*Semi-Groups and a Class of Singular Perturbation Problems**

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Introduction. We assume that A_1 and A_2 are infinitesimal generators of strongly continuous (class (C_0)) semi-groups of bounded linear operators on a Banach space B , and P is a 2×2 matrix with real or complex entries. It follows that $G = \text{diagonal}(A_i) + P$ is the infinitesimal generator of a class (C_0) semi-group on $B \times B$ for which we obtain an explicit representation as a Bochner integral with a Bessel function kernel.

In Section 2 we specialize to

$$G^\epsilon = \frac{1}{\epsilon^{1/2}} \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} + \frac{1}{\epsilon} \begin{bmatrix} -a & a \\ a & -a \end{bmatrix}$$

with A the infinitesimal generator of a group of class (C_0) , and $a > 0$. Each component of the solution of the abstract Cauchy problem $d\bar{u}/dt = G^\epsilon \bar{u}$ satisfies the abstract telegraph equation $\epsilon u_{tt} + 2au_t = A^2 u$: we use semi-group theory to show that this solution converges as $\epsilon \rightarrow 0$ to the solution of $2au_t = A^2 u$ and to obtain a rate of convergence. This is done as follows: the resolvent of G^ϵ , $(\lambda I - G^\epsilon)^{-1}$ is shown to converge at a certain rate to the resolvent of

$$G = \frac{1}{4a} \begin{bmatrix} A^2 & A^2 \\ A^2 & A^2 \end{bmatrix}.$$

The rate of convergence of the resolvents is translated into a rate of convergence of the semi-groups $\exp(G^\epsilon t)$ to the semi-group $\exp(Gt)$ (Theorem 1) via a modified version of the Trotter-Kato Theorem. This in turn implies that the solutions converge (Theorem 2).

In Section 3 we assume A is a self-adjoint operator on a Hilbert space H ; iG^ϵ is then the generator of a unitary group of class (C_0) . The corresponding second order equation, $\epsilon u_{tt} + 2aiu_t + A^2 u$ is a transformed version of the Klein-Gordon

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equation of relativistic quantum mechanics, and the results of Section 2 carry over to establish a rate of convergence to the solution of the abstract Schrödinger Equation $2\alpha iu_t + A^2u = 0$.

Convergence in the Hilbert space case is contained in the recent general results of Friedman [1], who extended the work of Smoller [2]. The rate of convergence obtained here is new in the abstract setting, having been given previously for A^2 replaced by a second order elliptic differential operator by Zlamal [3]. Griego and Hersh [4] have shown weak convergence in the Banach space case using probabilistic arguments. Kac [5] considered a stochastic process in which a particle moving in one dimension alternates randomly between velocities v and $-v$ according to a Poisson process, and obtained the telegraph equation by averaging. The model was extended to a random alternation governed by a Markov chain between n velocity states by Pinsky [6], who proved a singular perturbation result for the equations governing the process. Griego and Hersh gave the model its natural setting by replacing the n velocity states by n semi-groups and showed by averaging the "random evolution" one gets a vector semi-group, a special case being the semi-group with generator G considered in Section 1.

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Section 1. Representation of a semi-group by a Bessel function kernel. In this section we assume that $T_1(t)$, $T_2(t)$ are two commuting semi-groups of class (C_0) acting on a Banach space B and $P = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is a matrix with complex entries. We let A_1 , A_2 denote the infinitesimal generators of $T_1(t)$, $T_2(t)$, respectively. According to Theorem 13.2.1. of Hille-Phillips [7; p. 389], the matrix operator diagonal $(A_i) + P$ is the infinitesimal generator of a semi-group of class (C_0) , $S(t)$ on the Banach space $B \times B$. We proceed to obtain an explicit representation for $S(t)$.

In general we have from the above theorem that if A is the infinitesimal generator of a class (C_0) semi-group $T(t)$ and Q is a bounded linear operator, then $A + Q$ is the infinitesimal generator of a class (C_0) semi-group $S(t)$ given by

$$S(t) = \sum_{n=0}^{\infty} S_n(t)$$

where

$$S_0(t) = T(t),$$

$$S_n(t) = \int_0^t T(u)Q S_{n-1}(t-u) du = (TQ * S_{n-1})(t) \quad n \geq 1,$$

and $*$ denotes convolution.

In order to apply this we define

$$L_1(t) = e^{at}T_1(t), \quad L_2(t) = e^{bt}T_2(t).$$

Then

$$L(t) = \begin{bmatrix} L_1(t) & 0 \\ 0 & L_2(t) \end{bmatrix}$$

has generator

$$\begin{bmatrix} A_1 + a & 0 \\ 0 & A_2 + b \end{bmatrix}$$

so that we take Q to be $\begin{bmatrix} 0 & a \\ \beta & 0 \end{bmatrix}$.

$$S_1 = LQ*L = \begin{bmatrix} 0 & \alpha L_1 \\ \beta L_2 & 0 \end{bmatrix} * \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} = \begin{bmatrix} 0 & \alpha L_1 * L_2 \\ \beta L_2 * L_1 & 0 \end{bmatrix}$$

Using the fact that for commuting semi-groups convolution is a commutative operation, and denoting the n -fold convolution of an operator A with itself by A^{*n} we find by induction for $1 \leq n < \infty$,

$$\begin{aligned} S_{2n}(t) &= (\alpha\beta)^n \begin{bmatrix} L_1^{*n+1} * L_2^{*n} & 0 \\ 0 & L_1^{*n} * L_2^{*n+1} \end{bmatrix} (t), \\ S_{2n-1}(t) &= (\alpha\beta)^{n-1} \begin{bmatrix} 0 & \alpha L_1^{*n} * L_2^{*n} \\ \beta L_1^{*n} * L_2^{*n} & 0 \end{bmatrix} (t). \end{aligned}$$

