aligned}$$

We assume that $\mathbf{n}$ and ζ are such that

$$(3.23) \quad u_2(\epsilon) = R, \quad f_1(\epsilon) + if_2(\epsilon) = \int_0^L \exp i\Phi(\tau) d\tau,$$

for all ϵ in some neighborhood of $\epsilon = \mathbf{0}$, say $|\epsilon| < \epsilon_0$.

From the minimization property of $\{ \psi_0, \gamma_0 \}$, it follows that the minimum of $u_1(\epsilon)$ over all ϵ near $\mathbf{0}$ that satisfy (3.23) is attained at $\epsilon = \mathbf{0}$. If the functions (3.22) are continuously differentiable at $\epsilon = \mathbf{0}$, the multiplier rule for ordinary functions implies that there are real constants $\lambda_1, \lambda_2, \nu_1, \nu_2$ (not all zero) such that

$$(3.24) \quad \frac{\partial}{\partial \epsilon_k} [\lambda_1 u_1(\epsilon) + \lambda_2 u_2(\epsilon) + \nu_1 f_1(\epsilon) + \nu_2 f_2(\epsilon)]|_{\epsilon=0} = 0.$$

We show that this is the case.

Let us denote one of the triples $(\eta_k, \zeta_k, \epsilon_k)$ by (η, ζ, ϵ) . By the Mean Value Theorem

$$(3.25) \quad \begin{aligned} & \{U_1[\psi_0 + \epsilon\eta, \gamma_0 + \epsilon\zeta] - U_1[\psi_0, \gamma_0]\}/\epsilon \\ &= \int_0^L [\Omega_{\psi'}(\psi'_0(\tau) + \epsilon\theta(\tau)\eta'(\tau), \gamma_0(\tau) + \epsilon\theta(\tau)\zeta(\tau), \tau)\eta'(\tau) \\ & \quad + \Omega_{\gamma'}(\psi'_0(\tau) + \epsilon\theta(\tau)\eta'(\tau), \gamma_0(\tau) + \epsilon\theta(\tau)\zeta(\tau), \tau)\zeta(\tau)] d\tau, \end{aligned}$$

where $0 < \theta(\tau) < 1$. As $\epsilon \rightarrow 0$, the integrand of the righthand side of (3.25) approaches $\Omega_{\psi'}(\psi'_0(\tau), \gamma_0(\tau), \tau)\eta'(\tau) + \Omega_{\gamma'}(\psi'_0(\tau), \gamma_0(\tau), \tau)\zeta(\tau)$, pointwise. The inequality (3.21) implies that for $|\epsilon| \leq 1$, the integrand is uniformly less than

$$\begin{aligned} & \text{const } \{|\psi'_0(\tau) + \epsilon\tau(\tau)\eta'(\tau)|^\alpha + |\gamma_0(\tau) + \epsilon\theta(\tau)\zeta(\tau)|^\beta + 1\}(|\eta'| + |\zeta'|) \\ & \leq \text{const } \{|\psi'_0(\tau)|^\alpha + |\gamma_0(\tau)|^\beta + |\eta'(\tau)|^\alpha + |\zeta(\tau)|^\beta + 1\}, \end{aligned}$$

which is integrable. Thus, by Lebesgue's Dominated Convergence Theorem the limit of the integral of (3.25) exists, so that (3.25) becomes

$$(3.26) \quad \frac{\partial u_1}{\partial \epsilon} \Big|_{\epsilon=0} = \int_0^L [\Omega_{\psi'}(\psi'_0(\tau), \gamma_0(\tau), \tau)\eta'(\tau) + \Omega_{\gamma'}(\psi'_0(\tau), \gamma_0(\tau), \tau)\zeta(\tau)] d\tau.$$

