Existence, Uniqueness and Stability of Solutions of the Equation

$$u_{,tt} = \frac{\partial}{\partial x} (\sigma(u_x) + \lambda(u_x)u_{,tt})$$

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1. Introduction. This paper concerns an equation which arises in the study of nonlinear elasticity. Consider purely longitudinal motions of a homogeneous bar of uniform cross section and unit length. Let x denote the position of a cross section in a homogeneous rest configuration and let $u(x, t)$ denote the displacement, at time t , of the cross section. Let $\tau(x, t)$ denote the stress on the section at time t and ρ_0 be the density in the rest position: then the equation of motion is,

$$(1.1) \quad \rho_0 u_{,tt} = \tau_x .$$

One obtains a differential equation for u by making some constitutive assumptions about the form of τ . In classical elasticity one assumes that τ is given by an equation of the form

$$(1.2) \quad \tau(x, t) = \sigma(u_x(x, t)),$$

u_x being called the strain. Substitution of (1.2) into (1.1) yields a nonlinear wave equation for u . To this we append the condition that the ends are clamped, that is,

$$(1.3) \quad u(0, t) = u(1, t) = 0.$$

It was shown in [1] that the nonlinear problem above, with appropriate initial conditions, has no smooth solution. Thus classical nonlinear theory is in some sense an inadequate model. In recent years there has been developed a more general model, that of 'fading memory' [2]. We describe this briefly.

The central notion of fading memory is the replacement of (1.2) by the state-

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ment that τ is given in terms of a functional of the *past history* u_x^t of the strain. This is defined by,

$$(1.4) \quad u_x^t(x, s) = u_x(x, t - s) \quad s \in [0, \infty).$$

One assumes that, for each x , u_x^t belongs to a *history space* B , which is a Banach space. (A class of physically reasonable history spaces is discussed in [3]. They have the form of weighted L^p spaces, that is they have norms of the type,

$$(1.5) \quad \|u_x^t\|^p = \int_0^\infty |u_x^t(x, s)|^p k(s) ds.$$

If $k(s)$ tends to zero as s tends to infinity then (1.5) shows that the distant past is less heavily weighted.)

The constitutive equation of fading memory theory is,

$$(1.6) \quad \tau(x, t) = \hat{\sigma}(u_x^t),$$

where σ is a continuous functional on B . Substitution of (1.6) into (1.1) yields for u , the functional partial differential equation

$$(1.7) \quad u_{tt} = \frac{\partial}{\partial x} \hat{\sigma}(u_x^t).$$

The 'zero-dimensional' analog of this equation, that is the equation,

$$\ddot{y}(t) = \hat{\sigma}(y^t), \quad y^t(s) = y(t - s),$$

has been studied in [4] but (1.7) remains, as yet, too difficult to handle except in the linear case. Equation (1.7) does, however, suggest an approximation which *can* be treated. (This is the *slow flow* approximation suggested in [2].) Let φ_k denote the functions defined by $\varphi_k(s) = s^k$, $s \in [0, \infty)$. Then by Taylor's Theorem we have, provided that $u_{x,t}$ is bounded for all x and t ,

$$u_x^t = u_x(x, t)\varphi_0 - u_{xt}(x, t)\varphi_1 + \epsilon, \quad |\epsilon(s)| \leq ks^2.$$

If σ in (1.6) is a differentiable functional on B we have

$$\hat{\sigma}(\eta + h) = \hat{\sigma}(\eta) + \delta\hat{\sigma}(\eta | h) + o(\|h\|),$$

where $\delta\hat{\sigma}(\eta | \cdot)$ is a linear functional. Thus we obtain,

$$(1.8) \quad \hat{\sigma}(u_x^t) = \hat{\sigma}(u_x(x, t)\varphi_0) - u_{xt}(x, t) \delta\hat{\sigma}(u_x(x, t)\varphi_0 | \varphi_1) + o(\|\epsilon\|) + o(\|\varphi_1\|).$$

We obtain an approximate fading memory theory by neglecting the order terms in (1.8). Set,

$$\sigma(\xi) = \hat{\sigma}(\xi\varphi_0), \quad \lambda(\xi) = -\delta\hat{\sigma}(\xi\varphi_0 | \varphi_1).$$

Then the approximation to Equation (1.7) is,

$$(E) \quad u_{tt} = \frac{\partial}{\partial x} (\sigma(u_x) + \lambda(u_x)u_{xt}).$$

The validity of this approximation is open to question. Its complete investigation would require the solution of (1.7) hence is not possible. However one can at least verify that the model (E) is consistent and this is the content of this paper. The special case λ constant was studied in [5]. It is, however, inconsistent to allow nonlinearities in σ while assuming λ is a constant. The results of this paper remedy the inconsistency by allowing *both* σ and λ to be nonlinear.