Now

$$(L_1 * L_1)(t)f = \int_0^t L_1(u)L_1(t-u)f \, du = tL_1(t)f,$$

and in general

$$L_1^{*n}(t)f = \frac{t^{n-1}}{(n-1)!} L_1(t)f.$$

We shall write $\tilde{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ for elements of $B \times B$, where we identify $\tilde{f}_1 = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$ with the element f_1 of B . With this convention we have

$$S(t) = \begin{Bmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{Bmatrix} = \begin{Bmatrix} L_1(t) & 0 \\ 0 & L_2(t) \end{Bmatrix} + \begin{Bmatrix} S'_{11}(t) & S_{12}(t) \\ S_{21}(t) & S'_{22}(t) \end{Bmatrix},$$

where

$$\begin{aligned} S'_{11}(t) &= \sum_{n=1}^{\infty} \int_0^t (\alpha\beta)^n \frac{u^n}{n!} \frac{(t-u)^{n-1}}{(n-1)!} L_1(u)L_2(t-u) \, du, \\ S_{12}(t) &= \sum_{n=1}^{\infty} \int_0^t \alpha^n \beta^{n-1} \frac{(u(t-u))^{n-1}}{(n-1)!^2} L_1(u)L_2(t-u) \, du, \\ S_{21}(t) &= \sum_{n=1}^{\infty} \int_0^t \alpha^{n-1} \beta^n \frac{(u(t-u))^{n-1}}{(n-1)!^2} L_1(u)L_2(t-u) \, du, \end{aligned}$$

$$S'_{22}(t) = \sum_{n=1}^{\infty} \int_0^t (\alpha\beta)^n \frac{u^{n-1}(t-u)^n}{(n-1)!n!} L_1(u)L_2(t-u) du.$$

These integrals are to be understood symbolically because they must be taken in the strong operator topology. We shall, however, omit the elements of B whenever convenient.

The power series converge uniformly for $u \in [0, t]$ in the strong operator topology so that we may interchange summation and integration to get

$$(1) \quad S(t) = \begin{Bmatrix} L_1(t) & 0 \\ 0 & L_2(t) \end{Bmatrix} + \begin{Bmatrix} S'_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{Bmatrix},$$

where

$$S'_{11}(t) = (\alpha\beta)^{1/2} \int_0^t \left(\frac{u}{t-u} \right)^{1/2} I_1(2(\alpha\beta u(t-u))^{1/2}) L_1(u)L_2(t-u) du,$$

$$S_{12}(t) = \alpha \int_0^t I_0(2(\alpha\beta u(t-u))^{1/2}) L_1(u)L_2(t-u) du,$$

$$S_{21}(t) = \beta \int_0^t I_0(2(\alpha\beta u(t-u))^{1/2}) L_1(u)L_2(t-u) du,$$

$$S'_{22} = (\alpha\beta)^{1/2} \int_0^t \left(\frac{t-u}{u} \right)^{1/2} I_1(2(\alpha\beta u(t-u))^{1/2}) L_1(u)L_2(t-u) du.$$

Introducing the notation

$$K_1(u, t) = \left(\frac{u}{t-u} \right)^{1/2} I_1(2(\alpha\beta u(t-u))^{1/2}),$$

$$K_0(u, t) = I_0(2(\alpha\beta u(t-u))^{1/2}),$$

$$K(u, t) = \begin{bmatrix} (\alpha\beta)^{1/2} K_1(u, t) & \alpha K_0(u, t) \\ \beta K_0(u, t) & (\alpha\beta)^{1/2} K_1(t-u, u) \end{bmatrix},$$

$$M(u, t) = \begin{bmatrix} L_1(u)L_2(t-u) & 0 \\ 0 & L_1(u)L_2(t-u) \end{bmatrix},$$

we can write the semi-group compactly as

$$S(t)\tilde{f} = L(t)\tilde{f} + \int_0^t K(u, t)M(u, t)\tilde{f} du.$$

Section 2. Convergence in Banach space to solutions of an abstract heat equation. In this section we shall specialize the results of Section 1 and obtain a convergence theorem for the semi-groups to be considered.

Let $T(t)$ be a group of class (C_0) on a Banach space B ,

(0) $\|T(t)\| \leq M \exp(w|t|), -\infty < t < \infty$, with infinitesimal generator A .

Let for $t \geq 0$

$$T_1(t) = T(t), T_2(t) = T^{-1}(t) = T(-t), P = \begin{bmatrix} -a & a \\ a & -a \end{bmatrix}, \quad a > 0$$

and denote by $S(t)$ the vector semi-group with infinitesimal generator

$$\begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} + \begin{bmatrix} -a & a \\ a & -a \end{bmatrix}.$$

Using the representation developed in Section 1 we can prove

Lemma 1. $\|S(t)\|_m \leq 2M \exp [((a^2 + w^2)^{1/2} - a)t]$ where $\|\cdot\|_m$ denotes the matrix norm $\max_{1 \leq i \leq 2} \sum_{j=1}^2 \|a_{ij}\|$ compatible with the vector norm $\|f\|_{B \times B} = \sum_{i=1}^2 \|f_i\|$.

Proof. Let

$$S(t) = \begin{bmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{bmatrix}.$$

From equation (1) of the preceding section

$$S_{11}(t)f = e^{-at} \left(T(t) + a \int_0^t \left(\frac{u}{t-u} \right)^{1/2} I_1(2a(u(t-u))^{1/2}) T(u) T(u-t)f \, du \right).$$

On making the change of variables $y = 2u - t$ this becomes

$$S_{11}(t)f = e^{-at} \left(T(t) + \frac{1}{2} a \int_{-t}^t \left(\frac{t+y}{t-y} \right)^{1/2} I_1(a(t^2 - y^2)^{1/2}) T(y)f \, dy \right).$$

Then, since the kernel is non-negative

$$\|S_{11}(t)f\| \leq e^{-at} \left(\|T(t)f\| + \frac{a}{2} \int_{-t}^t \left(\frac{t+y}{t-y} \right)^{1/2} I_1(a(t^2 - y^2)^{1/2}) \|T(y)f\| \, dy \right).$$

To estimate this integral note that (0) implies

$$\|T(y)f\| \leq 2M \cosh(wy) \|f\|.$$

Substituting this in the preceding inequality and using the fact that $\cosh(wy)$ is even gives

$$\begin{aligned} \|S_{11}(t)f\| &\leq 2M \|f\| e^{-at} \left(\cosh(wt) \right. \\ &\quad \left. + a \int_0^t \frac{t}{(t^2 - y^2)^{1/2}} I_1(a(t^2 - y^2)^{1/2}) \cosh(wy) \, dy \right). \end{aligned}$$