Treating U_2 in the same way, we have

$$(3.27) \quad \begin{aligned} & 2\{U_2[\psi_0 + \epsilon\eta, \gamma_0 + \epsilon\zeta] - U_1[\psi_0, \gamma_0]\}/\epsilon \\ &= \int_0^L \Delta'(\gamma_0(\tau) + \epsilon\theta_1(\tau)\zeta(\tau))\zeta(\tau) \left\{ - \int_\tau^L [1 + \Delta(\gamma_0(\xi))] \right. \\ & \quad \left. + \int_0^\tau [1 + \Delta(\gamma_0(\xi)) + \epsilon\Delta'(\gamma_0(\xi) + \epsilon\theta_1(\xi)\zeta(\xi))\zeta(\xi)] \right\} \\ & \quad \cdot \sin[\psi_0(\tau) + \Phi(\tau) - \psi_0(\xi) - \Phi(\xi)] d\xi d\tau \\ & \quad + \int_0^L [1 + \Delta(\gamma_0(\tau) + \epsilon\zeta(\tau))] \int_0^\tau [1 + \Delta(\gamma_0(\xi) + \epsilon\zeta(\xi))] \\ & \quad \cdot \cos[\psi_0(\tau) + \Phi(\tau) - \psi_0(\xi) - \Phi(\xi)][\eta(\tau) - \eta(\xi)] d\xi d\tau, \end{aligned}$$

where $0 < \theta_1(\tau), \theta_2(\tau) < 1$. The properties of Δ enable us to show that the integrand of the first term on the righthand side of (3.27) is uniformly bounded above by an integrable function. Since this integrand is pointwise convergent we know that the limit of this integral as $\epsilon \rightarrow 0$ exists and we can obtain it by the Dominated Convergence Theorem. Since the indefinite integral appearing in the integrand of the second term is continuous and can be bounded by a continuous function independent of ϵ , we can readily bound the integrand of this term and again employ the Dominated Convergence Theorem. (Note that

the quadratic dependence on γ causes no trouble here.) In the limit as $\epsilon \rightarrow 0$, (3.27) becomes

$$(3.28) \quad 2 \frac{\partial u_2}{\partial \epsilon} \Big|_{\epsilon=0} = \int_0^L \Delta'(\gamma_0(\tau)) \zeta(\tau) \left\{ -\int_\tau^L + \int_0^\tau \right\} [1 + \Delta(\gamma_0(\xi))] \\ \cdot \sin [\psi_0 + \Phi(\tau) - \psi_0(\xi) - \Phi(\xi)] d\xi d\tau + \int_0^L [1 + \Delta(\gamma_0(\tau))] \int_0^\tau [1 + \Delta(\gamma_0(\xi))] \\ \cdot \cos [\psi_0(\tau) + \Phi(\tau) - \psi_0(\xi) - \Phi(\xi)] [\eta(\tau) - \eta(\xi)] d\xi d\tau.$$

The treatment of the functions $f_1(\epsilon)$ and $f_2(\epsilon)$ is analogous. Thus we can write (3.24) in the form

$$(3.29) \quad \int_0^L \{ \lambda_1 \Omega_\psi(\psi'_0(\tau), \gamma_0(\tau), \tau) \eta'(\tau) + H(\tau) \eta(\tau) + [\lambda_1 \Omega_\gamma(\psi'_0(\tau), \gamma_0(\tau), \tau) \\ + \Delta'(\gamma_0(\tau)) Z(\tau)] \zeta(\tau) \} d\tau = 0,$$

where

$$(3.30) \quad H(\tau) = [1 + \Delta(\gamma_0(\tau))] \left\{ (\lambda_2/2) \left(\int_0^\tau - \int_\tau^L \right) \right. \\ \cdot [1 + \Delta(\gamma_0(\xi))] \cos [\psi_0(\tau) + \Phi(\tau) - \psi_0(\xi) - \Phi(\xi)] d\xi \\ \left. - \nu_1 \sin [\psi_0(\tau) + \Phi(\tau)] + \nu_2 \cos [\psi_0(\tau) + \Phi(\tau)] \right\},$$

$$(3.31) \quad Z(\tau) = (\lambda_2/2) \left(\int_0^\tau - \int_\tau^L \right) [1 + \Delta(\gamma_0(\xi))] \sin [\psi_0(\tau) + \Phi(\tau) \\ - \psi_0(\xi) - \Phi(\xi)] d\xi + \nu_1 \cos [\psi_0(\tau) + \Phi(\tau)] + \nu_2 \sin [\psi_0(\tau) + \Phi(\tau)].$$

