A Sharp Maximum Principle for Degenerate Elliptic-Parabolic Equations

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For linear second order elliptic or parabolic equations, the “weak” maximum principle and the “strong” maximum principle are well known. In this paper, we introduce the *sharp* maximum principle for equations of degenerate elliptic-parabolic type; that is, for operators of the form

$$Lu = a^{ij}(x)u_{x_i x_j} + b^i(x)u_{x_i}, \quad a^{ij}(x)\xi_i \xi_j \geq 0.$$

Here the coefficients are smooth functions of $x = (x_1, x_2, \dots, x_n)$ in a connected open set $\omega \subset \mathbb{R}^n$; we use summation convention ($i, j = 1, 2, \dots, n$). Our exact assumptions will be described more precisely later on.

The sharp maximum principle deals with the following question: Suppose a point $P \in \omega$ is given. Then is it always possible to determine a subset $S(P; \omega)$ of ω such that, if $Lu \geq 0$ in $S(P; \omega)$ and u has a local maximum at P , relative to the set $S(P; \omega)$, then necessarily $u \equiv u(P) = \text{const.}$ throughout its closure $\bar{S}(P; \omega)$? The answer is yes; the sharp maximum principle gives a precise description of $S(P; \omega)$.

In connection with $S(P; \omega)$ (or its closure $\bar{S}(P; \omega)$), which we call the propagation set of P in ω , several interesting phenomena arise: For either elliptic or parabolic equations, the propagation set is well known [11], [13]; its determination in a neighborhood of P is a purely local question. But for general degenerate elliptic-parabolic equations, the propagation set has a global character. For example, a small perturbation of the boundary of the region ω , far away from P , can cause a drastic change in $S(P; \omega)$, even in a neighborhood of P . Also, a small perturbation of the coefficients of L can lead to a drastic change in $S(P; \omega)$. These global phenomena come about because $S(P; \omega)$ is determined by the trajectories of certain vector fields associated with the operator L . In the familiar cases of elliptic or parabolic equations, the global behavior of these trajectories is extremely simple. However, in the general case, their global behavior is nontrivial and can change drastically with respect to the perturbations mentioned above.

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Actually, we discuss two different situations: the *interior principle* and the *boundary principle*. The interior principle is the analogue, for degenerate elliptic-parabolic equations, of the “strong maximum principle” for parabolic equations due to Nirenberg [13]. The boundary principle is the corresponding analogue of the parabolic version of “Hopf’s Lemma” due to Vyborny [19] and Friedman [9].

There is an extensive literature on the maximum principle in its various forms. In addition to the works cited above, mention should be made of the original paper by Hopf [11], the extensive work of Aleksandrov [1], [2], [3], [4], [5], [6], the paper by Hartman and Sacksteder [10], the original work of Fichera [7], [8] on degenerate elliptic-parabolic equations, and especially the book by Protter and Weinberger [18] where many more references are listed. Naturally, we use in our proofs many of the same techniques as those in the works mentioned above.

We also give (in Section 7) an application of one of our techniques to the problem of unique continuation. The main theorems on the maximum principle are stated in Section 4 and proved in Section 6. Sections 1 and 2 are devoted to a description of $S(P; \omega)$; a number of examples are given in Section 3. In Section 5, we interpret our results in terms of the uniqueness question for the degenerate Dirichlet–Neumann Problem.

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1. Diffusion and drift trajectories. Let ω be a connected open set in $\mathbf{R}^n$. Consider an $n \times n$ positive semi-definite matrix $a(x) = [a^{ij}(x)]$ with $a^{ij}(x) \in C^2(\omega)$; $a^{ij}(x)\xi_i\xi_j \geq 0$ for all $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbf{R}^n$ and all $x \in \omega$. Then $a(x)$ has a symmetric positive semi-definite square root $\alpha(x) = [\alpha^{ik}(x)]$; $a^{ij}(x) = \alpha^{ik}(x)\alpha^{jk}(x)$. Phillips and Sarason [17] have shown that each $\alpha^{ik}(x)$ is then uniformly Lipschitz continuous on any relatively compact subset of ω . This Lipschitz continuity of $\alpha(x)$ is all that is actually needed in what follows. However, we prefer to state our assumptions about the smoothness of the coefficients directly in terms of the original matrix $a(x)$.

Let $X_k(x)$ denote the k -th column vector,

$$X_k(x) = (\alpha^{1k}(x), \alpha^{2k}(x), \dots, \alpha^{nk}(x)),$$

of the matrix $\alpha(x)$. Each $X_k(x)$ defines a Lipschitz continuous vector field in ω . Then through any point $P \in \omega$, there passes a unique trajectory $\{x = x(t)\}$ of the vector field X_k ; that is,

$$\begin{aligned} \frac{dx(t)}{dt} &= X_k(x(t)), \\ (1.1) \quad x(0) &= P. \end{aligned}$$

Consider the curve $\Gamma: \{x = x(t); t_1 \leq t \leq t_2\}$ parametrized by any finite segment

$(t_1 < t_2)$ of such a trajectory. We define Γ to be a *diffusion trajectory* in ω if the following conditions are satisfied:

- (1.2) (i) $x(t) \in \omega$, $t_1 \leq t \leq t_2$,
 (ii) $X_k(x(t)) \neq 0$, $t_1 \leq t \leq t_2$.

These conditions are clearly independent of the particular parametrization of Γ . We adopt the convention that *diffusion trajectories are not oriented*; they may be traversed in either direction. Note that a diffusion trajectory corresponding to X_k passes through P if and only if $X_k(P) \neq 0$. Here $k = 1, 2, \dots, n$. Thus if $\text{rank } \alpha(P) = l$, there are at least l (perhaps more) distinct diffusion trajectories which pass through P .

Let the coefficients $b^i(x) \in C^1(\omega)$, and consider the vector field $X_0(x) = (\beta^1(x), \beta^2(x), \dots, \beta^n(x))$ whose i -th component is given by

$$\beta^i(x) = b^i(x) - a_{x_i}^{ii}(x).$$

Then $X_0(x)$ is Lipschitz continuous in ω . Again, this Lipschitz continuity of $X_0(x)$ is all that is actually needed; for simplicity, we assume $b^i \in C^1$ and $a^{ii} \in C^2$. The direction defined by $X_0(P)$ we call the *drift direction* at P . We define a *drift trajectory* in ω in exactly the same way we defined a diffusion trajectory; that is, by (1.1) and (1.2) with $k = 0$. However, the *drift trajectories are oriented*; they point in the drift direction. Thus a drift trajectory passes through P if and only if $X_0(P) \neq 0$, in which case it emanates from P in the direction $X_0(P)$.

The importance of the vector field X_0 stems from the fact that the expression $(b^i - a_{x_i}^{ii})\varphi_{x_i}$ is invariant under changes of coordinates.