The expression in parentheses may be evaluated using a table of Laplace transforms [8; p. 9], or by using the series definitions of I_1 and $\cosh$ and integrating term by term. One finds

$$\cosh (wt) + a \int_0^t \frac{t}{((t^2 - y^2)^{1/2})} I_1(a(t^2 - y^2)^{1/2}) \cosh (wy) dy \\ = \cosh ((a^2 + w^2)^{1/2}t),$$

yielding the estimate

$$||S_{11}(t)|| \leq 2Me^{-at} \cosh ((a^2 + w^2)^{1/2}t).$$

The identical estimate is valid for $||S_{22}(t)||$ since

$$S_{22}(t)f = e^{-at} \left(T(-t) + \frac{1}{2}a \int_{-t}^t \left(\frac{t+y}{t-y} \right)^{1/2} I_1(a(t^2 - y^2)^{1/2}) T(-y)f dy \right),$$

and

$$||T(-y)f|| \leq 2M \cosh (wy) ||f||.$$

Using the same procedure we find

$$||S_{12}(t)f|| = ||S_{21}(t)f|| \leq 2Ma ||f|| e^{-at} \int_0^t I_0(a(t^2 - y^2)^{1/2}) \cosh (wy) dy \\ = 2M \frac{a}{(a^2 + w^2)^{1/2}} e^{-at} \sinh ((a^2 + w^2)^{1/2}t).$$

Combining these estimates

$$||S(t)||_m \leq 2Me^{-at} \left(\cosh ((a^2 + w^2)^{1/2}t) + \frac{a}{(a^2 + w^2)^{1/2}} \sinh ((a^2 + w^2)^{1/2}t) \right) \\ \leq 2Me^{-at} \exp ((a^2 + w^2)^{1/2}t),$$

which proves Lemma 1.

We wish to consider the family of semi-groups $S^\epsilon(t)$ depending on the positive parameter ϵ whose infinitesimal generator is

$$G^\epsilon = \frac{1}{\epsilon^{1/2}} \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} + \frac{1}{\epsilon} \begin{bmatrix} -a & a \\ a & -a \end{bmatrix}.$$

Now $S^\epsilon(t)$ can be obtained from $S(t)$ by replacing $T(t)$ by $T(t/\epsilon^{1/2})$ and a by a/ϵ . Recalling that $||T(t/\epsilon^{1/2})|| \leq M \exp (w |t|/\epsilon^{1/2})$ by hypothesis, we can obtain an estimate for $||S^\epsilon(t)||$ by replacing w by $w/\epsilon^{1/2}$ and a by a/ϵ in the preceding inequality. This yields

$$||S^\epsilon(t)||_m \leq 2M \exp \left(\frac{1}{\epsilon} ((a^2 + \epsilon w^2)^{1/2} - a)t \right).$$

An elementary calculation shows that $(1/\epsilon)[(a^2 + \epsilon w^2)^{1/2} - a] \leq w^2/2a$ for all $\epsilon > 0$. Therefore

$$(1) \quad ||S^\epsilon(t)||_m \leq 2M \exp (w^2t/2a).$$

We shall now calculate the resolvent of the infinitesimal generator G^ϵ of the semi-group $S^\epsilon(t)$, and show that it converges as $\epsilon \rightarrow 0$ to a bounded linear operator

whose restriction to the "diagonal" subspace $V = \{f: f \in B\}$ is the resolvent of a closed linear operator.

Formally we have

$$(\lambda I - G^\epsilon)^{-1} = \begin{bmatrix} \epsilon\lambda + \epsilon^{1/2}A + a & a \\ a & \epsilon\lambda - \epsilon^{1/2}A + a \end{bmatrix} (\epsilon\lambda^2 + 2a\lambda - A^2)^{-1}.$$

By hypothesis the spectrum of A is contained in the strip $-w < \operatorname{Re}(\lambda) < w$. It follows from the Spectral Mapping Theorem that the spectrum of A^2 is contained in the half-plane $\{\lambda: \operatorname{Re}(\lambda) < w^2\}$. Consequently, for $\lambda > w^2/2a$, $(\epsilon\lambda^2 + 2a\lambda - A^2)^{-1}$ exists as a bounded linear operator, and the preceding equation gives the resolvent of G^ϵ . It is apparent that

$$\lim_{\epsilon \rightarrow 0} (\lambda I - G^\epsilon)^{-1} = \begin{bmatrix} a & a \\ a & a \end{bmatrix} (2a\lambda - A^2)^{-1} \equiv D.$$

In fact,

$$\begin{aligned} \|(\lambda I - G^\epsilon)^{-1} - D\| &\leq \left\| \begin{bmatrix} a & a \\ a & a \end{bmatrix} \right\|_m \|(\epsilon\lambda^2 + 2a\lambda - A^2)^{-1} - (2a\lambda - A^2)^{-1}\| \\ &\quad + \left\| \begin{bmatrix} \epsilon\lambda + \epsilon^{1/2}A & 0 \\ 0 & \epsilon\lambda - \epsilon^{1/2}A \end{bmatrix} (\epsilon\lambda^2 + 2a\lambda - A^2)^{-1} \right\|_m. \end{aligned}$$

By the resolvent equation

$$\begin{aligned} (\epsilon\lambda^2 + 2a\lambda - A^2)^{-1} - (2a\lambda - A^2)^{-1} &= -\epsilon\lambda^2(\epsilon\lambda^2 + 2a\lambda - A^2)^{-1}(2a\lambda - A^2)^{-1} \\ &= \frac{\epsilon\lambda^2}{2u} [(-u - A)^{-1} - (u - A)^{-1}](2a\lambda - A^2)^{-1}, \end{aligned}$$

where

$$u = (\epsilon\lambda^2 + 2a\lambda)^{1/2} > b = (2a\lambda)^{1/2}.$$

By the Hille-Yosida-Phillips Theorem

$$\|(\alpha - A)^{-1}\| \leq \frac{M}{|\alpha| - w} \quad \text{for } \alpha \text{ real, } |\alpha| > w.$$