An integration by parts yields

$$(3.32) \quad \int_0^L H(\tau) \eta(\tau) d\tau = \eta(L) \int_0^L H(\tau) d\tau - \int_0^L \left[\int_0^\tau H(\xi) d\xi \right] \eta'(\tau) d\tau,$$

which we substitute into (3.29). If $\eta(L)$ is not fixed at 0, the arbitrariness of η implies that

$$(3.33) \quad \int_0^L H(\tau) d\tau = 0.$$

We now invoke the Fundamental Lemma of the Calculus of Variations in the following two forms [1]: If $A(\tau)$ and $B(\tau)$ are in $L_1(0, L)$ and if

$$\int_0^L A(\tau) \eta'(\tau) d\tau = 0, \quad \int_0^L B(\tau) \zeta(\tau) d\tau = 0,$$

for all piecewise smooth $\eta(\tau)$ with compact support on $(0, L)$ and for all piecewise continuous $\zeta(\tau)$, then

$A(\tau) = \text{constant function, a.e., } B(\tau) = 0, \text{ a.e.}$

Applying this result to (3.29) and using (2.3), we have

$$(3.34) \quad \lambda_1 W_{\psi'}(\psi'_0(\tau), \Delta(\gamma_0(\tau)), \tau) = \int_0^\tau H(\xi) d\xi + \text{const, a.e.,}$$

$$(3.35) \quad \lambda_1 W_\delta(\psi'_0(\tau), \Delta(\gamma_0(\tau)), \tau) = -Z(\tau), \text{ a.e.}$$

The multiplier λ_1 does not vanish, for if it did, (3.34) and (3.35) would imply that

$$(3.36) \quad \lambda_2 \left(\int_0^\tau - \int_\tau^L \right) [1 + \Delta(\gamma_0(\xi))] \exp \{-i[\psi_0(\xi) + \Phi(\xi)]\} d\xi + i\nu_1 + \nu_2 = 0,$$

which can be differentiated to yield

$$(3.37) \quad \lambda_2 [1 + \Delta(\gamma_0(\tau))] \exp \{-i[\psi_0(\tau) + \Phi(\tau)]\} = 0, \text{ a.e.,}$$

whence it follows that $\lambda_2 = 0$. Equation (3.36) would then imply that $\nu_1 = 0$, $\nu_2 = 0$ which contradicts the property that $\lambda_1, \lambda_2, \nu_1, \nu_2$ cannot vanish simultaneously. We therefore set $\lambda_1 = 1$.

The righthand sides of (3.34) and (3.35) are continuous functions of τ . We ignore the fact that they depend upon τ through $\psi_0(\tau)$ and $\gamma_0(\tau)$. Since $W(\psi', \delta, \tau)$ is a thrice continuously differentiable function of its arguments whose Hessian matrix is positive definite and since $\Delta'(\gamma) > 0$, we can solve (3.34) and (3.35) for $\psi'_0(\tau)$ and $\gamma_0(\tau)$:

$$(3.38) \quad \psi'_0(\tau) = G_1(\tau, \int_0^\tau H(\xi) d\xi, Z(\tau)),$$

$$\gamma_0(\tau) = G_2(\tau, \int_0^\tau H(\xi) d\xi, Z(\tau)),$$

where G_1 and G_2 are differentiable functions of their arguments. Since the arguments of G_1 and G_2 are continuous functions of τ , $\psi'_0(\tau)$ and $\gamma_0(\tau)$ are continuous. (By the uniqueness of the distributional derivative, $\psi'_0(\tau)$ is the classical derivative of $\psi_0(\tau)$.) Thus the arguments of G_1 and G_2 are differentiable functions of τ so that $\psi_0(\tau) \in C^2[0, L]$, $\gamma_0(\tau) \in C^1[0, L]$.