2. The propagation set. Two distinct points P and Q in ω are connected by a diffusion trajectory if there is some diffusion trajectory in ω on which they both lie. P and Q are connected by a drift trajectory, running from P to Q , if they both lie on the same drift trajectory in ω , and its orientation is from P to Q . We say P and Q are connected by a finite chain of propagation paths, running from P to Q , if there is a finite sequence of points $P = P_0, P_1, P_2, \dots, P_k = Q$ in ω such that, for $j = 1, 2, \dots, k$, each pair $\{P_{j-1}, P_j\}$ is connected by either a diffusion trajectory in ω or else by a drift trajectory in ω , running from P_{j-1} to P_j .

Consider a point $P \in \omega$. Then $S(P; \omega)$, the *propagation set* of P in ω , is defined as follows: By convention, we always assume that $P \in S(P; \omega)$. For any other point Q in ω , $Q \in S(P; \omega)$ if and only if P and Q can be connected by a finite chain of propagation paths in ω , running from P to Q . Note that $S(P; \omega) = \{P\}$ if and only if there are no diffusion trajectories and no drift trajectory passing through P .

$\bar{S}(P; \omega)$ shall always be the closure of $S(P; \omega)$ in the relative topology of ω . The following lemma will be needed later.

Lemma 1.

- (a) If $P^* \in S(P; \omega)$, then $S(P^*; \omega) \subset S(P; \omega)$.
 (b) If $P^* \in \bar{S}(P; \omega)$, then $\bar{S}(P^*; \omega) \subset \bar{S}(P; \omega)$.

Proof. Part (a) is trivial. In order to establish part (b), it will suffice to show that $S(P^*; \omega) \subset \bar{S}(P; \omega)$, for the desired result then follows by taking closures. Thus what must be proved is that $Q \in S(P^*; \omega)$ implies $Q \in \bar{S}(P; \omega)$. In fact, it will suffice to demonstrate this implication in the special case where there is only one link in the chain of propagation paths which connects P^* to Q . For, assuming that has been demonstrated, let Q be a general point in $S(P^*; \omega)$. Let $P^* = Q_0, Q_1, Q_2, \dots, Q_k = Q$ be the sequence of points which form the ends of the various links in the chain. Then $Q_1 \in \bar{S}(P; \omega)$ because there is only one link in the chain between P^* and Q_1 . But $Q_2 \in S(Q_1; \omega)$, so $Q_2 \in \bar{S}(P; \omega)$ because there is only one link in the chain connecting Q_1 to Q_2 . Repeating this argument k times, we find $Q \in \bar{S}(P; \omega)$.

Finally, to demonstrate $Q \in \bar{S}(P; \omega)$ in the case of one link between P^* and Q , let P_m^* be a sequence of points in $S(P; \omega)$ such that $P_m^* \rightarrow P^*$. Consider the trajectory which passes through P_m^* of the same vector field as the one whose trajectory connects P^* to Q . These trajectories depend continuously on their initial conditions because of the Lipschitz continuity. Hence for m sufficiently large, there is a trajectory which connects P_m^* to some point Q_m with $\|Q_m - Q\| = O(\|P_m^* - P^*\|)$, and such that conditions (1.2) are satisfied. This means that $Q_m \in S(P; \omega)$. Since $Q_m \rightarrow Q$, it follows that $Q \in \bar{S}(P; \omega)$. The proof of the lemma is complete.

3. Examples. 1°. *Elliptic equations.* Let the matrix $a(x)$ be positive definite in ω . Then for any point P in ω , $\bar{S}(P; \omega) = \omega$. Since then $\text{rank } \alpha(x) \equiv n$, the intuitive reason for this is that one can move in any direction by using diffusion trajectories; in fact, the drift trajectories are not needed.

A formal proof is easily given: Suppose there is some point $Q \in \omega - \bar{S}(P; \omega)$. Connect P and Q by any smooth curve Γ contained in ω . Traversing Γ from Q to P , there is a first point P' on Γ which is in $\bar{S}(P; \omega)$. Back up along Γ a distance, say, one-third the distance between P' and $\partial\omega$ to some point Q' . Then $Q' \in \omega - \bar{S}(P; \omega)$ and there is a convex neighborhood of Q' which contains P' and is contained in ω . Let $\delta > 0$ be the distance from Q' to $\bar{S}(P; \omega)$. Then there is some point $P^* \in \bar{S}(P; \omega)$ such that $\|Q' - P^*\| = \delta$. Consider the vector τ from Q' to P^* . For any diffusion trajectory $\{x = x(t)\}$, with $x(0) = P^*$, we must have $(d/dt) \|x(t) - Q'\|^2 = 0$ at $t = 0$ because, otherwise, since $S(P^*; \omega) \subset \bar{S}(P; \omega)$, we could move a short distance along $\{x = x(t)\}$ and find a point in $\bar{S}(P; \omega)$ closer to Q' than P^* (recall that diffusion trajectories can be traversed in either direction). This means that $(\tau, X_k(P^*)) = 0$ for $k = 1, 2, \dots, n$. Since $\text{rank } \alpha(P^*) = n$, we obtain $\tau = 0$ and hence $Q' = P^* \in \bar{S}(P; \omega)$. This contradiction completes the proof.

2°. *Parabolic equations.* Let the operator L have the form

$$Lu = \sum_{i,j=2}^n a^{ij} u_{x_i x_j} + \sum_{i=2}^n b^i u_{x_i} - u_{x_1},$$

where $[a^{ij}]$ is positive definite in ω . Then for any $P \in \omega$, $\tilde{S}(P; \omega)$ consists of all points Q in ω which can be connected to P by a path, running from P to Q and contained in ω , along which the x_1 -coordinate does not increase. Intuitively, one can move in any direction in $(x_2, x_3, \dots, x_n)$ -space by using diffusion trajectories, and, by using drift trajectories, one can move in the negative x_1 -direction since the first component of the drift direction is $b^1 - a_{x_1}^{1j} \equiv -1$. We omit the formal proof, which is similar to the one given above for the elliptic case.

3°. *First order equations.* Let $Lu = b^i(x)u_{x_i}$ where $\sum_{i=1}^n (b^i(x))^2 \neq 0$ in ω . Then for any $P \in \omega$, $\tilde{S}(P; \omega)$ is the curve in ω parametrized by the drift trajectory $\{x = x(t); t \geq 0\}$ defined by: $x(0) = P$, $\dot{x}_i(t) = b^i(x(t))$ ($i = 1, 2, \dots, n$).

