On the Geometry of Dissociative Gas Flows

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Part-I Variation of Flow Quantities Along Stream Lines, their Principal Normals and Binormals

1. Introduction. For gaining some insight into the physics of gas flows with any new model, we have the following main recourses:

- (i) to investigate some of the classical gas flows with this new model,
- (ii) to study the geometry of fluid flow with this new model.

While concentrating on this aspect of physics of gas flow through our first recourse, we usually investigate flows over some simple geometries like the wedge, the cone, sharper pointed nose type bodies *etc.* and see how in these simple cases the flows are modified with the introduction of the new model.

Through our second recourse we seek answers to the following questions:

(i) to what extent the forms of the stream lines, constant pressure surfaces and some other curves and surfaces, allowed by the solution of the system of equations governing the flow, are determined by the equations,

(ii) to what extent the prescription of these curves and surfaces determine the properties of the flow.

So far as the ideal dissociating gas model attributed to Lighthill [5] is concerned, much has been contributed on this aspect through our first recourse [2, 3, 6]. However, the second recourse, which is quite an important one, does not seem to have been exploited. Accordingly, we have taken up the study of ideal dissociating gas flows through the second recourse. The complete study is divided into two parts. In Part-I, which is the subject of present investigation, we have obtained the variation of flow quantities along stream lines, their principal normals and binormals in steady, three dimensional, ideal dissociative gasdynamics. Part-II (which will be communicated separately), deals with the determination of stream line pattern for the plane steady flow of an ideal dissociating gas. The detailed plan of Part-I is as follows.

After formulation of the various dynamical and kinematical relations connecting the flow quantities with the geometrical parameters of the stream line trajectories, the expressions for the tangent, principal normal and binormal vectors and the curvature and torsion of the stream lines have been obtained

in terms of the velocity components, pressure, density and the mass-fraction variable. This is followed by the determination of the pressure gradient along the stream lines, their principal normals and binormals. It turns out that the dissociating character of the gas decreases the pressure gradient along the stream line. But, as in the conventional gas dynamical case, the pressure remains constant along the binormals. Further, from the relations connecting the vorticity components with the gradients of dissociation variable and entropy of the system, we conclude that the effect of dissociation is to decrease the vorticity components. Finally, the expressions for the vorticity components in terms of the curvature of the stream line and the velocity gradients along the stream line and their principal normals and binormals are obtained.

2. Equations of Motion. The fundamental equations of motion for a steady, three dimensional, ideal dissociating gas flow, referred to a rectangular co-ordinate system are as follows:

continuity

$$(2.1) \quad \rho_{,\alpha} u_{\alpha} + \rho u_{\alpha,\alpha} = 0,$$

momentum

$$(2.2) \quad \rho u_{\alpha} u_{\alpha,\alpha} + p_{,\alpha} = 0,$$

energy

$$(2.3) \quad \rho h_{,\alpha} u_{\alpha} - p_{,\alpha} u_{\alpha} = 0,$$

state

$$(2.4) \quad p = (1 + q)\rho RT \quad (\text{Thermal}),$$

$$(2.5) \quad h = (4 + q)RT + Dq \quad (\text{Caloric}),$$

reaction rate

$$(2.6) \quad q_{,\alpha} u_{\alpha} = C\rho T^{-n} \left[(1 - q) e^{-D/RT} - \frac{\rho}{\rho_D} q^2 \right].$$

In equations (2.1)–(2.6) ρ denotes the density, p the pressure, u_{α} the components of the flow velocity, h the enthalpy, q the mass-fraction, T the temperature, R the gas constant, D the dissociation energy per unit mass, ρ_D is a characteristic density and C is a constant with the dimensions of (temperature)/(density) (time). (For the introduction of C and some more details about the rate equation we refer to [3].).

The internal energy e for the ideal dissociating gas can be expressed as [2]

$$(2.7) \quad e = 3RT + qD,$$

where qD is the energy absorbed by dissociation.