We can accordingly differentiate (3.34) and (3.35) with respect to τ :

$$(3.39) \quad \frac{d}{d\tau} [W_{\psi'}(\psi'_0(\tau), \Delta(\gamma_0(\tau)), \tau)] = H(\tau),$$

$$(3.40) \quad \begin{aligned} \frac{d}{d\tau} [W_\delta(\psi'_0(\tau), \Delta(\gamma_0(\tau)), \tau)] = & -[\psi'_0(\tau) + \Phi'(\tau)] \\ & \cdot \left\{ (\lambda_2/2) \left(\int_0^\tau - \int_\tau^L \right) [1 + \Delta(\gamma_0(\xi))] \cos [\psi_0(\tau) + \Phi(\tau) - \psi_0(\tau) - \Phi(\xi)] d\xi \right. \\ & \left. - \nu_1 \sin [\psi_0(\tau) + \Phi(\tau)] + \nu_2 \cos [\psi_0(\tau) + \Phi(\tau)] \right\}. \end{aligned}$$

By (3.30), the righthand side of (3.40) is just

$$[\psi'_0(\tau) + \Phi'(\tau)]H(\tau)/[1 + \Delta(\gamma_0(\tau))].$$

Thus (3.39) and (3.40) imply that

$$(3.41) \quad [\psi'_0(\tau) + \Phi'(\tau)] \frac{d}{d\tau} [W_{\psi'}(\psi_0(\tau), \Delta(\gamma_0(\tau)), \tau)] \\ + [1 + \Delta(\gamma_0(\tau))] \frac{d}{d\tau} [W_{\delta}(\psi_0(\tau), \Delta(\gamma_0(\tau)), \tau)] = 0.$$

Since $(\psi_0(\tau), \gamma_0(\tau))$ are solutions of (3.39) and since $H(\tau)/[1 + \Delta(\gamma_0(\tau))]$ is differentiable, we obtain the following equation from (3.39) by means of (3.30), (3.31), (3.35):

$$(3.42) \quad \frac{d}{d\tau} \left\{ [1 + \Delta(\gamma_0(\tau))]^{-1} \frac{d}{d\tau} W_{\psi'}(\psi'_0(\tau), \Delta(\gamma_0(\tau)), \tau) \right\} \\ = \lambda_2 [1 + \Delta(\gamma_0(\tau))] + [\psi'_0(\tau) + \Phi'(\tau)] W_{\delta}(\psi'_0(\tau), \Delta(\gamma_0(\tau)), \tau).$$

By identifying λ_2 with $-q$ we see that the equations (3.41) and (3.42) satisfied by $\{\psi_0(\tau), \gamma_0(\tau)\}$ are equivalent to the system (1.4), (1.6), (1.7). (For the minimizer of Theorem 3.2, the value of q is determined by the side condition $U_2[\psi, \gamma] = R$. The value of q is specified for the minimizer of Theorem (3.1).)

We have only to verify that the boundary conditions of (2.8)–(2.10) that are absent from (3.3a)–(3.3c) are satisfied. The conditions $M(a) = W_{\psi'}(\psi'_0(a), \gamma_0(a), a) = 0$ are obtained from (3.29) by setting $\zeta = 0$ in (3.29) and integrating (3.29) by parts to get

$$\int_0^L \left[-\lambda_1 \frac{d}{d\tau} \Omega_{\psi'}(\psi'_0(\tau), \gamma_0(\tau), \tau) + H(\tau) \right] \eta(\tau) d\tau \\ + [\eta(\tau) \Omega_{\psi'}(\psi'_0(\tau), \gamma_0(\tau), \tau)]_0^L = 0.$$

The arbitrariness of η then implies $\Omega_{\psi'}(\psi'_0(a), \gamma_0(a), a) \equiv W_{\psi'}(\psi'_0(a), \Delta(\gamma_0(a)), a) = 0$ provided η is not defined to be zero at a .

For the ring problem (3.33) holds so that (3.34) implies

$$(3.43) \quad W_{\psi'}(\psi'_0(L), \Delta(\gamma_0(L)), L) = W_{\psi'}(\psi'_0(0), \Delta(\gamma_0(0)), 0).$$

From (3.31) and (3.3a) we have that

$$(3.44) \quad Z(L) - Z(0) = \lambda_2 \int_0^L [1 + \Delta(\gamma_0(\xi))] \\ \cdot \sin [\psi_0(0) + \Phi(0) - \psi_0(\xi) - \Phi(\xi)] d\xi.$$

But the right side of (3.44) vanishes by (2.7) and (1.14) so that (3.35) implies

$$(3.45) \quad W_{\delta}(\psi'_0(L), \Delta(\gamma_0(L)), L) = W_{\delta}(\psi'_0(0), \Delta(\gamma_0(0)), 0).$$