4°. *The operator* $Lu = u_{xx} + xu_y$. First, consider this example in the square $\omega = \{-2 < x < 0, 0 < y < 2\}$, and let $P = (-1, 1)$. Then $\tilde{S}(P; \omega)$ is the rectangle $\{-2 < x < 0, 0 < y \leq 1\}$. Now let P be as before, but perturb the right-hand side of the square by a small amount $\epsilon > 0$ so that ω becomes $\{-2 < x < \epsilon, 0 < y < 2\}$. Then $\tilde{S}(P; \omega)$ consists of the entire rectangle $\{-2 < x < \epsilon, 0 < y < 2\}$. Thus an arbitrarily small perturbation of the region, far away from P , can cause $\tilde{S}(P; \omega)$ to change drastically near P . In this example, one can always diffuse across ω (in the x -direction). In the first case, one can only drift down. But in the second case, one can also drift up in the small strip $0 < x < \epsilon$ of ω ; that is enough to go anywhere.

5°. *The operator* $Lu = y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} - xu_x - yu_y$. In this example, let ω be the unit disc about the origin. If $P = (0, 0)$, then $\tilde{S}(P; \omega) = \{P\}$. For any other point $P \neq (0, 0)$ in ω , $\tilde{S}(P; \omega)$ consists of the circle drawn through P with center at the origin. Indeed, when written in polar coordinates, the operator L becomes $Lu = u_{\theta\theta}$. Hence there are no drift trajectories, and the diffusion runs along circles about the origin.

6°. *The operator* $Lu = y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} - (1 + \epsilon)xu_x - (1 + \epsilon)yu_y$. Again, let ω be the unit disc about the origin and $P \neq (0, 0)$ be a point in ω . If $\epsilon > 0$, then $\tilde{S}(P; \omega)$ is the closed disc which is centered about the origin and has P on its boundary. If $\epsilon < 0$, then $\tilde{S}(P; \omega)$ is the annulus $\{\|P\| \leq r < 1, 0 \leq \theta < 2\pi\}$. In polar coordinates, the L of this example is $Lu = u_{\theta\theta} - \epsilon ru_r$. Therefore, when $\epsilon > 0$, the drift points radially inward; when $\epsilon < 0$, the drift points radially outward. Thus $\tilde{S}(P; \omega)$ can be unstable with respect to small perturbations of the coefficients of L .

4. Statement of the main theorems. Our results shall be stated only for the case where u has a maximum and u is a subsolution ($Lu \geq 0$). Each theorem has, obviously, a reflection which covers the case where u has a minimum and u is a supersolution ($Lu \leq 0$). The assumptions we make about the smoothness of the coefficients are designed to cover the general situation. In particular cases (elliptic equations, parabolic equations, etc.), the details of the proofs show that these smoothness assumptions can often be relaxed.

The hypotheses needed for Theorem 1, which for later reference we call the *interior assumptions*, are as follows: ω is open and connected in $\mathbf{R}^n$, $P \in \omega$, $a^{ij} \in C^2(\omega)$,

$b^i \in C^1(\omega)$, $u \in C^2(\omega)$, and $a^{ii}(x)\xi_i\xi_i \geq 0$ for all $\xi \in \mathbb{R}^n$ and all $x \in \omega$. Here $Lu = a^{ii}u_{x_i x_i} + b^i u_{x_i}$.

Theorem 1. (the interior principal). *If $Lu \geq 0$ in $S(P; \omega)$,*

$$\sup_{S(P; \omega)} u \leq M < \infty,$$

and $u(P) = M$, then $u \equiv M$ in $\bar{S}(P; \omega)$.

As is well known, theorems of the above type can be reformulated in various ways. We give one such reformulation: Let $c(x) \in C(\omega)$ be a function such that $c(x) \leq 0$ in ω .

Theorem 1'. *If $Lu + cu \geq 0$ in $S(P; \omega)$,*

$$\sup_{S(P; \omega)} u \leq M < \infty,$$

$u(P) = M$, and $M > 0$, then $u \equiv M$ in $\bar{S}(P; \omega)$.

Consider an open set $\Omega \subset \mathbb{R}^n$ and a point $P \in \partial\Omega$. $\partial\Omega$ has the interior sphere property at P if there exists an open ball $B \subset \Omega$ with $P \in \partial B$. If τ is any direction which points into such a ball B at P , we say τ points into Ω at P . Let ω be an open neighborhood of P , and consider $\bar{S}(P; \omega)$. We say the direction τ points into $\Omega \cap \bar{S}(P; \omega)$ at P , if τ points into Ω at P , and if τ is tangent at P to some (small) curve contained in $\bar{\Omega} \cap \bar{S}(P; \omega)$. If the coefficients of L satisfy the interior assumptions in ω , then $S(P; \omega)$ is well-defined. However, it is of interest to make the less stringent requirement that $a^{ii}(x)\xi_i\xi_i \geq 0$ merely for $x \in \Omega \cap \omega$. In that case, Phillips and Sarason [17] have given an example which shows that the $a^{ik}(x)$ need not be Lipschitz continuous at P , even for entire analytic $a^{ii}(x)$. Therefore, in order to define $S(P; \omega)$ as before, we need additional hypotheses in the case where the semi-definiteness of $a(x)$ is known to hold only in $\Omega \cap \omega$.

With the above remarks in mind, we now state the *boundary assumptions* needed for Theorem 2: Ω has the interior sphere property at $P \in \partial\Omega$, ω is an open connected neighborhood of P , $a^{ii} \in C^2(\omega)$, $b^i \in C(\omega)$, $a^{ii}(x)\xi_i\xi_i \geq 0$ for all $\xi \in \mathbb{R}^n$ and all $x \in \Omega \cap \omega$, and the vector fields $X_k(x)$ ($k = 0, 1, 2, \dots, n$) have Lipschitz continuous extensions to ω . Then $S(P; \omega)$ and $\bar{S}(P; \omega)$ can be defined just as before. Finally, we assume that $u \in C^2(\Omega \cap \omega)$ and $u \in C(\bar{\Omega} \cap \omega)$.

Theorem 2 (boundary principle). *If $Lu \geq 0$ in $\Omega \cap S(P; \omega)$, $u < M$ in $\Omega \cap \bar{S}(P; \omega)$, and $u(P) = M$, then $D_\tau u(P) < 0$ for any directional derivative τ which points into $\Omega \cap \bar{S}(P; \omega)$ at P .*

Remark. If $D_\tau u(P)$ does not exist in the ordinary sense, then it should be interpreted as the limit superior of $\Delta u / \Delta \tau$ along the curve in $\bar{\Omega} \cap \bar{S}(P; \omega)$ tangent to τ at P .