Equation (E) occurs in the more well-known setting of one-dimensional, compressible, viscous flow of a fluid, treated from the Lagrangian point of view.

In the next section we state our results. These are uniqueness, existence and stability theorems for solutions of the mixed initial-boundary value problem for (E). The proofs are mainly more complicated versions of those in [5] and appear in detail in [6]. Thus we present only an outline with emphasis on the new aspects. These appear in Section 3.

2. Statement of Results. We study Equation (E), that is,

$$(E) \quad u_{tt} = \frac{\partial}{\partial x} (\sigma(u_x) + \lambda(u_x)u_{xt}),$$

with the additional conditions,

$$(A) \quad u(0, t) = u(1, t) = 0,$$

$$(B) \quad u(x, 0) = f(x),$$

$$(C) \quad u_t(x, 0) = g(x).$$

The results parallel those of [5] in that we establish the uniqueness, existence and stability of solutions of the problem defined by (E), (A), (B), (C).

We assume throughout that f and g belong to $C^4[0, 1]$ and $C^2[0, 1]$ respectively. Also σ is to belong to $C^2(-\infty, \infty)$ and λ to $C^1(-\infty, \infty)$. We impose on σ and λ the following conditions. (Stronger assumptions on σ are made in [6]. It turns out that these are not necessary.)

$$(*) \quad \sigma(0) = 0, \quad \sigma'(\xi) > 0,$$

$$(**) \quad 0 < \lambda_0 \leq \lambda(\xi) \leq \lambda_1 \quad \text{for all } \xi,$$

where λ_0 and λ_1 are constants.

Remark. Conditions (*) and the first inequality in (**) are physically reasonable. There seems to be no physical reason why λ need be bounded above but without this condition we cannot obtain our global stability results.

We follow the notation and terminology of [5]. For functions Λ and Ω which are in $C^2[0, 1]$ we set,

$$(2.1) \quad J(\Lambda, \Omega) = \sum_{i=0}^2 \max_{x \in [0, 1]} |\Lambda^{(i)}(x)| + \max_{x \in [0, 1]} |\Omega^{(i)}(x)|.$$

For any positive number T we denote by S_T the strip,

$$S_T = \{(x, t) \mid 0 \leq x \leq 1, 0 \leq t \leq T\}.$$

Then if $U \in C^2(S_T)$ we write

$$(2.2) \quad |||U|||(t) = \sum_{\substack{i=0 \\ (i,k) \neq (2,2)}}^2 \sum_{k=0}^i \max_{x \in [0,1]} \left| \frac{\partial^i U}{\partial x^{i-k} \partial t^k} \right|.$$

An (f, g) displacement in S_T will be a function u such that:

- (i) All derivatives appearing in (E) are continuous in S_T ,
- (ii) u satisfies (E), (A), (B), (C).

Theorem (1). *There exists M such that for any (f, g) displacement u ,*

$$(2.3) \quad |||u|||(t) \leq M \quad \text{on any } S_T.$$

The constant M depends only on λ_0, λ_1 and $J(f, g)$ and, for fixed (λ_0, λ_1) , M tends to zero as J tends to zero. Moreover any (f, g) displacement satisfies,

$$(2.4) \quad \lim_{t \rightarrow \infty} |||u|||(t) = 0.$$

Theorem (2). *There exists a unique (f, g) displacement in S_T for any T .*

Remarks. (1) It is to be emphasized that the constant M in Theorem (1) is independent of T .

(2) Theorem (1) is slightly weaker than that in [5] in that boundedness and stability of u_{tt} is omitted. It would be possible to obtain such a result but this requires additional computations which were not included in [6].

(3) Physical considerations require that in the applications in elasticity we must have $u_x > -1$ (positivity of density). Theorem (1) shows that we can always insure this condition by making $J(f, g)$ small.