Since W has period L in τ , we can write (3.43) and (3.45) as

$$(3.46) \quad \begin{aligned} W_{\psi'}(\psi'_0(L), \Delta(\gamma_0(L)), 0) &= W_{\psi'}(\psi'_0(0), \Delta(\gamma_0(0)), 0), \\ W_{\delta}(\psi'_0(L), \Delta(\gamma_0(L)), 0) &= W_{\delta}(\psi'_0(0), \Delta(\gamma_0(0)), 0), \end{aligned}$$

which we regard as a system for determining $\psi'_0(L)$ and $\Delta(\gamma_0(L))$ when $\psi'_0(0)$ and $\Delta(\gamma_0(0))$ are given. Since the Jacobian is just the nonzero Hessian determinant of W , the solution of this system is unique and must therefore be

$$(3.47) \quad \psi'_0(L) = \psi'_0(0), \quad \Delta(\gamma_0(L)) = \Delta(\gamma_0(0)).$$

This completes the proof. $\square$

Remarks. The multipliers ν_1 and ν_2 appearing in this proof are related to the components of shear at the ends of the arch.

In [2], condition (3.21) was replaced by

$$|W_{\psi'}(\psi', \delta, \tau)| + |W_{\delta}(\psi', \delta, \tau)| \leq \text{const} (|\psi'|^\alpha + |\delta|^\beta + 1),$$

which was suitable for incompressible arches but inconsistent with the requirement that $W_{\delta}(\psi', \delta, \tau) \rightarrow -\infty$ as $\delta \rightarrow -1$. However the unboundedness of W_{δ} as $\delta \rightarrow -1$ could have been made the basis of an a posteriori verification of the continuity condition. Such an approach is far less satisfactory than that used here.

4. Existence of nontrivial solutions. Among the boundary value problems under consideration, we now seek those that possess readily identifiable or "trivial" solutions. In particular, we seek those problems for which $\psi(\tau) \equiv 0$ is a solution. Assuming that (1.11) holds, we find that, if $\psi = 0$ is a solution, then our system (1.4), (1.6), (1.7) reduces to

$$(4.1) \quad \Phi'(\tau)W_{\delta}(0, \delta(\tau), \tau) + q[1 + \delta(\tau)] = 0,$$

$$(4.2) \quad W_{\delta}(0, \delta(\tau), \tau) = \text{const.}$$

Thus if $\psi = 0$ is to be a solution, the reference curvature $\Phi'(\tau)$ and the strain energy function must be so related that $1 + \delta(\tau)$, with $\delta(\tau)$ the solution of (4.2), is proportional to $\Phi'(\tau)$. The most general strain energy function with this property is of the form

$$(4.3) \quad W(\psi', \delta, \tau) = \Phi'(\tau)H(f(\tau)\psi', (1 + \delta)/\Phi'(\tau)),$$

where $f(\tau)$ is an arbitrary C^3 function, $\Phi'(\tau)$ is positive, and $H(u, v)$ is an arbitrary C^3 function so adjusted that the requirements on W are met. We assume that $y(0) = y(L)$ so that (1.20) becomes

$$(4.4) \quad \int_0^L [1 + \delta(\tau)] \cos \Phi(\tau) d\tau = \int_0^L \cos \Phi(\tau) d\tau, \quad \int_0^L [1 + \delta(\tau)] \sin \Phi(\tau) d\tau = 0.$$

If $1 + \delta(\tau) = c\Phi'(\tau)$, then (4.4) implies that

$$(4.5) \quad c[\sin \Phi(L) - \sin \Phi(0)] = \int_0^L \cos \Phi(\tau) d\tau, \quad c[\cos \Phi(L) - \cos \Phi(0)] = 0.$$

Since c cannot vanish by (1.18), we have that

$$(4.6) \quad \Phi(L) = \Phi(0) + 2\pi j, \quad \int_0^L \exp i\Phi(\tau) d\tau = 0.$$

Thus those of our boundary value problems that admit this trivial solution have a closed loop as a reference shape (whose tangent angle may change by any integral multiple of 2π in a complete circuit).