By strengthening the hypothesis, one can obtain another (weaker) formulation of Theorem 2:

Theorem 2'. *If $Lu \geq 0$ in $\Omega \cap \omega$, $u < M$ in $\Omega \cap \omega$, $u(P) = M$, and if either the hyperplane tangent to $\partial\Omega$ (i.e., tangent to B) at P is noncharacteristic or else the drift direction at P points into Ω (i.e., into B), then $D_\tau u(P) < 0$ for any direction τ which points into Ω (i.e., into B) at P .*

5. Application to the degenerate Dirichlet-Neumann Problem. The maximum principle is one of the most important tools in the theory of second order elliptic and parabolic equations; its many applications are well known (see, for example, the book [18]). In particular, it provides a routine method for proving the uniqueness of classical solutions to the standard elliptic and parabolic boundary value problems. Here we discuss briefly that routine application to a class of boundary value problems for degenerate elliptic-parabolic equations which was introduced by Fichera [7], [8]. We do this not out of any inordinate interest in the uniqueness of classical solutions, but because we wish to illustrate how the global behavior of the diffusion and drift trajectories may be connected with the uniqueness question. Indeed, several authors (see [12], [14], [15], [16], [17]) have discussed the uniqueness and regularity of *weak solutions* to Fichera's Problem. However, the authors mentioned above have all dealt with operators of the form $Lu + cu$ where c is assumed to be strictly negative or even sufficiently large negative. Here we treat the case $c \equiv 0$ which, in spite of the fact that our solutions are assumed to be classical, is geometrically a more delicate question.

Let Ω be an open connected region in $\mathbb{R}^n$. For simplicity, we assume that Ω is bounded and that its boundary $\partial\Omega$ is smooth. At interior points of Ω , let the coefficients of L satisfy the interior assumptions; at boundary points of Ω , let the coefficients of L satisfy the boundary assumptions. The outer unit normal to $\partial\Omega$ is denoted by $\nu = (\nu_1, \nu_2, \dots, \nu_n)$. Following Fichera, we divide $\partial\Omega$ into three parts:

Σ_3 = the set of noncharacteristic boundary points; i.e., those where $a^{ij}\nu_i\nu_j > 0$.

Σ_2 = the set of characteristic boundary points where the drift direction points outward; i.e., those where $a^{ij}\nu_i\nu_j = 0$ and $(b^i - a_{x_i}^{ij})\nu_j > 0$.

$\Sigma_1 = \partial\Omega - \Sigma_2 - \Sigma_3$.

Let $\overset{\circ}{\Sigma}_1$ denote the interior of Σ_1 in the relative topology of the boundary.

In Section 2, the sets $S(P; \Omega)$ and $\bar{S}(P; \Omega)$ (with $\Omega = \omega$) were defined only for the case where $P \in \Omega$ and Ω is open. This was done so as not to obscure our arguments with extraneous detail. Actually, for the application at hand, we need to make slight modifications of our definitions; namely, for $P \in \Omega \cup \overset{\circ}{\Sigma}_1$, we define analogous sets $S(P; \Omega \cup \overset{\circ}{\Sigma}_1)$ and $\bar{S}(P; \Omega \cup \overset{\circ}{\Sigma}_1)$ as follows: $P \in S(P; \Omega \cup \overset{\circ}{\Sigma}_1)$ by convention. For any other point $Q \in \Omega \cup \overset{\circ}{\Sigma}_1$, Q is in $S(P; \Omega \cup \overset{\circ}{\Sigma}_1)$ if and only if P and Q can be connected by a finite chain of propagation paths, running from P to Q , which are all contained in $\Omega \cup \overset{\circ}{\Sigma}_1$. $\bar{S}(P; \Omega \cup \overset{\circ}{\Sigma}_1)$ denotes the closure of $S(P; \Omega \cup \overset{\circ}{\Sigma}_1)$ in the relative topology of $\Omega \cup \overset{\circ}{\Sigma}_1$.

Next, we observe that Theorem 1 remains true with $S(P; \omega)$ and $\bar{S}(P; \omega)$ replaced by $S(P; \Omega \cup \overset{\circ}{\Sigma}_1)$ and $\bar{S}(P; \Omega \cup \overset{\circ}{\Sigma}_1)$. Indeed, suppose $Lu \geq 0$ in $S(P; \Omega \cup \overset{\circ}{\Sigma}_1)$,

$$\sup_{S(P; \Omega \cup \overset{\circ}{\Sigma}_1)} u \leq M,$$

and $u(P) = M$. We want to show that $u \equiv M$ in $\tilde{S}(P; \Omega \cup \overset{\circ}{\Sigma}_1)$. Suppose the contrary, that there is some point $Q \in \tilde{S}(P; \Omega \cup \overset{\circ}{\Sigma}_1)$ such that $u(Q) < M$. Then by continuity of u , there is a point $Q' \in S(P; \Omega \cup \overset{\circ}{\Sigma}_1)$ such that $u(Q') < M$. Traverse the chain of propagation paths connecting P and Q' backward toward P . There is a first point P' at which $u = M$. First, suppose $P' \in \Omega$, and consider $S(P'; \omega)$ where ω is a small neighborhood of P' such that $\omega \subset \Omega$. Then $u \equiv M$ in $\tilde{S}(P'; \omega)$ by Theorem 1. Next, suppose $P' \in \overset{\circ}{\Sigma}_1$, and consider a sufficiently small neighborhood ω of P' . Then $S(P'; \omega)$ and $\tilde{S}(P'; \omega)$ are both contained in $\Omega \cup \overset{\circ}{\Sigma}_1$. This follows from the fact that on $\omega \cap \partial\Omega$, the drift direction cannot point outward, and all the diffusion directions are actually tangential (see (6.6)) to $\partial\Omega$. Thus $u \equiv M$ in $\tilde{S}(P'; \omega)$, again by Theorem 1. Hence in either case, P' is not the first point where $u = M$ after all. This contradiction completes the proof.

The Dirichlet Problem. This is the problem of solving $Lu = f$ in Ω with prescribed data $u = g$ on $\Sigma_2 \cup \Sigma_3$. We assume we are dealing with classical solutions; that is, $u \in C^2(\Omega \cup \overset{\circ}{\Sigma}_1)$ and $u \in C(\bar{\Omega})$. To prove uniqueness, suppose u_1 and u_2 are two solutions corresponding to the same data, and set $u = u_1 - u_2$. Then $u = 0$ on $\Sigma_2 \cup \Sigma_3$ and $Lu = 0$ on $\Omega \cup \overset{\circ}{\Sigma}_1$ by continuity. We must show that $u \equiv 0$ on $\bar{\Omega}$. Assume the contrary, say, that $u > 0$ somewhere. Then u must attain its maximum $= M > 0$ at some point $P \in \bar{\Omega}$. The possibility $P \in \Sigma_2 \cup \Sigma_3$ can not occur. The remaining possibility $P \in \Omega \cup \overset{\circ}{\Sigma}_1$ implies that $u \equiv M > 0$ on $\tilde{S}(P; \Omega \cup \overset{\circ}{\Sigma}_1)$. In order to derive a contradiction from that, we must have some point in $\Sigma_2 \cup \Sigma_3$ which is a limit point of points in $\tilde{S}(P; \Omega \cup \overset{\circ}{\Sigma}_1)$. Thus we arrive at a sufficient condition for uniqueness in the Dirichlet problem.