Note. The author has become aware of some work by C. M. Dafermos [7] which relates to our results. Dafermos considers the more general equation,

$$(2.5) \quad u_{tt} = \frac{\partial}{\partial x} \sigma(u_x, u_{xt}).$$

The solution is to satisfy (B) and (C) but 'stress' boundary conditions, $\sigma = 0$ at $x = 0$ and $x = 1$. His function σ satisfies the conditions,

$$(***) \quad \sigma_\eta(\xi, \eta) \geq \lambda_0 > 0; \quad |\sigma_\xi(\xi, \eta)| \leq N(\sigma_\eta(\xi, \eta))^{1/2}.$$

Under these conditions he proves existence and uniqueness and local stability; that is he shows that u and certain of its derivatives are bounded, in terms of the initial data, on bounded time intervals. He also obtains partial asymptotic stability by showing that u_t and σ tend to zero as t tends to infinity. It is pointed out that u_x can actually become infinite in certain cases.

Conditions (***) for our equation become,

$$(2.6) \quad \lambda(\xi) \geq \lambda_0 > 0,$$

$$(2.7) \quad |\sigma'(\xi) + \lambda'(\xi)\eta| \leq N\lambda(\xi).$$

The first of these will be satisfied under our assumptions but the second will not be except in the case λ constant. We have replaced (2.7) with the condition that σ' be positive and thereby obtain stronger stability results. We obtain *global* bounds, holding for all t , and asymptotic stability for *all* derivatives.

3. Outline of Proofs. The first step in the proof of Theorem (1) consists of two energy inequalities. Let u be an (f, g) displacement on S_∞ . We multiply (E) by u_t and integrate over $(0, 1) \times (t_1, t_2)$. The result is,

$$(3.1) \quad \|u_t\|^2(t_2) + 2 \int_0^1 \int_{u_x(x, t_1)}^{u_x(x, t_2)} \sigma(\xi) d\xi dx \\ + 2 \int_{t_1}^{t_2} \int_0^1 \lambda(u_x(\xi, \tau)) u_{xt}^2(\xi, \tau) d\xi d\tau = \|u_t\|^2(t_1).$$

Next we multiply by u_{xx} and integrate. We obtain, after some computations,

$$(3.2) \quad \int_0^1 \lambda^2 u_{xx}^2(x, t_2) + \int_{t_1}^{t_2} \int_0^1 \sigma'(u_x) \lambda(u_x) u_{xx}^2(x, \tau) dx d\tau \\ = 2 \int_0^1 \lambda^2 u_{xx}^2(x, t_1) + 2 \int_0^1 [\lambda u_t u_{xx}(x, t_2) - \lambda u_t u_{xx}(x, t_1)] dx \\ + 2 \int_0^1 \int_{t_1}^{t_2} \lambda u_{tx}^2(x, \tau) d\tau dx.$$

We set,

$$(3.3) \quad E_0(a) = \inf_{|\xi| \leq a} \sigma'(\xi), \quad E_1(a) = \sup_{|\xi| \leq a} \sigma'(\xi).$$

Then from (3.1) we deduce, as in [5], the estimates,

$$(3.4) \quad \|u_t\|^2(t) \leq M_1^2, \quad \int_0^t \|u_{x\tau}\|^2(\tau) d\tau \leq M_2^2,$$

where double bars denote L_2 -norms and

$$M_1^2 = J(f, g)^2 + E_1(J(f, g))J(f, g)^2, \quad M_2^2 = \lambda_0^{-1} M_1^2.$$

Next we obtain from (3.2), again as in [5], the further estimates,

$$(3.5) \quad \|u_{xx}\|^2(t_2) \leq M_3^2, \quad \int_0^t \|u_{xx}\|^2(\tau) d\tau \leq M_4^2,$$

where M_3 and M_4 are constants depending only on λ_0, λ_1 and $J(f, g)$.

From the first of estimates (3.5) we deduce that,

$$(3.6) \quad |u|(t) \leq |u_x|(t) \leq M_3, \quad (|u|(t) = \max_{[0,1]} |u(x, t)|).$$

It follows from the second of the estimates (3.6) that in any further computations

one can always assume $\sigma^{(k)}(u_x)$, $k = 0, 1, 2$ and $\lambda^{(k)}(u_x)$, $k = 0, 1$, all bounded uniformly in x and t .