To avoid the artificiality and computational difficulty inherent in the general case (4.3), we restrict our attention to the initially circular homogeneous ring characterized by

$$(4.7) \quad \Phi'(\tau) = 1, \quad W_\tau = 0, \quad L = 2\pi.$$

Then the uniform circular state of radius r

$$(4.8) \quad \psi(\tau) = 0, \quad \delta(\tau) = r - 1, \quad M = 0, \quad N = qr,$$

is a solution of the ring problem provided

$$(4.9) \quad qr = W_\delta(0, r - 1).$$

The lefthand side of (4.9) for several values of q and a typical form for $W_\delta(0, r - 1)$ are sketched as functions of r in Fig. 2. Since $W_\delta(0, r - 1)$ is finite for positive r by (3.21), there is a unique pressure q given by (4.9) corresponding to each circular configuration of radius r . For each nonpositive q , there is a unique radius r satisfying (4.9) since

$$(4.10) \quad W_{\delta\delta}(0, r - 1) - q > 0,$$

for $q \leq 0$. Relation (3.4) implies that $W_\delta(0, r - 1)$ behaves for large r as $(r - 1)^{\beta-1}$. Fig. 2 indicates that if $\beta \leq 2$, there are no solutions of (4.9) for q exceeding some critical value, whereas for $\beta > 2$, there are solutions for each q . In the latter case, there is a unique r for each q provided

$$(4.11) \quad rW_{\delta\delta}(0, r - 1) - W_\delta(0, r - 1) > 0.$$

The existence of a nontrivial solution for the initially circular uniform ring is guaranteed by

Theorem 4.1. *Let the conditions of Theorem 3.3 hold. If $q \in Q$, where Q is a certain set of real numbers to be defined below, then there exists at least one non-circular buckled configuration for the initially circular uniform ring.*

Proof. The second variation of $V[\psi, \delta]$ at the solution (4.8), (4.9) is given by

$$(4.12) \quad V^{(2)} = V_1^{(2)} - qV_2^{(2)},$$

with

FIGURE 2.

$$(4.13) \quad V_1^{(2)} = \int_0^{2\pi} [W_{\psi', \psi'}(0, r-1) \eta'(\tau)^2 + W_{\delta\delta}(0, r-1) \zeta(\tau)^2] d\tau,$$

$$(4.14) \quad V_2^{(2)} = \int_0^{2\pi} \zeta(\tau) \int_0^\tau \zeta(\xi) \sin(\tau - \xi) d\xi d\tau \\ + r \int_0^{2\pi} \int_0^\tau [\eta(\tau) - \eta(\xi)][\zeta(\tau) + \zeta(\xi)] \cos(\tau - \xi) d\xi d\tau.$$

Here $\eta(\tau)$ and $\zeta(\tau)$ are arbitrary smooth functions of period 2π , satisfying

$$(4.15) \quad \int_0^{2\pi} \eta(\tau) d\tau = 0,$$

$$(4.16) \quad \int_0^{2\pi} [\zeta(\tau) + ir\eta(\tau)] \exp i\tau d\tau = 0.$$

Now the boundary value problem linearized about the solution (4.8) has the eigenfunction pair

$$(4.17) \quad \psi_n = (\sin n\tau)/n, \quad \delta_n = -\rho_n \cos n\tau,$$

where n is an integer greater than 1,

$$(4.18) \quad \rho_n = W_{\psi', \psi'}(0, r_n - 1)/r_n W_{\delta\delta}(0, r_n - 1),$$

and the critical radii r_n satisfy

$$(4.19) \quad (1 - n^2)W_{\psi', \psi'}(0, r_n - 1) = r_n W_{\delta\delta}(0, r_n - 1)[r_n + \rho_n].$$

The corresponding critical pressures q_n (which are negative) are then found from (4.9). For a given n , there may be zero, one, or several roots of (4.19). (We shall illustrate this below). In general, no conclusion about the ordering of r_m and r_n can be obtained from the ordering of m and n , although (4.19) does imply that if $r_m = r_n$, then $m = n$.