Dirichlet uniqueness criterion. For every $P \in \Omega \cup \overset{\circ}{\Sigma}_1$, the closure of $\tilde{S}(P; \Omega \cup \overset{\circ}{\Sigma}_1)$ in $\mathbb{R}^n$ has non-empty intersection with $\Sigma_2 \cup \Sigma_3$.

The above criterion can be generalized in various ways. For example, one can allow isolated points P which fail to satisfy the criterion. An example is given by the operator L of the previous example 6°, with $\epsilon < 0$, in the unit disc. Then the boundary of the disc is of the type Σ_2 , and every point P in the disc satisfies the criterion except for the isolated point $P = (0, 0)$. Uniqueness still holds, however, since the criterion is satisfied throughout any annulus.

Another type of generalization is obtained by prescribing interior data at *stagnation points*; that is, at points where all the coefficients of L simultaneously vanish. As an example, consider the operator L of the previous example 6°, with $\epsilon > 0$, in $\Omega =$ the unit disc. Then the boundary of the disc is of type $\overset{\circ}{\Sigma}_1$, and for any point $P \in \Omega \cup \overset{\circ}{\Sigma}_1$, $\tilde{S}(P; \Omega \cup \overset{\circ}{\Sigma}_1)$ contains the origin. Hence, for this example, a solution of $Lu = f$ is uniquely determined by prescribing the value of u at the origin. Note that here Σ_2 and Σ_3 are empty.

The Dirichlet-Neumann Problem. This is just like the Dirichlet Problem except that Σ_3 is decomposed,

$$\Sigma_3 = \Sigma_3(D) + \Sigma_3(N),$$

into two disjoint sets; Dirichlet data $u = g$ is prescribed on $\Sigma_2 \cup \Sigma_3(D)$, and Neumann data $D_\tau u = h$ is prescribed on $\Sigma_3(N)$. Here D_τ denotes differentiation in some nontangential direction τ which may vary on $\Sigma_3(N)$. Arguing as before and using Theorem 2' to rule out the possibility of u attaining its maximum at a point $P \in \Sigma_3(N)$, we obtain a sufficient condition for uniqueness in the Dirichlet–Neumann Problem.

Dirichlet–Neumann uniqueness criterion. For every $P \in \Omega \cup \overset{\circ}{\Sigma}_1$, the closure of $\tilde{S}(P; \Omega \cup \overset{\circ}{\Sigma}_1)$ in $\mathbb{R}^n$ has non-empty intersection with $\tilde{\Sigma}_2 \cup \tilde{\Sigma}_3(D)$.

6. Proof of the main theorems. Until mentioned otherwise, we work throughout this section under the *interior assumptions* listed just before the statement of Theorem 1. Recall that our definitions include these conventions: ω is open in $\mathbb{R}^n$, $P \in \omega$, $P \in S(P; \omega)$, $\tilde{S}(P; \omega)$ is the closure of $S(P; \omega)$ in the relative topology of ω , and $u \in C^2(\omega)$. In the following lemma, we consider the restriction of u to $\tilde{S}(P; \omega)$.

Lemma 2. If, relative to $\tilde{S}(P; \omega)$, u has a local maximum at some point $P^* \in \tilde{S}(P; \omega)$, then $Lu \leq 0$ at P^* .

Proof. Introduce the first order differential operators $D_k = \alpha^{ik}(x)(\partial/\partial x_i)$ ($k = 1, 2, \dots, n$) and $D_0 = \beta^i(x)(\partial/\partial x_i)$ corresponding to the vector fields X_k ($k = 1, 2, \dots, n$) and X_0 . Then at any given point, each D_k is either zero or else it represents differentiation along a diffusion trajectory through that point. Similarly, D_0 is either zero or else it represents differentiation along the drift trajectory. According to Lemma 1, $\tilde{S}(P^*; \omega) \subset \tilde{S}(P; \omega)$. Hence the maximum of u at P^* is also a local maximum relative to the set $\tilde{S}(P^*; \omega)$. Therefore, at P^* we have $D_0 u \leq 0$, $D_k u = 0$, and $D_k^2 u \leq 0$ for $k = 1, 2, \dots, n$.

Suppose the α^{ik} have first derivatives. Then

$$\begin{aligned}
 (6.1) \quad Lu &= (a^{ii}u_{x_i})_{x_i} + (b^i - a_{x_i}^i)u_{x_i} \\
 &= (\alpha^{ik}\alpha^{ik}u_{x_i})_{x_i} + \beta^i u_{x_i} \\
 &= \alpha^{ik}(\alpha^{ik}u_{x_i})_{x_i} + \alpha_{x_i}^{ik}\alpha^{ik}u_{x_i} + \beta^i u_{x_i} \\
 &= D_k D_k u + \alpha_{x_i}^{ik} D_k u + D_0 u \\
 &\leq 0,
 \end{aligned}$$

at P^* . In order to obtain the result in general, replace the derivatives with respect to the x_i in (6.1) by corresponding difference quotients taken along the relevant diffusion trajectories. By the Lipschitz continuity of α^{ik} , the difference quotients corresponding to $\alpha_{x_i}^{ik}$ are uniformly bounded. Hence, in the limit one obtains the desired inequality.

The next lemma is a generalization of the well known lemmas of Hopf [11] and Nirenberg [13].

Lemma 3. Let B be an open ball with $\bar{B} \subset \omega$. Suppose $Lu \geq 0$ in $S(P; \omega)$

and

$$\sup_{S(P; \omega)} u = M < \infty.$$

If $u < M$ in $B \cap \bar{S}(P; \omega)$ but $u = M$ at some point $P_1 \in \partial B \cap \bar{S}(P; \omega)$, then the tangent hyperplane to B at P_1 is characteristic for L .

Proof. By replacing B by another open ball B_0 contained in B and tangent to B at P_1 , we can assume that $u < M$ in $\bar{B}_0 \cap \bar{S}(P; \omega) - \{P_1\}$ and $u(P_1) = M$. Let $P_0 = (x_1^0, x_2^0, \dots, x_n^0)$ be the center of B_0 and $P_1 = (x_1^1, x_2^1, \dots, x_n^1)$. What must be shown is that

$$(6.2) \quad a^{ii}(P_1)(x_i^1 - x_i^0)(x_i^1 - x_i^0) = 0.$$

Assume the contrary, that (6.2) is positive. Since $P_1 \in \omega$, there is then a sufficiently small open ball B_1 centered at P_1 , with $\bar{B}_1 \subset \omega$, such that $a^{ii}(x)(x_i - x_i^0)(x_i - x_i^0)$ is bounded away from zero on $\bar{B}_1$. Set $\partial_1 B_1 = \partial B_1 \cap \bar{B}_0$ and $\partial_2 B_1 = \partial B_1 - \partial_1 B_1$. Since $\partial_1 B_1$ is closed and $\partial_1 B_1 \subset \omega$, it follows that $\partial_1 B_1 \cap \bar{S}(P; \omega)$ is closed. But $u < M$ on $\partial_1 B_1 \cap \bar{S}(P; \omega)$. Hence $u \leq M - \eta$ there for some small $\eta > 0$. Also, $u \leq M$ on $\partial_2 B_1 \cap \bar{S}(P; \omega)$.