Thus $\|(\epsilon\lambda^2 + 2a\lambda - A^2)^{-1}\| \leq [M/u(u - w)] \leq [M/b(b - w)]$. And, letting $\epsilon \rightarrow 0$, we obtain

$$\|(2a\lambda - A^2)^{-1}\| \leq \frac{M}{b(b - w)}.$$

Therefore,

$$(2) \quad \left\| \begin{bmatrix} a & a \\ a & a \end{bmatrix} \right\|_m \|(\epsilon\lambda^2 + 2a\lambda - A^2)^{-1} - (2a\lambda - A^2)^{-1}\| \leq \epsilon \frac{M^2\lambda}{(b - w)^2}.$$

Now $A(u^2 - A^2)^{-1} = \frac{1}{2} [(-u - A)^{-1} + (u - A)^{-1}]$ so that $\|A(u^2 - A^2)^{-1}\| \leq M/(b - w)$. Consequently, for the second term we have the estimate

$$(3) \quad \left\| \begin{bmatrix} \epsilon\lambda + \epsilon^{1/2}A & 0 \\ 0 & \epsilon\lambda - \epsilon^{1/2}A \end{bmatrix} (\epsilon\lambda^2 + 2a\lambda - A^2)^{-1} \right\|_m \leq \epsilon \frac{M\lambda}{b(b-w)} + \epsilon^{1/2} \frac{M}{b-w}.$$

Combining (2) and (3) gives

$$(4) \quad \|(\lambda I - G')^{-1} - D\|_m \leq \epsilon \frac{M^2\lambda}{(b-w)^2} + \epsilon \frac{M\lambda}{b(b-w)} + \epsilon^{1/2} \frac{M}{b-w}.$$

It may be verified by multiplication that $D = \begin{bmatrix} a & a \\ a & a \end{bmatrix} (2a\lambda - A^2)^{-1}$ is the resolvent of the restriction of $G = (1/4a) \begin{bmatrix} A^* & A^* \\ A^* & A^* \end{bmatrix}$ to the "diagonal" V of $B \times B$.

According to the Hille-Yosida-Phillips Theorem the estimate $\|S'(t)\|_m \leq 2M \exp(w^2 t/2a)$ implies

$$\|(\lambda I - G')^{-n}\|_m \leq \frac{2M}{\left(\lambda - \frac{w^2}{2a}\right)^n} \quad \lambda > \frac{w^2}{2a}, \quad n \geq 1.$$

Going to the limit yields

$$\|D^{-n}\|_m \leq \frac{2M}{\left(\lambda - \frac{w^2}{2a}\right)^n} \quad \lambda > \frac{w^2}{2a}, \quad n \geq 1.$$

Noting that G maps the subspace V into itself we see again from the Hille-Yosida-Phillips Theorem that the restriction of G to V is the generator of a class (C_0) semi-group such that

$$(5) \quad \|\exp(Gt)\|_m \leq 2M \exp(w^2 t/2a).$$

From this we can deduce that A^2 is the generator of a semi-group of class (C_0) . In fact, $2aG$ generates a class (C_0) semi-group $\tilde{Q}(t)$ such that $\|\tilde{Q}(t)\|_m \leq 2M \exp(w^2 t)$. Using the representation

$$\tilde{Q}(t) = \begin{bmatrix} Q(t) & Q(t) \\ Q(t) & Q(t) \end{bmatrix},$$

we have from the semi-group property that

$$2Q(t)Q(s) = Q(t+s), \quad \lim_{t \rightarrow 0^+} \|Q(t)f - \frac{1}{2}f\| = 0 \quad \text{for all } f \in B,$$

$$\lim_{t \rightarrow 0^+} \frac{1}{t} [Q(t)g - \frac{1}{2}g] = \frac{1}{2}A^2g \quad \text{for } g \in \mathfrak{D}(A^2), \quad \text{and}$$

$$\|Q(t)\| \leq M \exp(w^2 t).$$

Thus $Q_1(t) = 2Q(t)$ is a semi-group of class (C_0) with generator A^2 . This last result can also be obtained directly from an example in Dunford and Schwartz [9; pp. 639–640].

We can now use a form of the Trotter–Kato Theorem to obtain convergence of the family of semi-groups $\{\exp(G^*t)\}$ to the semi-group $\exp(Gt)$. For two semi-groups $\exp(S_1t)$, $\exp(S_2t)$ of class (C_0) we have the identity [10; p. 501] valid for all $f \in B$:

$$(6) \quad (\lambda - S_1)^{-1}[\exp(S_2t) - \exp(S_1t)](\lambda - S_2)^{-1}f \\ = \int_0^t \exp(S_1(t-s))[(\lambda - S_2)^{-1} - (\lambda - S_1)^{-1}] \exp(S_2s)f \, ds.$$

Remark. The same identity holds if we only assume that the restriction of $\exp(S_2t)$ to B_1 is a (C_0) semi-group mapping B_1 into itself, where B_1 is a closed subspace of B , provided we assume $f \in B_1$, as may be seen by examining the proof of (6). This remark applies equally to the two formulas which follow.

Now assume that $w \in \mathfrak{D}(S_2^2)$ and replace f by $(\lambda - S_2)^2w$ in (6). Using the algebraic identity

$$(\lambda - S_1)^{-1}[\exp(S_2t) - \exp(S_1t)](\lambda - S_2)w = [\exp(S_2t) - \exp(S_1t)]w \\ + (\exp(S_1t)[(\lambda - S_2)^{-1} - (\lambda - S_1)^{-1}] \\ - [(\lambda - S_2)^{-1} - (\lambda - S_1)^{-1}] \exp(S_2t))(\lambda - S_2)w$$

and transposing, we obtain from (6)

$$(7) \quad [\exp(S_2t) - \exp(S_1t)]w = -(\exp(S_1t)[(\lambda - S_2)^{-1} - (\lambda - S_1)^{-1}] \\ - [(\lambda - S_2)^{-1} - (\lambda - S_1)^{-1}] \exp(S_2t))(\lambda - S_2)w \\ + \int_0^t \exp(S_1(t-s))[(\lambda - S_2)^{-1} - (\lambda - S_1)^{-1}] \exp(S_2s)(\lambda - S_2)^2w \, ds.$$