Since the pair $\{\psi_n, \delta_n\}$ satisfy the conditions on $\{\eta, \zeta\}$, we can set $\eta = \psi_n$, $\zeta = \delta_n$ in (4.12) to obtain

$$(4.20) \quad V^{(2)}/\pi \equiv V_n^{(2)}(r)/\pi = [W_{\psi', \psi'}(0, r - 1) + W_{\delta\delta}(0, r - 1)\rho_n^2] \\ + (q/n^2 - 1)\rho_n(\rho_n + 2r).$$

Let Q_n be the set of all q for which (4.20) is negative. (We define Q_n to be empty if no r_n exists for the integer n .) Thus $V^{(2)} < 0$ when

$$q \in Q \equiv \bigcup_{n=2}^{\infty} Q_n.$$

For $q \in Q$, the trivial solution cannot provide V with even a local minimum. The minimizer pair $\{\psi_0, \delta_0\}$ of Theorem 3.1, known to be classical solutions by Theorem 3.3, therefore cannot be the trivial solution if $q \in Q$. $\square$

Following the arguments of this last proof, we have the

Corollary. *If for some $n \geq 2$, there is an r^* such that $V_{(n)}^{(2)}(r^*) < 0$ and $dV_{(n)}^{(2)}(r)/dr > 0$ for $r \leq r^*$, then Q can be taken to be the semi-infinite interval $(-\infty, q^*)$ where $q^* = \sup_n [W_{\delta\delta}(0, r^* - 1)/r^*]$.*

These general results can be readily translated into sharp criteria for specific materials or classes of materials. *E.g.*, by the differentiation of (4.20) subject to (4.9), we obtain that $dV_{(n)}^{(2)}/dr < 0$ for $q \leq 0$ ($r \leq 1$) provided

$$(4.21) \quad W_{\psi', \psi', \delta}(0, r - 1) + W_{\delta\delta\delta}(0, r - 1)\rho_n^2 > 0, \quad r \leq 1.$$

Our conditions on W as $\delta \rightarrow -1$ suggest that this is a physically reasonable assumption.

Example 1. $W = c\psi'^2/2 + \delta + \delta^2/2 - \ln(1 + \delta)$, $c > 0$. Equation (4.19) assumes the form

$$(4.22) \quad (n^2 - 1)c = r(1 - r^2)(1 + r^2 + c)/(1 + r^2).$$

Since the righthand side has a crude upper bound of $2 + c$ for $r \in [0, 1]$, we see that (4.22) has no roots r_n for c sufficiently large. For a fixed c that is small enough there are at least two roots for small enough n , but for large enough n there will not be any roots. Thus for any fixed c there is but a finite number of roots, and there can be more than one root for a given n . Condition (4.21) is satisfied. Equation (4.20) becomes

$$(4.23) \quad V_n^{(2)}/\pi = c + \rho_n^2 \left(\frac{n^2}{n^2 - 1} \right) + \frac{\rho_n^2}{r^2} \left(\frac{n^2 - 2}{n^2 - 1} \right) \frac{2r\rho_n}{n^2 - 1} - \frac{1}{r} \frac{\rho_n}{n^2 - 1}.$$

If c is small enough then (4.22) has a root r_n close to 1 and ρ_n is then approximately $c/2$. Since the coefficient of $1/r^2$ in (4.23) contains c^2 and the coefficient of $1/r$ contains just c , we can choose r such that for small enough c , $V_n^{(2)}/\pi$ is negative. In this case the hypotheses of the Corollary to Theorem 4.1 are satisfied; thus there is a q^* such that there is at least one buckled state for all $q < q^*$.

Example 2. $W = c\psi'^2/2 + \delta + \delta^2/2 + (1 + \delta)^{-2}/2$.
Equation (4.19) becomes

$$(4.24) \quad (n^2 - 1)c = \frac{1}{r} (1 - r^4) \left(1 + \frac{cr^2}{r^4 + 3} \right).$$

The right side of (4.24) is a nonnegative function for $r \in (0, 1]$ that equals 0 at $r = 1$ and approaches $+\infty$ as $r \rightarrow 0$. Since this function is continuous on $(0, 1]$, equation (4.24) has at least one solution for each c and each n . If $c < 1$, the slope of the right side of (4.24) is negative so that for each n there is exactly one root r_n . Since (4.21) holds, we can use the same argument as in Example 1 to show that for sufficiently small c , the hypotheses of the Corollary to Theorem 4.1 are satisfied; thus there is a q^* such that there is at least one buckled state for all $q < q^*$.