Let $R > 0$ be the radius of B_0 . The function

$$h = \exp \left\{ -\gamma \sum_{i=1}^n (x_i - x_i^0)^2 \right\} - \exp \{-\gamma R^2\}, \quad (\gamma > 0)$$

is positive inside B_0 , zero on ∂B_0 , and negative outside $\bar{B}_0$. It follows that

$$\exp \left\{ \gamma \sum_{i=1}^n (x_i - x_i^0)^2 \right\} Lh = 4\gamma^2 [a^{ii}(x)(x_i - x_i^0)(x_i - x_i^0)] + O(\gamma).$$

Since the term in brackets is bounded away from zero, γ can be chosen so large that $Lh > 0$ on $\bar{B}_1$.

Now consider the function $v = u + \delta h$ where $\delta > 0$ is a constant chosen so small that $v < M$ on $\partial_1 B_1 \cap \bar{S}(P; \omega)$. But $v < M$ also on $\partial_2 B_1 \cap \bar{S}(P; \omega)$ and $v = u = M$ at $P_1 \in B_1 \cap \bar{S}(P; \omega)$. It follows that the maximum of v on the closed set $\bar{B}_1 \cap \bar{S}(P; \omega)$ is attained at some point $P^* \in B_1 \cap \bar{S}(P; \omega)$. Thus v has a local maximum relative to $\bar{S}(P; \omega)$ at P^* . Hence $Lv \leq 0$ at P^* by Lemma 2. But $Lu \geq 0$ at P^* by continuity. Therefore, $Lv = Lu + \delta Lh \geq \delta Lh > 0$ at P^* . This gives the desired contradiction; the proof of the lemma is complete.

The next lemma is concerned with the same situation as Lemma 3, but gives additional information.

Lemma 4. Let B be an open ball with $\bar{B} \subset \omega$. Suppose $Lu \geq 0$ in $S(P; \omega)$ and

$$\sup_{S(P; \omega)} u = M < \infty.$$

If $u < M$ in $B \cap \bar{S}(P; \omega)$ but $u = M$ at some point $P_1 \in \partial B \cap \bar{S}(P; \omega)$, then the drift direction at P_1 cannot point into B .

Proof. If there is no drift direction at P_1 , then there is nothing to prove. Otherwise, the nonzero vector $\beta = (\beta^1, \beta^2, \dots, \beta^n)$, where

$$(6.3) \quad \beta^i = b^i(P_1) - a_{x_i}^{ii}(P_1),$$

points in the drift direction. Let $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ be the unit inner normal to B at P_1 . Then the assertion of the lemma is that $(\beta, \xi) > 0$ cannot hold. Note that $Lu \geq 0$ at P_1 by continuity. Under the assumption that $(\beta, \xi) > 0$, we shall derive the contradiction that $Lu < 0$ at P_1 .

According to Lemma 3, we have

$$(6.4) \quad a^{ij}(P_1)\xi_i\xi_j = 0.$$

It is convenient to make a local change of coordinates near P_1 so that in the new coordinate system P_1 is the origin, $\xi = (1, 0, 0, \dots, 0)$, and the boundary ∂B near P_1 corresponds to the hyperplane $x_1 = 0$ with B on the side $x_1 > 0$. We denote the coefficients of L in the new coordinate system again by a^{ij} and b^i . Then (6.4) remains true in the new coordinate system. It is also important to note that the drift direction β is still given by (6.3) and that the invariant condition $(\beta, \xi) > 0$ becomes

$$(6.5) \quad \beta^1 = b^1(0) - a_{x_1}^{11}(0) > 0.$$

In this new coordinate system, consider the open ball B_0 of radius $R > 0$ centered at the point $(R, 0, 0, \dots, 0)$. The point P_1 (now the origin) lies on its boundary. Let B_1 be a sufficiently small open ball of radius $\epsilon > 0$ centered at P_1 such that $\bar{B}_1 \subset \omega$. A convenient choice of R and ϵ will be made later. For the moment, let us assume that this choice has been made and hence R and ϵ are fixed. Consider the open domain $D = B_1 \cap B_0$. Set $\partial_1 D = \partial D \cap \bar{B}_0$ and $\partial_2 D = \partial D - \partial_1 D$. Since $u < M$ on the closed set $\partial_1 D \cap \bar{S}(P; \omega)$, then $u \leq M - \eta$ there for some small $\eta > 0$. On the set $\partial_2 D \cap \bar{S}(P; \omega) - \{P_1\}$ we have $u < M$.

Now consider the function

$$h = R^2 - (x_1 - R)^2 - \sum_{i=2}^n x_i^2,$$

which is zero on ∂B_0 and positive in B_0 . If $D_0 = \beta^i(\partial/\partial x_i)$ denotes differentiation in the drift direction, then we have $D_0 h = 2\beta^1 R > 0$ at P_1 because of (6.5). With proper choices of R and ϵ , one can achieve that $Lh > 0$ in $\bar{B}_1$: First observe that, by (6.4),

$$(6.6) \quad 0 = a^{ii}(0)\xi_i\xi_i = \alpha^{ik}(0)\alpha^{ik}(0)\xi_i\xi_i = \sum_{k=1}^n \left[\sum_{i=1}^n \alpha^{ik}(0)\xi_i \right]^2.$$

Thus each of the operators $D_k = \alpha^{ik}(\partial/\partial x_i)$ represents, at P_1 , differentiation in a direction tangential to ∂B_0 . Therefore each $D_k h = 0$ at P_1 because h vanishes on ∂B_0 . The magnitude of $D_k D_k h$ at P_1 depends only on the convexity of the function h , and that is independent of R . Write Lh in the form (6.1). Choose $R > 0$ so large that the positive term $D_0 h = 2\beta^1 R$ dominates $D_k D_k h$. For fixed R ,

the remaining term $\alpha_{x_i}^{ik} D_k h$ is $O(\epsilon)$ in $\bar{B}_1$. Hence for $\epsilon > 0$ chosen sufficiently small, $Lh > 0$ in $\bar{B}_1$.