Now assume that $\|S_1(t)\| \leq C \exp(\alpha t)$, $\|S_2(t)\| \leq C \exp(\alpha t)$, and $\|(\lambda - S_2)^{-1} - (\lambda - S_1)^{-1}\| \leq K(\lambda)$. Then (7) yields the estimate

$$(8) \quad \|[\exp(S_2t) - \exp(S_1t)]w\| \leq 2C \exp(\alpha t)K(\lambda) \|(\lambda - S_2)w\| \\ + Ct \exp(\alpha t)K(\lambda) \|(\lambda - S_2)^2w\|.$$

To apply this we take

$$S_2 = G \equiv \frac{1}{4a} \begin{bmatrix} A^2 & A^2 \\ A^2 & A^2 \end{bmatrix}, \quad S_1 = G^*,$$

and

$$w = \tilde{g} = \begin{bmatrix} g \\ g \end{bmatrix} \in \mathfrak{D}(A^4) \times \mathfrak{D}(A^4) \subset V.$$

According to the remark, equation (8) is valid for these choices. Combining the results of equations (1), (4), and (5) with (8) yields

$$(9) \quad \|(\exp(G^\epsilon t) - \exp(Gt))\tilde{g}\|_{B \times B} \leq \epsilon^{1/2} K(t+1) \exp(tw^2/2a) \left(\sum_{k=0}^2 \|A^{2k}g\| \right)$$

where we have chosen a convenient value for $\lambda > w^2/2a$ and absorbed the dependence on λ into the constant K . We can state this as a theorem:

Theorem 1. *On the subspace V of $B \times B$, $\exp(G^\epsilon t)$ converges strongly to $\exp(Gt)$, uniformly for t in a bounded interval.*

Proof. The preceding inequality shows that the conclusions of the theorem hold for each $\tilde{g} \in \mathfrak{D}(A^4) \times \mathfrak{D}(A^4)$. $\mathfrak{D}(A^4)$ is dense in B in virtue of A being an infinitesimal generator of a class (C_0) semi-group. Hence $D(A^4) \times D(A^4)$ is dense in V , and the convergence on all of V follows from the boundedness of $\exp(G^\epsilon t)$ and $\exp(Gt)$ for each t . The uniform boundedness of the families $\exp(G^\epsilon t)$ such intervals, completing the proof.

For $g \in \mathfrak{D}(A^2)$, $\tilde{u}^\epsilon(t) = \exp(G^\epsilon t)\tilde{g}$ and $\tilde{u}(t) = \exp(Gt)\tilde{g}$ are, respectively, the unique solutions of the systems

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} u_1^\epsilon \\ u_2^\epsilon \end{pmatrix} &= \begin{pmatrix} \frac{1}{\epsilon^{1/2}} \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} + \frac{1}{\epsilon} \begin{bmatrix} -a & a \\ a & -a \end{bmatrix} \end{pmatrix} \begin{pmatrix} u_1^\epsilon \\ u_2^\epsilon \end{pmatrix}, & \tilde{u}^\epsilon(0) &= \tilde{g}, \\ \frac{d}{dt} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \frac{1}{4a} \begin{bmatrix} A^2 & A^2 \\ A^2 & A^2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, & \tilde{u}(0) &= \tilde{g}. \end{aligned}$$

It is apparent that $u_1(t) = u_2(t) = u(t)$, say, and that $u(t)$ satisfies

$$(10) \quad \begin{aligned} u_\epsilon &= \frac{1}{2a} A^2 u, \\ u(0) &= g \in D(A^2). \end{aligned}$$

Hence, Theorem 1 implies that each component $u_i^\epsilon(t)$ of $\tilde{u}^\epsilon(t)$ converges strongly to the solution $u(t)$ of (10). If we let $\tilde{V}^\epsilon(t) = \exp(G^\epsilon t)\tilde{h}$, $h \in D(A^2)$, then a solution of the initial value problem

$$\epsilon w_{i\epsilon}' = A^2 w^\epsilon - 2a w_i^\epsilon, w^\epsilon(0) = g, w_i^\epsilon(0) = Ah, g, h \in \mathfrak{D}(A^2)$$

is given by

$$w^\epsilon = \frac{1}{2}(u_1^\epsilon + u_2^\epsilon) + \frac{1}{2}\epsilon^{1/2}(V_1^\epsilon - V_2^\epsilon).$$

Now assume that $g \in D(A^4)$. If we write

$$w^\epsilon - u = \frac{1}{2}[(u_1^\epsilon - u) + (u_2^\epsilon - u)] + \frac{1}{2}\epsilon^{1/2}[(v_1^\epsilon - u) - (v_2^\epsilon - u)],$$

then it follows readily from (9) that $\|w^\epsilon - u\| = O(\epsilon^{1/2})$. However, by a closer

examination this can be improved to $\|w^\epsilon - u\| = O(\epsilon)$. We write

$$(\lambda - G)^{-1} - (\lambda - G^\epsilon)^{-1} \equiv R_1 + R_2,$$

where

$$R_1 \equiv \epsilon \lambda^2 \begin{bmatrix} a & a \\ a & a \end{bmatrix} (\epsilon \lambda^2 + 2a\lambda - A^2)^{-1} (2a\lambda - A^2)^{-1} \\ - \epsilon \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} (\epsilon \lambda^2 + 2a\lambda - A^2)^{-1},$$

$$R_2 \equiv -\epsilon^{1/2} \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} (\epsilon \lambda^2 + 2a\lambda - A^2)^{-1}.$$

We first estimate $(u_1^\epsilon - u) + (u_2^\epsilon - u)$ using the identity (7) again with $S_1 = G^\epsilon$, $S_2 = G$, and $w = \tilde{g}$, only now we regard (7) as two equations and add them. The terms involving R_1 can be estimated by $\epsilon C_1(t+1) \exp(tw^2/2a) \sum_{k=0}^2 \|A^{2k}g\|$ just as in (8). In matrix form the terms involving R_2 are

$$(11a) \quad \exp(G^\epsilon t) R_2 (\lambda - G) \tilde{g},$$

$$(11b) \quad -R_2 \exp(G^\epsilon t) (\lambda - G) \tilde{g},$$

$$(11c) \quad \int_0^t \exp(G^\epsilon(t-s)) R_2 \exp(Gs) (\lambda - G)^2 \tilde{g} ds.$$