From the above set of equations one can easily deduce that

$$(2.8) \quad h + \frac{1}{2}v^2 = h_0 ; \quad v^2 = u_a u_a .$$

Equation (2.8) shows that the total enthalpy remains constant along the stream lines but will, in general, vary from stream line to stream line.

From equations (2.4) and (2.5) we can write

$$(2.9) \quad h = h(p, \rho, q),$$

which yields

$$(2.10) \quad h_{,i} = \frac{\partial h}{\partial p} p_{,i} + \frac{\partial h}{\partial \rho} \rho_{,i} + \frac{\partial h}{\partial q} q_{,i} .$$

Substituting the values of $\rho_{,i}$, $p_{,i}$ and $h_{,i}$ respectively from equations (2.1)–(2.3) in (2.10), we get

$$(2.11) \quad \frac{1}{a_f^2} u_{a,a} u_a u_a - u_{a,a} = \left\{ \frac{D(1+q)}{3a_f^2} - \frac{3}{(1+q)(4+q)} \right\} q_{,a} u_a ,$$

where a_f is the frozen speed of sound given by

$$a_f^2 = \gamma(q) \cdot p / \rho = \left(\frac{4+q}{3} \right) \frac{p}{\rho} .$$

3. Variation of Pressure Along and Perpendicular to the Stream Lines. The differential equations for a stream line can be represented by proportionality

$$(3.1) \quad dx_1 : dx_2 : dx_3 = u_1 : u_2 : u_3 .$$

If λ^i , μ^i and ν^i respectively be the components of unit tangent, principal normal and binormal vectors to the stream line at a point, we then have, under usual notations

$$(3.2) \quad \frac{dx^i}{ds} = \lambda^i = \frac{u^i}{v} ,$$

and hence

$$(3.3) \quad \frac{d^2 x^i}{ds^2} = \frac{u_{i,j} \lambda_j}{v} - \frac{u_\rho u_j u_{\rho,j} \lambda_i}{v^3} .$$

Further with the help of the Frenet Formulae [1] the equation (3.3) yields

$$(3.4) \quad K \mu^i = \frac{u_{i,j} u^j}{v^2} - \frac{u_{\rho,j} u_\rho u_j \lambda_i}{v^3} .$$

Using equation (3.2) in (3.4), we get

$$(3.5) \quad \mu^i = -\frac{1}{K} \left[\frac{p_{,i}}{\rho v^2} + \frac{u_{\rho,j} u_\rho u_j \lambda_i}{v^3} \right] .$$

Substituting for $u_{i,j} u_j$ from (2.11) in (3.5) we get

$$(3.6) \quad K^4 = \frac{1}{v^2} \left[\frac{p_{,i} p_{,i}}{\rho^2} - \frac{a_f^4}{v^2} \left\{ u_{i,i} + \left(\frac{D(1+q)}{3a_f^2} - \frac{3}{(1+q)(4+q)} \right) q_{,i} u_i \right\}^2 \right].$$

Further using equations (2.6) and (2.11) in the equations resulting through the multiplication of (3.5) by λ_i , μ_i and ν_i we get, respectively the following equations:

$$(3.7) \quad \frac{1}{\rho v} p_{,i} \lambda_i = -\frac{1}{M_f^2} \left[u_{i,i} + \left\{ \frac{D(1+q)}{3a_f^2} - \frac{3}{(1+q)(4+q)} \right\} \cdot C \rho T^{-n} \left\{ (1-q) e^{-D/R T} - \frac{\rho}{\rho_D} q^* \right\} \right],$$

$$(3.8) \quad p_{,i} \mu_i = -K \rho v^*,$$

$$(3.9) \quad p_{,i} \nu_i = 0,$$

where M_f is the frozen Mach number defined by $M_f = v/a_f$. Equation (3.9) indicates that the pressure remains constant along the binormals to the stream lines. More appropriately the trajectories of the binormals lie on the surface of the constant pressure.