Set $v = u + \delta h$ where $\delta > 0$ is a constant chosen so small that $v < M$ on $\partial_1 D \cap \bar{S}(P; \omega)$. Note that $v < M$ also on $\partial_2 D \cap \bar{S}(P; \omega) - \{P_1\}$, since $h = 0$ there. In $D \cap \bar{S}(P; \omega)$, v cannot have a local maximum relative to $\bar{S}(P; \omega)$. Indeed, if P^* were such a maximum, we would have $Lv \leq 0$ at P^* by Lemma 2. But at P^* , $Lv = Lu + \delta Lh \geq \delta Lh > 0$, which is a contradiction. Thus the maximum of v on the closed set $\bar{D} \cap \bar{S}(P; \omega)$ occurs exactly at P_1 . Since $S(P_1; \omega) \subset \bar{S}(P; \omega)$, it follows that $D_0 v \leq 0$ at P_1 . Therefore $D_0 u \leq -\delta D_0 h < 0$ at P_1 . However, at P_1 , u has a local maximum relative to $\bar{S}(P_1; \omega)$. Hence $D_k u = 0$ and $D_k D_k u \leq 0$ at P_1 . Writing Lu in the form (6.1), we obtain $Lu \leq D_0 u < 0$ at P_1 . This is the desired contradiction; the proof of the lemma is complete.

Proof of Theorem 1. Our aim is to prove that $u \equiv M$ in $\bar{S}(P; \omega)$. Assume the contrary, that there is a point $Q \in \bar{S}(P; \omega)$ at which $u < M$. By continuity of u , there is then a point $Q' \in S(P; \omega)$ such that $u(Q') < M$. Now P and Q' are connected by a finite chain of propagation paths, running from P to Q' , in which each link is either a diffusion trajectory or a drift trajectory. Traversing these trajectories backward from Q' to P , there is a first point P' (perhaps $P' = P$) at which $u(P') = M$. We shall derive a contradiction by showing that P' is not, after all, the first such point.

It will suffice to confine our attention to a small neighborhood U of P' such that $\bar{U} \subset \omega$. Let the trajectory which emanates from P' and forms a link in the chain connecting P' to Q' be given by $\{x = x(t); t \geq 0\}$. Without loss of generality, take P' to be the origin. Then there are two cases to consider, depending on whether $\{x = x(t); t \geq 0\}$ is a diffusion trajectory or a drift trajectory. Thus, explicitly,

$$(6.7) \quad \begin{aligned} \frac{dx(t)}{dt} &= X(x(t)), \\ x(0) &= 0, \end{aligned}$$

for t sufficiently small, where the vector field $X(x)$ is either $X_0(x)$ or else one of the $X_k(x)$. In the latter case, we shall suppose that $k = 1$. In either case, we have, by the definitions of drift and diffusion trajectories, that $\|X(0)\| \neq 0$. Therefore we may assume that $\|X(x)\| \neq 0$ for $x \in U$. Consider the relatively closed set

$$\mathfrak{N} = \{x \in \omega \mid u(x) = M\} \cap \bar{S}(P; \omega).$$

Then $x(0) = 0 \in \mathfrak{N}$ but $x(t) \notin \mathfrak{N}$ for (small) $t > 0$. If

$$\rho(t) = \inf_{y \in \mathfrak{N}} \|x(t) - y\|$$

denotes the distance from the point $x(t)$ to the set $\mathfrak{N}$, then we have $\rho(0) = 0$ but $\rho(t) > 0$ for (small) $t > 0$. In order to complete the proof, it will suffice to show, since $\mathfrak{N}$ is relatively closed, that actually $\rho(t) \equiv 0$ for (small) $t \geq 0$.

The Lipschitz continuity of $X(x)$ implies that $\|x(t + \delta) - x(t)\| = O(\delta)$. But the function representing the distance from a point to a set is also Lipschitz

continuous. Hence $\rho(t)$ is Lipschitz continuous. Using the Lipschitz continuity of $X(x)$ again, we can choose U so small that the direction of the vector field X varies throughout U at most by a small angle, say ϵ . In fact, if such a neighborhood U has diameter $O(\rho)$, we have $\epsilon = O(\rho)$. Let $B(t)$ be the open ball of radius $\rho(t)$ centered at $x(t)$. For t sufficiently small, $\overline{B(t)} \subset U$. Then $u < M$ in $B(t) \cap \bar{S}(P; \omega)$ but $u = M$ for at least one point $P_1(t) \in \partial B(t) \cap \bar{S}(P; \omega)$. We wish to examine the distribution on $\partial B(t)$ of all such points $P_1(t)$. The two cases must be considered separately.

First, consider the case $X = X_0$ of a drift trajectory. According to Lemma 4, $X_0(P_1(t))$ cannot point into $B(t)$. But throughout U , the direction of X_0 varies by no more than ϵ from the direction $X_0(0)$. Therefore all such points $P_1(t)$ must lie on the "down-drift" hemisphere, enlarged by an amount $\epsilon\rho(t)$, formed by cutting $\partial B(t)$ with the hyperplane normal to $X_0(0)$ which passes through $x(t)$ (see Figure 1).

Second, consider the case $X = X_1$ of a diffusion trajectory. According to Lemma 3, the hyperplane tangent to $B(t)$ at $P_1(t)$ must be characteristic. Let ξ be the unit inner normal to $\partial B(t)$ at the point $P_1(t)$. Then (6.4) holds, and, by a calculation like (6.6), we see that ξ must be orthogonal to the vector

$$X_1(P_1(t)) = (\alpha^{11}(P_1(t)), \dots, \alpha^{n1}(P_1(t))).$$

But, again, the direction of X_1 varies from the direction $X_1(0)$ by no more than ϵ . Hence all such points $P_1(t)$ must lie in a narrow band of width $2\epsilon\rho(t)$ on $\partial B(t)$ centered about the intersection of $\partial B(t)$ with the hyperplane normal to $X_1(0)$ which passes through $x(t)$ (see Figure 1).

Having established the distribution of the points $P_1(t)$ on $\partial B(t)$, we can now estimate $\rho(t + \delta) - \rho(t)$ for small $\delta > 0$. Namely, let δ be so small that $0 < \Delta < \rho(t)$ where $\Delta = \|x(t + \delta) - x(t)\|$. Consider the ball $B(t + \delta)$ of radius $\rho(t + \delta)$ centered at $x(t + \delta)$ (see Figure 2). A very small ball centered at $x(t)$ is contained in $B(t + \delta)$ and hence does not intersect $\mathfrak{M}$. As the radius of that small ball increases, the first opportunity it has to intersect $\mathfrak{M}$ is at a point of the type A shown in Figure 2. Therefore we obtain the lower bound $\rho(t) \geq a$ so that $\rho(t + \delta) - \rho(t) \leq \rho(t + \delta) - a = b$. However, elementary trigonometry shows that $b = O(\Delta)[O(\Delta) + O(\rho(t))]$. But $\Delta = O(\delta)$ and $0 < \Delta < \rho(t)$. Hence $b = O(\delta)O(\rho(t))$, and we obtain,

Second, consider the case where the tangent hyperplane to B at P is characteristic. Then, as we have seen in the proof of Lemma 4, all of the diffusion trajectories are tangent to B at P . Hence there is nothing to prove unless the drift direction at P points into B . But if that is the case, we can repeat the proof of Lemma 4, with $P_1 = P$, down to the point where it has been shown that the maximum of v , relative to $\bar{D} \cap \bar{S}(P; \omega)$, occurs exactly at P . Once again $D_\tau h > 0$ at P provided τ points into B . Hence as before, $D_\tau u \leq -\delta D_\tau h < 0$ at P .