If we denote the elements of $\exp(G^\epsilon t)$ by $S_{ij}^\epsilon(t)$, then it follows from (1) of Section 1 that $S_{12}^\epsilon(t) \equiv S_{21}^\epsilon(t)$. Also let $A_1 \equiv (\lambda - 1/2aA^2)$ and $B \equiv A(\epsilon\lambda^2 + 2a\lambda - A^2)^{-1}$. After multiplying out (11a), (11b), and (11c) and adding the components of each we obtain for the three terms

$$(11a') \quad -\epsilon^{1/2} (S_{11}^\epsilon(t) - S_{22}^\epsilon(t)) B A_1 g$$

$$(11b') \quad 0$$

$$(11c') \quad -\epsilon^{1/2} \int_0^t (S_{11}^\epsilon(t-s) - S_{22}^\epsilon(t-s)) B \exp(A^2 s/2a) A_1 g ds$$

where in (11c') we have used the fact that $\exp(Gs) |_{\mathcal{V}}$ has the representation

$$\begin{bmatrix} \exp(A^2 s/2a) & 0 \\ 0 & \exp(A^2 s/2a) \end{bmatrix}.$$

Since B is bounded and $\text{Range } B \subset D(A)$, to show $\|u_1^\epsilon - u + u_2^\epsilon - u\| = O(\epsilon)$ it will suffice to show $\|[S_{11}^\epsilon(t) - S_{22}^\epsilon(t)]f\| = O(\epsilon^{1/2})$ for all $f \in D(A)$. Now from Section 1 we have

$$(12) \quad [S_{11}(t) - S_{22}(t)]f = \exp(-at)[T(t) - T(-t)]f \\ + a \int_{-t}^t \left(\frac{t+y}{t-y}\right)^{1/2} I_1(a(t^2 - y^2)^{1/2}) [T(y) - T(-y)]f dy$$

$$\begin{aligned}
&= \exp(-at)[T(t) - T(-t)]f \\
&\quad + a \int_0^t \left(\frac{y^2}{t^2 - y^2} \right)^{1/2} I_1(a(t^2 - y^2)^{1/2}) [T(y) - T(-y)]f dy.
\end{aligned}$$

Since $f \in D(A)$ the formula $T(t)f - f = \int_0^t T(u)Af du$ is valid and leads to $[T(t) - T(-t)]f = \int_0^t [T(u) + T(-u)]Af du$ from which we obtain the estimate (using $\|T(t)\| \leq M \exp(w|t|)$)

$$(13) \quad \|[T(t) - T(-t)]f\| \leq 2M \|Af\|/w \sinh wt.$$

Inserting this into (12) after taking norms yields

$$\begin{aligned}
\|[S_{11}(t) - S_{22}(t)]f\| &\leq 2M \|Af\|/w \exp(-at) \\
&\quad \cdot \left(\sinh wt + a \int_0^t \left(\frac{y^2}{t^2 - y^2} \right)^{1/2} I_1(a(t^2 - y^2)^{1/2}) \sinh wy dy \right) \\
&= 2M \|Af\|/w \exp(-at) \left(\frac{w^2}{a^2 + w^2} \right)^{1/2} \sinh((a^2 + w^2)^{1/2}t)
\end{aligned}$$

as may be seen by termwise integration. Making the substitutions $a \rightarrow a/\epsilon$, $w \rightarrow w/\epsilon^{1/2}$, and $A \rightarrow A/\epsilon^{1/2}$ we obtain

$$\begin{aligned}
\|[S'_{11}(t) - S'_{22}(t)]f\| &\leq 2M \|Af\| \left(\frac{\epsilon}{a^2 + \epsilon w^2} \right)^{1/2} \exp(-at/\epsilon) \sinh((a^2 + \epsilon w^2)^{1/2}t/\epsilon) \\
&\leq 2M \|Af\| \epsilon^{1/2}/a \exp(w^2t/2a).
\end{aligned}$$

This shows the terms involving R_2 can also be estimated by $\epsilon C_2(t+1) \exp(w^2t/2a) \sum_{k=0}^2 \|A^{2k}g\|$ for a constant C_2 . Combining the estimates for R_1 and R_2 gives

$$(14) \quad \|u_1^\epsilon - u + u_2^\epsilon - u\| \leq \epsilon(C_1 + C_2)(t+1) \exp(w^2t/2a) \sum_{k=0}^2 \|A^{2k}g\|.$$

It remains to estimate $\frac{1}{2}\epsilon^{1/2}(V_1^\epsilon - V_2^\epsilon)$. Since

$$\begin{pmatrix} V_1^\epsilon \\ V_2^\epsilon \end{pmatrix} = \exp(S^\epsilon t) \begin{pmatrix} h \\ h \end{pmatrix}$$

we have

$$(15) \quad \|V_1^\epsilon - V_2^\epsilon\| = \|[S'_{11}(t) - S'_{22}(t)]h\| \leq 2M \|Ah\| \epsilon^{1/2}/a \exp(w^2t/2a).$$

The above results can be summarized as

Theorem 2. *The semi-group solution of*

$$\epsilon w_{it}^\epsilon + 2aw_t^\epsilon = A^2w^\epsilon \quad w^\epsilon(0) = g, \quad w_t^\epsilon(0) = Ah, \quad g, h \in \mathfrak{D}(A^2)$$

converges strongly to the unique solution of

$$2au_t = A^2u \quad u(0) = g.$$

If we assume additionally that $g \in D(A^4)$ we have the estimate

$$\|w^\epsilon - u\| \leq \epsilon C(t+1) \exp(w^2 t/2a)(\|Ah\| + \sum_{k=0}^2 \|A^{2k}g\|).$$

Proof. The estimate follows by combining (14) and (15). Strong convergence follows from the estimate as in Theorem 1. Uniqueness holds for the first order equation because A^2 is the infinitesimal generator of a class (C_0) semi-group.