From equations (3.2) and (3.5) the components of the binormal vector can be expressed as

$$(3.10) \quad \nu_i = e_{ijk} \lambda_j \mu_k = -\frac{1}{\rho K v^3} e_{ijk} u_j p_{,k}$$

where e_{ijk} is the usual permutation symbol having the value one or minus one according as ijk is an even or odd permutation of 123 and having the value zero when two of the indices are equal.

Substituting for ν_i from (3.10) in the Frenet's Formula given below

$$(3.11) \quad \frac{d\nu_i}{ds} = -\tau \mu^i,$$

and using equations (2.1), (2.2), (2.6) and (2.11) we get the following relation connecting the torsion τ of the stream lines with the parameters of the dissociating gas, viz.,

$$(3.12) \quad \tau \mu^i = -\frac{1}{\rho K^2 v^6} \left[u_j p_{,k} \left\{ K a_f^2 (3 - M_f^2) u_{i,i} + v^* K_{,i} u_i \right. \right. \\ \left. \left. - 3K \rho C T^{-n} a_f^2 \cdot \left((1-q) e^{-D/R T} - \frac{\rho}{\rho_D} q^* \right) \right. \right. \\ \left. \left. \cdot \left(\frac{D(1+q)}{3a_f^2} - \frac{3}{(1+q)(4+q)} \right) \right\} - K v^* p_{,k} \rho u_{,k} u_i \right] e_{ijk}.$$

4. Gradients. For our future work we need the gradient of λ_k in the rectangular coordinate system x^i . If d/ds , d/dn_1 and d/dn_2 denote, respectively the differentiation along the stream line, the principal normal and the binormal

of the stream line, then the desired gradient of λ_k can be represented by vector sum of its gradients in the system of axes determined by λ , μ and ν as

$$(4.1) \quad \lambda_{k,i} = \frac{d\lambda_k}{ds} \lambda_i + \frac{d\lambda_k}{dn_1} \mu_i + \frac{d\lambda_k}{dn_2} \nu_i .$$

In order to evaluate the quantities occurring in the right hand side of the equation (4.1), we consider the congruence determined by (i) the principal normal (ii) the binormal, to the stream line, in such a way that the former have μ as the tangent vector, λ as principal normal vector, ν as the binormal vector, while latter, have ν as the tangent vector, λ as the principal normal vector and μ as the binormal vector.

If K' and τ' be the curvature and torsion of the curves in the congruence determined by the principal normal and K'' and τ'' be the similar quantities for the curves in the congruence determined by the binormal of the stream lines, the Frenet-Formulae yield

$$(4.2) \quad \begin{cases} \frac{d\lambda_k}{ds} = K\mu_k \\ \frac{d\lambda_k}{dn_1} = -K'\mu_k + \nu_k\tau' \\ \frac{d\lambda_k}{dn_2} = \mu_k\tau'' - \nu_kK'' \end{cases}$$

From equation (4.1) and (4.2) we get

$$(4.3) \quad \lambda_{k,i} = K\mu_k\lambda_i + (\tau'\nu_k - K'\mu_k)\mu_i + (\tau''\mu_k - K''\nu_k)\nu_i .$$

Multiplying both sides of the equation (4.3) by $e_{ijk}\lambda_i$, $e_{ijk}\mu_i$, $e_{ijk}\nu_i$ in turn and making use of the following identities

$$(4.4) \quad \begin{cases} e_{ijk}\mu_i\nu_k = \lambda_i , \\ e_{ijk}\lambda_i\mu_k = \nu_i , \\ e_{ijk}\lambda_i\mu_k = -\mu_i , \\ e_{ijk}\lambda_i\mu_i\nu_k = 1 \end{cases}$$

and the skew symmetric properties of e_{ijk} , we get

$$(4.5) \quad \begin{cases} e_{ijk}\lambda_{k,i}\lambda_i = (\tau' - \tau''), \\ e_{ijk}\lambda_{k,i}\mu_i = 0, \\ e_{ijk}\lambda_{k,i}\nu_i = K. \end{cases}$$