An approach analogous to that used in this section was employed in [10]. Because the ring in that paper was inextensible, their results exhibited a much simpler structure than our examples. Conclusions about the number and distribution of eigenvalues for these examples are analogous to those obtained in [8] for straight rods under end thrust. (It should be pointed out, however, that the material response assumed in [8] is distinct from the response defined in (1.4).)

A thorough treatment of bifurcation and the qualitative nature of solutions for the ring problem is given in [4].

5. Conclusion. The problems we have treated are typical of conservative equilibrium problems in nonlinear elasticity. Such problems may be formulated as variational problems with the equations of differential geometry as side conditions. Our approach consisted of several steps:

- a. The exploitation of the integrability of the geometric equations to remove

the position coordinates as dependent variables, dispose of the differential equations as side conditions, and introduce constraints in integral form at the expense of recasting the integrand of the functional to be minimized in a non-standard form. This new formulation proved convenient for the subsequent analysis. (This approach was first employed in [10] and further developed in [2].)

b. The treatment of the continuity condition (1.13) as a constitutive restriction to eliminate this inequality from the boundary value problems. We adopted the view that no real material could violate this condition and constructed, by means of (2.1) and the assumption that (3.1) is positive definite, a class of materials that satisfy (1.18) for any finite stress.

c. The use of convexity of Ω in ψ' and γ to prove the existence and regularity of the solutions of the boundary value problems. The variable γ , introduced to deal with the continuity condition, is such that the convexity of Ω in ψ' and γ implies the convexity of W in ψ' and δ , which guarantees physically reasonable behavior and is analogous to conditions proposed for problems of three dimensional elasticity.

d. The use of the direct methods of the calculus of variations to treat two types of existence problems and a modification of the classical theory of Tonelli to prove regularity. By means of steps a, b, c we formulated the problem in a manner suitable for the application of these variational methods.

e. The use of the negativity of the second variation at the trivial solution to show that there exists at least one nontrivial solution.

Acknowledgment. The author is indebted to Professor Joseph B. Keller for a critical reading of this manuscript. The research leading to this paper was begun at New York University where it was supported by the Office of Naval Research. Reproduction in whole or in part is permitted for any purpose of the United States Government. The work was completed while the author was a senior visiting fellow of the Science Research Council of Great Britain at the Mathematical Institute of the University of Oxford.

REFERENCES

- [1] N. I. AKHIEZER, *The calculus of variations* (translated from Russian by A. H. Frink), Blaisdell, New York, 1962.
- [2] S. ANTMAN, Equilibrium states of nonlinearly elastic rods, *J. Math. Anal. Appl.*, **23** (1968) 459–470.
- [3] S. ANTMAN, General solutions for plane extensible elasticae having nonlinear stress-strain laws, *Quart. Appl. Math.*, **26** (1968) 35–47.
- [4] S. ANTMAN, The shape of buckled nonlinearly elastic rings, *Z.A.M.P.*, **21** (1970) 422–438.
- [5] M. S. BERGER, On von Kármán's equations and the buckling of a thin elastic plate I, *Comm. Pure Appl. Math.*, **20** (1967) 687–719.
- [6] M. S. BERGER & P. FIFE, On von Kármán's equations and the buckling of a thin elastic plate II, *Comm. Pure Appl. Math.*, **21** (1968) 227–241.
- [7] K. O. FRIEDRICHs & J. J. STOKER, The nonlinear boundary value problem of the buckled plate, *Am. J. Math.*, **63** (1941) 839–888.

- [8] J. M. GREENBERG, On the equilibrium configurations of compressible slender bars, *Arch. Rational Mech. Anal.*, **27** (1967) 181–194.
- [9] W. PIECHOCKI, On the nonlinear theory of thin elastic spherical shells, *Arch. Mech. Stos.*, **21** (1969) 81–102.
- [10] I. TADJBAKHSH & F. ODEH, Equilibrium states of elastic rings, *J. Math. Anal. Appl.*, **18** (1967) 59–74.
- [11] M. M. VAINBERG, *Variational methods for the study of nonlinear operators* (translated from Russian by A. Feinstein), Holden-Day, San Francisco, 1964.
- [12] J. H. WOLKOWISKY, Existence of buckled states of circular plates, *Comm. Pure Appl. Math.*, **20** (1967) 549–560.

University of Oxford

Date communicated: NOVEMBER 19, 1969