Section 3. Convergence in Hilbert space to solutions of an abstract Schrödinger equation. Let A be a self-adjoint operator on a Hilbert space H and let $a \neq 0$ be a real number. Then the matrix operator

$$\begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} + \begin{bmatrix} -a & a \\ a & -a \end{bmatrix}$$

on $H \times H$ is self-adjoint. It is known that if L is of the form $L = iT$, where T is self-adjoint, then L generates a group of unitary operators. Consequently,

$$iG^\epsilon = \frac{i}{\epsilon^{1/2}} \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} + \frac{i}{\epsilon} \begin{bmatrix} -a & a \\ a & -a \end{bmatrix}$$

generates a unitary group on $H \times H$ for each $\epsilon > 0$. For the resolvent of iG^ϵ we have

$$\begin{aligned} (\lambda - iG^\epsilon)^{-1} &= -i(-i\lambda - G^\epsilon)^{-1} \quad \text{for } \lambda \text{ real, } \lambda \neq 0 \\ &= -i \begin{bmatrix} -i\epsilon\lambda + \epsilon^{1/2}A + a & a \\ a & -i\epsilon\lambda - \epsilon^{1/2}A + a \end{bmatrix} (-\epsilon\lambda^2 - 2ia\lambda - A^2)^{-1}, \end{aligned}$$

as in the preceding section. Now A^2 is self-adjoint as a consequence of A being self-adjoint, and hence $(-\epsilon\lambda^2 - 2ia\lambda - A^2)^{-1}$ exists since $\epsilon\lambda^2 + 2ia\lambda$ has non-zero imaginary part.

As before denoting $i \begin{bmatrix} a & a \\ a & -a \end{bmatrix} (2ia\lambda + A^2)^{-1}$ by D we have

$$\begin{aligned} (1) \quad & \|(\lambda - iG^\epsilon)^{-1} - D\|_{H \times H} \\ & \leq \left\| \begin{bmatrix} a & a \\ a & a \end{bmatrix} [(\epsilon\lambda^2 + 2ia\lambda + A^2)^{-1} - (2ia\lambda + A^2)^{-1}] \right\| \\ & \quad + \left\| \begin{bmatrix} -i\epsilon\lambda + \epsilon^{1/2}A & 0 \\ 0 & -i\epsilon\lambda - \epsilon^{1/2}A \end{bmatrix} (\epsilon\lambda^2 + 2ia\lambda + A^2)^{-1} \right\|. \end{aligned}$$

By the resolvent equation

$$(\epsilon\lambda^2 + 2ia\lambda + A^2)^{-1} - (2ia\lambda + A^2)^{-1} = -\epsilon\lambda^2(\epsilon\lambda^2 + 2ia\lambda + A^2)^{-1}(2ia\lambda + A^2)^{-1}.$$

For a general self-adjoint operator T with resolvent $R(\zeta, T)$ the following estimate holds

$$\|R(\zeta, T)\| = 1/\text{dist}(\zeta, \tau(T)) \leq |\text{Im } \zeta|^{-1},$$

where $\tau(T)$ denotes the spectrum of T . Consequently, for λ real, $\lambda \neq 0$,

$$\| -\epsilon\lambda^2(\epsilon\lambda^2 + 2ia\lambda + A^2)^{-1}(2ia\lambda + A^2)^{-1} \|_H \leq \epsilon\lambda^2/(2a\lambda)^2$$

and

$$\left\| \begin{bmatrix} a & a \\ a & a \end{bmatrix} [(\epsilon\lambda^2 + 2ia\lambda + A^2)^{-1} - (2ia\lambda + A^2)^{-1}] \right\|_{H \times H} \leq \frac{\epsilon}{|2a|}.$$

According to the functional calculus as developed in Riesz and Nagy [11; pp. 341-351], $(-i\epsilon\lambda + \epsilon^{1/2}A)(\epsilon\lambda^2 + 2ia\lambda + A^2)^{-1}$ defines a bounded linear operator such that

$$\|(-i\epsilon\lambda + \epsilon^{1/2}A)(\epsilon\lambda^2 + 2ia\lambda + A^2)^{-1}f\|^2 = \int_{-\infty}^{\infty} \left| \frac{-i\epsilon\lambda + \epsilon^{1/2}x}{\epsilon\lambda^2 + 2ia\lambda + x^2} \right|^2 d(E_x f, f)$$

where $\{E_x\}$ is the spectral family of A . It follows that

$$\begin{aligned} & \|(-i\epsilon\lambda + \epsilon^{1/2}A)(\epsilon\lambda^2 + 2ia\lambda + A^2)^{-1}\| \\ & \leq \max_{x \in (-\infty, \infty)} \left| \frac{-i\epsilon\lambda + \epsilon^{1/2}x}{\epsilon\lambda^2 + 2ia\lambda + x^2} \right| = \left(\frac{\epsilon}{4|a\lambda|} \right)^{1/2}, \end{aligned}$$

fixing $\lambda \neq 0$ and taking ϵ small that $2|a| > \epsilon|\lambda|$. The identical estimate holds for $\|(-i\epsilon\lambda - \epsilon^{1/2}A)(\epsilon\lambda^2 + 2ia\lambda + A^2)^{-1}\|$. Thus

$$\begin{aligned} (3) \quad & \left\| \begin{bmatrix} -i\epsilon\lambda + \epsilon^{1/2}A & 0 \\ 0 & -i\epsilon\lambda - \epsilon^{1/2}A \end{bmatrix} (\epsilon\lambda^2 + 2ia\lambda + A^2)^{-1} \right\|_{H \times H} \\ & \leq \left(\frac{\epsilon}{4a\lambda} \right)^{1/2}. \end{aligned}$$

Combining (2) and (3) with (1) yields

$$\|(\lambda I - iG^*)^{-1} - D\|_{H \times H} \leq \frac{\epsilon}{|2a|} + \left(\frac{\epsilon}{4a\lambda} \right)^{1/2}.$$

It can be shown exactly as in Section 2 that $i \begin{bmatrix} a & a \\ a & a \end{bmatrix} (2ia\lambda + A^2)^{-1}$ is the resolvent of the restriction of $iG = i/4a \begin{bmatrix} A^4 & A^4 \\ A^4 & A^4 \end{bmatrix}$ to the diagonal V of $H \times H$. Since A^2 is self-adjoint, the restriction of iG to V is the infinitesimal generator of a group of unitary operators of class (C_0) . We have

$$\|\exp(iG^*t)\| \leq 1, \quad \|\exp(iGt)\| \leq 1.$$

Consequently, the method of Section 2 applies to yield the result

$$\|(\exp(iG^*t) - \exp(iGt))\tilde{g}\|_{H \times H} \leq \epsilon^{1/2}K(t+1) \left(\sum_{k=0}^2 \|A^{2k}g\|_H \right)$$

where K is a constant, and $\tilde{g} = \begin{pmatrix} g \\ g \end{pmatrix}$ with $g \in D(A^4)$.