Contraction in j and k in (4.3), yields

$$(4.6) \quad \lambda_{k,k} = -(K' + K'').$$

Differentiating (3.2) and using (2.11) and (4.6) we get

$$(4.7) \quad (M_f^* - 1)u_{k,k} - \left\{ \frac{D(1+q)}{3a_f^2} - \frac{3}{(1+q)(4+q)} \right\} q_{,i} u_i = -(K' + K'')vM_f^*.$$

Equations (3.7) and (4.7) yield

$$(4.8) \quad \frac{dp}{ds} = \frac{\rho v^*(K' + K'')}{(M_f^2 - 1)} - \frac{\rho v^*}{(M_f^2 - 1)} \left\{ \frac{D(1+q)}{3a_f^2} - \frac{3}{(1+q)(4+q)} \right\} \frac{dq}{ds}.$$

From equation (4.8) we conclude that the effect of dissociation is to decrease the pressure gradient.

5. Vorticity Components. From (2.8), (2.4) and (2.5) we get

$$p = \frac{(1+q)}{2(4+q)} \cdot \rho \{ (2h_0 - v^*) - 2qD \}.$$

Differentiation of above with respect to x^i yields

$$(5.1) \quad p_{,i} = \frac{p}{\rho} \rho_{,i} - \left\{ \log \frac{\gamma(q)}{\gamma(q) - 1} \right\}_{,q} q_{,i} - \rho \frac{\gamma(q) - 1}{\gamma(q)} D q_{,i} - \rho \frac{\gamma(q) - 1}{\gamma(q)} u_{i,i} u_i.$$

Substituting the value of $p_{,i}$ from (2.2) in (5.1), and simplifying, we get

$$(5.2) \quad e_{ijk} \hat{\omega}_k u_j = -\frac{1}{\rho} \frac{1}{\gamma(q) - 1} p_{,i} + \frac{\gamma(q)}{\gamma(q) - 1} \frac{p}{\rho^2} \rho_{,i} + \left\{ \frac{a_f^2}{3\gamma(q)(\gamma(q) - 1)^2} - D \right\} q_{,i},$$

where $\hat{\omega}_k$, the vorticity components in (5.2), are given by

$$(5.3) \quad \hat{\omega}_k = e_{kim} u_{m,i}.$$

Further $p = p(\rho, s, q)$, where s is the entropy of the system, yields

$$p_{,i} = \left(\frac{\partial p}{\partial \rho} \right)_{s,q} \rho_{,i} + \left(\frac{\partial p}{\partial s} \right)_{\rho,q} S_{,i} + \left(\frac{\partial p}{\partial q} \right)_{\rho,s} q_{,i}.$$

Using the above relation in (5.2), we get

$$(5.4) \quad e_{ijk} \hat{\omega}_k u_j = - \left[\frac{(\partial p / \partial S)_{\rho,q}}{\rho \gamma(q) - 1} S_{,i} + \delta q_{,i} \right],$$

where δ is given by

$$(5.5) \quad \delta = \frac{a_f^2}{\gamma(q) - 1} \left\{ \frac{D\gamma(q) - 1}{a_f^2} + \frac{(2 - \gamma)}{3\rho\gamma(q)\gamma(q) - 1} \right\}.$$

From equation (5.4) we remark that the effect of dissociation is to decrease the vorticity components.