Note that we cannot assert that u has a local maximum at P relative to $\bar{S}(P; \omega)$ because we do not know that

$$\sup_{S(P; \omega)} u \leq M.$$

Indeed, this is the primary feature that distinguishes the interior principle from the boundary principle.

If we assume more, for example, that $Lu \geq 0$ in B and $u < M$ in B , then it is clear from the above proof that we obtain more; namely, that $D_\tau u < 0$ at P for *any* direction which points into B at P . This completes the proof of Theorems 2 and 2'.

7. Unique continuation. The technique used to prove Theorem 1 is reasonably general; it can be applied to situations in which there are appropriate analogues of Lemma 3 and/or Lemma 4. In this section, we show how it can be applied to study the propagation of the support of a solution of $Lu = 0$. As a corollary, we obtain a result on the unique continuation of solutions.

Analogous to $S(P; \omega)$, we define $C(P; \omega)$, the *continuation set* of P in ω , as follows: $C(P; \omega)$ is defined in exactly the same way as $S(P; \omega)$ except that, for $C(P; \omega)$, *only diffusion trajectories are allowed* in the propagation chain that connects P to a point $Q \in C(P; \omega)$. Again, $\bar{C}(P; \omega)$ denotes the closure of $C(P; \omega)$ in the relative topology of ω . Since diffusion trajectories can be traversed in either direction, the following lemma is the analogue of Lemma 1.

Lemma 1'.

- (a) If $P^* \in C(P; \omega)$, then $C(P^*; \omega) = C(P; \omega)$.
- (b) If $P^* \in \bar{C}(P; \omega)$, then $\bar{C}(P^*; \omega) \subset \bar{C}(P; \omega)$.

We also need the following assumption, which is the analogue of Lemma 3.

Assumption. Let B be an open ball with $\bar{B} \subset \omega$. Suppose u is a solution of $Lu = 0$ in ω . If $u \equiv 0$ in B but u fails to vanish identically in some (small) neighborhood of a point $P_1 \in \partial B$, then the tangent hyperplane to B at P_1 is characteristic for L .

Remark 1. This assumption amounts to the assertion that the solution of the local noncharacteristic Cauchy problem for L is unique. Therefore, in particular, the above assumption is valid if the coefficients of L are analytic by the classical Holmgren Uniqueness Theorem.

Remark 2. Here and in the theorems below, " u is a solution of $Lu = 0$ " can be taken in any desired sense; e.g., in the sense of distributions.

Remark 3. A similar analogue of Lemma 4 is not true, in general, even if L has constant coefficients. For example, one can easily construct a solution u of the heat equation such that $u \equiv 0$ for $t < 0$, but such that u does not vanish identically in any neighborhood of the origin.

Under the above assumption, we obtain the following theorems.

Theorem 3. Let u be a solution of $Lu = 0$ in ω and $P \in \omega$. If $P \in \text{supp } u$, then $\bar{C}(P; \omega) \subset \text{supp } u$.

Theorem 3'. Let u be a solution of $Lu = 0$ in ω and $A \neq \emptyset$ be an open set in ω . If $u \equiv 0$ in A , then $u(P) = 0$ for all points $P \in \omega$ such that $\bar{C}(P; \omega) \cap A \neq \emptyset$.

Example. Consider a solution u of $Lu = 0$ in ω where L is the operator of the previous example 5°, and ω is the rectangle $\{-2 < x < 2, -2 < y < 2\}$. If u vanishes identically in a small neighborhood of the point $(-1, 0)$, then u must also vanish at the point $(1, 0)$. But if ω is changed to the rectangle $\omega = \{-2 < x < 2, \frac{1}{2} < y < \frac{1}{2}\}$, and the points $(-1, 0)$ and $(1, 0)$ remain as before, the same assertion is no longer true (this can be seen by elementary examples).

Proofs. First we show that Theorem 3' is an immediate corollary of Theorem 3: Assume the contrary, that $u(P) \neq 0$. Then $P \in \text{supp } u$, so $\bar{C}(P; \omega) \subset \text{supp } u$ by Theorem 3. This means that u cannot vanish identically in any neighborhood of any point of $\bar{C}(P; \omega)$, which contradicts the assumption that $\bar{C}(P; \omega) \cap A$ is not void.

In order to prove Theorem 3, we argue as in the proof of Theorem 1: Assume the contrary, that there is some point $Q \in \bar{C}(P; \omega) \subset \omega$ with $Q \notin \text{supp } u$. Then u vanishes identically in some neighborhood of Q . Hence there exists a point $Q' \in \bar{C}(P; \omega)$ such that $Q' \notin \text{supp } u$. Traversing the finite chain of diffusion trajectories from Q' to P , we encounter a first point $P' \in \text{supp } u$. Now use exactly the same argument as in the proof of Theorem 1 but with the set $\mathfrak{M}$ replaced by $\text{supp } u$ and Lemma 3 replaced by our assumption. The conclusion is that P' cannot be the first such point after all. This contradiction completes the proof.

Added in Proof. The author has recently learned of the work:

JEAN-MICHEL BONY, Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptique dégénérés, *Ann. Inst. Fourier, Grenoble* **19** (1969) 277–304.

The above paper takes a somewhat different point of view from the one expressed here, but its results and techniques are very closely related to those of the present paper.

The author has also recently become aware of a gap in the proof given for Lemma 4—namely that the magnitude of $D_k D_l h$ at P_1 may not be independent of R , as claimed. However, Lemma 4 remains valid under a variety of restrictive hypotheses (with a different proof required in each case). In particular, Lemma 4 is easy to prove under the very restrictive assumption that special coordinates can be introduced such that $a_{i,j}^{1,j}(P_1) = 0$, so that $\beta^1(P_1) = b^1(P_1)$ (see, for

example, the proof of Theorem 3.2 in the above paper of Bony). In this connection the part of the proof of Theorem 1 which yields propagation of the maximum along diffusion trajectories is independent of Lemma 4, and hence can be done first as an aid to proving Lemma 4. A more precise statement of the ultimate generality of Lemma 4 is obviously needed. The author hopes that in the near future someone may be able to clarify this point.

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