Denoting by w^ϵ and u the unique solutions of

$$(4) \quad \epsilon w_{ii}^\epsilon + 2ia w_i^\epsilon + A^2 w^\epsilon = 0 \quad w^\epsilon(0) = g^\epsilon D(A^4), \quad w_i^\epsilon(0) = iAh, \quad h \in D(A^2)$$

and

$$(5) \quad 2iau_t + A^2 u = 0 \quad u(0) = g$$

we wish to show the following estimate obtains:

$$(6) \quad \|w'(t) - u(t)\| \leq \epsilon C(t+1) \left(\|Ah\| + \sum_{k=0}^2 \|A^{2k}g\| \right).$$

The proof given in Section 2 is not valid here because in estimating $\| [S_{11}(t) - S_{22}(t)]f \|$ we used the fact that $I_1(a(t^2 - y^2)^{1/2})$ is non-negative, which is no longer true when a is replaced by ia . Instead we may use the representation of Stone's Theorem:

$$T(t) = \int_{-\infty}^{\infty} \exp(ixt) dE_x,$$

where $\{E_x\}$ is the spectral family of A , yielding

$$[T(t) - T(-t)]f = \int_{-\infty}^{\infty} \sinh(ixt) dE_x f.$$

Inserting this into (12), Section 2 and interchanging the order of integration (permissible because

$$\begin{aligned} \int_0^t \int_{-\infty}^{\infty} \left(\frac{y^2}{t^2 - y^2} \right)^{1/2} |I_1(ia(t^2 - y^2)^{1/2})| |\sinh(ixy)| d(E_x f, g) dy \\ \leq 2 \|f\| \|g\| \int_0^t \left(\frac{y^2}{t^2 - y^2} \right)^{1/2} |I_1(ia(t^2 - y^2)^{1/2})| dy < \infty \end{aligned}$$

for all $f, g \in H$) we find

$$\begin{aligned} [S_{11}(t) - S_{22}(t)]f &= \exp(-iat) \int_{-\infty}^{\infty} [\sinh(ixt) \\ &\quad + ia \int_0^t \left(\frac{y^2}{t^2 - y^2} \right)^{1/2} I_1(ia(t^2 - y^2)^{1/2}) \sinh(ixy) dy] dE_x f \\ &= \exp(-iat) \int_{-\infty}^{\infty} \frac{x}{(x^2 + a^2)^{1/2}} \sinh(i(x^2 + a^2)^{1/2}t) dE_x f. \end{aligned}$$

Making the substitutions $E_x \rightarrow E_x^*$, $a \rightarrow a/\epsilon$, and taking norms:

$$\begin{aligned} \| [S_{11}^*(t) - S_{22}^*(t)]f \|^2 &= \epsilon^2 \int_{-\infty}^{\infty} \frac{x^2}{a^2 + x^2} \left| \sinh \left(\frac{i}{\epsilon} (a^2 + \epsilon^2 t)^{1/2} \right) \right|^2 d \|E_x^* f\|^2 \\ &\leq 4(\epsilon/a)^2 \int_{-\infty}^{\infty} x^2 d \|E_x^* f\|^2 = 4(\epsilon/a)^2 \|Af\|^2/\epsilon. \end{aligned}$$

Therefore $\| [S_{11}^*(t) - S_{22}^*(t)]f \| \leq 2\epsilon^{1/2}/a \|Af\|$. The rest of the proof given in Section 2 is valid and thus we obtain (6).

The uniqueness for equation (5) is a consequence of the fact that iA^2 generates

a group. The uniqueness for (6) can be shown with the assumption that $A^2 w^\epsilon$ is strongly differentiable in t . Since A is closed it follows that

$$\frac{d}{dt}(Aw^\epsilon) = A\left(\frac{dw^\epsilon}{dt}\right).$$

Taking the inner product of both sides of (4) with w_i^ϵ and adding gives

$$\epsilon(w_{ii}^\epsilon, w_i^\epsilon) + \epsilon(w_i^\epsilon, w_{ii}^\epsilon) + (A^2 w^\epsilon, w_i^\epsilon) + (w_i^\epsilon, A^2 w^\epsilon) = 0.$$

Using the differentiability assumption this becomes

$$\epsilon \frac{d}{dt}(w_i^\epsilon, w_i^\epsilon) + \frac{d}{dt}(Aw^\epsilon, Aw^\epsilon) = 0.$$

Hence $\|w^\epsilon\| = \text{constant} = 0$ for zero initial data implying $w^\epsilon = 0$.

As an example we consider the relativistic Klein–Gordon Equation of quantum mechanics. For a single free particle it is

$$-\hbar^2 u_{ii} = -\hbar^2 c^2 \Delta u + m^2 c^4 u$$

where m is the mass of the particle and Δ is the Laplacian in 3 space dimensions [12; p. 308]. Let $u^\epsilon(x, t) = u(x, t) \cdot \exp(imc^2 t/\hbar)$. Then $u^\epsilon(x, t)$ satisfies

$$\frac{\hbar^2}{2mc^2} u_{ii}^\epsilon - i\hbar u_t^\epsilon = \frac{\hbar^2}{2m} \Delta u^\epsilon.$$

To apply the results of this section we define Δ for C^2 functions g such that $\Delta g \in L^2(E^3)$. $-(\hbar^2/2m)\Delta$ so defined has a unique self-adjoint extension and we take A to be its unique positive self-adjoint square root [10; p. 281]. Hence Theorem 2 asserts that

$$\|u^\epsilon(x, t) - u^0(x, t)\| \leq C \frac{\hbar^2}{mc^2} (t + 1),$$

where $u^0(x, t)$ is the solution of the well-known Schrödinger Equation for a free particle obtained by omitting the first term in (7). This establishes rigorously that the Klein–Gordon Equation has the correct non-relativistic limit.

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