Substituting (3.2) in (5.4), we get

$$(5.6) \quad e_{ijk} \hat{\omega}_k \lambda_i = -\frac{1}{v} \left[\frac{(\partial p / \partial S)_{\rho, q}}{\rho \gamma(q) - 1} S_{,i} + \delta q_{,i} \right].$$

Now, multiplying both sides of (5.6) by λ_i , μ_i and ν_i in turn and using the skew symmetric property of e_{ijk} and the equations (5.4), we get

$$(5.7) \quad \frac{dS}{ds} + \rho \delta \left\{ \frac{\gamma(q) - 1}{(\partial p / \partial S)_{\rho, q}} \right\} \cdot \frac{dq}{ds} = 0,$$

$$(5.8) \quad \hat{\omega}_i \mu_i = -\frac{\delta}{v} \frac{dq}{dn_s} - \frac{(\partial p / \partial S)_{\rho, q}}{v \rho \gamma(q) - 1} \cdot \frac{dS}{dn_s},$$

$$(5.9) \quad \hat{\omega}_i \nu_i = -\frac{\delta}{v} \frac{dq}{dn_1} - \frac{(\partial p / \partial S)_{\rho, q}}{v \rho \gamma(q) - 1} \cdot \frac{dS}{dn_1}.$$

It is worth pointing out here that the result (5.7), is merely another version of (2.3).

Substituting (3.2) in (5.3)

$$(5.10) \quad \hat{\omega}_i = v e_{ijk} \lambda_{k,i} + e_{ijk} \frac{\lambda_k}{v} u_{\rho} u_{\rho, i}.$$

Now multiplying both sides of (5.10) by λ_i , μ_i and ν_i , in turn and using the identities (4.4), we arrive at the following relations:

$$(5.11) \quad \begin{aligned} \hat{\omega}_i \lambda_i &= v e_{ijk} \lambda_{k,i} \lambda_i, \\ \hat{\omega}_i \mu_i &= v e_{ijk} \lambda_{k,i} \mu_i + \frac{u_{\rho} u_{\rho, i} \nu_i}{v}, \\ \hat{\omega}_i \nu_i &= -v e_{ijk} \lambda_{k,i} \nu_i - \frac{u_{\rho} u_{\rho, i} \mu_i}{v}. \end{aligned}$$

Substituting (4.5) in (5.11) we get

$$(5.12) \quad \begin{aligned} \hat{\omega}_i \lambda_i &= v(\tau' - \tau''), \\ \hat{\omega}_i \mu_i &= \frac{dv}{dn_2}, \\ \hat{\omega}_i \nu_i &= vK - \frac{dv}{dn_1}. \end{aligned}$$

For finding the explicit expression for the vorticity components we solve equations (5.12) and that gives

$$(5.13) \quad \hat{\omega}_i = v(\tau' - \tau'') \lambda_i + \frac{dv}{dn_2} \mu_i + v \left(K - \frac{dv}{dn_1} \right) \nu_i.$$

Further, combining equations (5.7)–(5.9) we get

$$(5.14) \quad \hat{\omega}_i \mu_i = \frac{dv}{dn_2} = -\frac{\delta}{v} \frac{dq}{dn_2} - \frac{(\partial p / \partial S)_{\rho, q}}{\rho v \gamma(q) - 1} \frac{dS}{dn_2},$$

$$(5.15) \quad \hat{\omega}_i \nu_i = vK - \frac{dv}{dn_1} = -\frac{\delta}{v} \frac{dq}{dn_1} - \frac{(\partial p / \partial S)_{\rho, q}}{v \rho \gamma(q) - 1} \frac{dS}{dn_1}.$$

For the two dimensional case, the equations (5.14) and (5.15) reduce to

$$(5.16) \quad \hat{\omega} = vK - \frac{dv}{dn_1} = -\frac{\delta}{v} \frac{dq}{dn_1} - \frac{(\partial p / \partial S)_{\rho, q}}{\rho v \gamma(q) - 1} \frac{dS}{dn_1}.$$

As a necessary check to our calculations we deduced, from the above equation, the expression for vorticity in plane steady flow of a conventional gas and it tallied with the one derived in [4].

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