Computations analogous to those of [5] serve to show now that u and u_x both tend to zero as t tends to infinity.

The next step represents one of the major differences between the present paper and the work of [5]. We want to derive a pointwise bound for u_t . When λ is a constant one can consider (E) as an inhomogeneous heat equation for $v = u_t$, with forcing term equal to $\sigma'(u_x)u_{xx}$. (3.5) gives a bound for u_t . When λ is a function of u_x , Equation (E) becomes the more general parabolic equation

$$(3.7) \quad v_t(x, t) = A(x, t)v_{xx}(x, t) + B(x, t)v'_x(x, t) + k(x, t),$$

where,

$$A(x, t) = \lambda(u_x(x, t)), B(x, t) = \lambda'(u_x(x, t))u_{xx}(x, t), k(x, t) = \sigma'(u_x(x, t))u_{xx}(x, t).$$

The usual bounds [8] for solutions v of (3.7) require bounds for the Hölder norms of A and k and the sup norm of B . This information we do not have. We can, however, use a very powerful result of [9]. This has to do with parabolic equations which can be written in conservation form, that is,

$$(3.8) \quad v_t = \frac{\partial}{\partial x}(av + b),$$

$$(3.9) \quad v(0, t) = v(1, t) = 0,$$

$$(3.10) \quad v(x, 0) = \psi(x).$$

It is assumed that $a(x, t) \geq \lambda_0 > 0$. Let $k_0 = \max_{0 \leq x \leq 1} |\psi(x)|$. Then the result is the following.

Lemma. Suppose that

$$\lambda_0^{-1} \|b\|_{L_\infty(0,1)} \leq C \quad \text{for all } t \geq 0.$$

Let v be a solution of (3.8)–(3.10) such that

$$\|v\|(t) \leq M_1 \quad \text{for all } t \geq 0.$$

Define the constant M by

$$M^{4/3} = M_1^2 2^{4/3} (2C/a_0)^{2/3}.$$

Then

$$(3.11) \quad |v(x, t)| \leq k_0 + 2M.$$

We can apply this result to our situation. Write (E) as,

$$(3.12) \quad v_t = \frac{\partial}{\partial x}(\lambda(u_x)v_x + \sigma(u_x)).$$

Then we have $\lambda(u_x) \geq \lambda_0$ and $\|v\|(t) \leq M_1$ by (3.4). Also $k_0 = \max |v(x, 0)| \leq$

$J(f, g)$. Finally we have, by (3.6),

$$\lambda_0^{-1} \|\sigma\|_{L_\infty[0,1]} \leq \lambda_0^{-1} \max_{[0,1]} |\sigma(x, t)| \leq \lambda_0^{-1} E_1(M_3)M_3.$$

Thus we deduce the estimate

$$(3.13) \quad |v|(t) = |u_t|(t) \leq M_5.$$

Once an estimate for u_t is obtained, we can immediately derive an estimate for u_{xx} . We write (E) in the form,

$$(3.14) \quad u_{tt} = \frac{\sigma'(u_x)}{\lambda(u_x)} (\lambda(u_x)u_{xx}) + \frac{\partial}{\partial t} (\lambda(u_x)u_{xx}).$$

We consider (3.14) as an ordinary differential equation for λu_{xx} . Solving this equation and integrating by parts to replace u_{tt} by u_t yields the estimate,

$$(3.15) \quad |u_{xx}|(t) \leq M_6.$$

The calculations are essentially the same as in [5].

The boundedness of u_{xx} implies the Hölder continuity in x of $\lambda(u_x)$. We are then able to return to Equation (3.7) and use more nearly standard results on parabolic equations. In [6] we indicate modifications of the usual theorems which eliminate the requirement of Hölder continuity in t of the quantity $\lambda(u_x)$. These modifications and results analogous to those for the heat equation in [5] enable us to complete the proof of Theorem (1).

The proof of Theorem (2) is technically much more complicated than that in [5]. It is no longer possible simply to use the contractive mapping principle in spaces of continuous functions. One must replace that idea by the use of the Schauder Theorem on the Hölder spaces usually associated with parabolic problems. Details can be found in [6].

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