Decomposition of Semi-Groups of Partial Isometries

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§1. Introduction. The Jordan canonical form tells us that, given a nilpotent operator N on a finite dimensional Hilbert space F , we can find a new inner product on F such that N^k is a partial isometry for all k (*i. e.* N is a power partial isometry in the terminology of [1]). Moreover, in this new inner product, F splits into a direct sum of subspaces F_k , on each of which N acts as a truncated shift (see [1]) of index, say, i_k . That is, $F_k = \sum^{i_k} \oplus G$ and $N|_{F_k}$ has the matrix representation

$$\left\{ \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 0 \\ I_G & 0 & \cdot & \cdot & 0 \\ \vdots & & & & \vdots \\ 0 & \cdot & \cdot & \cdot & I_G & 0 \end{pmatrix} \right\}_{i_k}$$

I_G being the identity on G .

All this is, of course, known to every undergraduate mathematician, even though the terminology and point of view may seem a trifle strange. In this paper, we deal with an analogous situation in Hilbert space. For a nilpotent operator, we substitute a nilpotent semi-group of operators. We do make, however, an enormous simplification by assuming from the outset that our semi-groups are composed of partial isometries. In other words, we don't consider at all the similarity theory of such semi-groups.

It turns out that the decomposition, described above in the finite dimensional case, has a quite reasonable analog in the separable case. The space decomposes into a direct integral of Hilbert spaces and the semi-group decomposes into a direct integral of semi-groups of partial isometries. Each differential semi-group has a certain index which is a real number and, most important, is a truncated shift. All these terms will be made precise below.

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§2. Definitions and Some Easy Lemmas. All semi-groups considered in this paper will be assumed to be strongly continuous, one-parameter semi-groups defined on a Banach space X . We always assume the value of the semi-group at zero is the identity. If $t \rightarrow T_t$ is the semi-group, we will denote it succinctly by T . The symbol " H " will always indicate a Hilbert space. H is not assumed separable unless explicitly stated.

Definition 1. A semi-group T is nilpotent if there is a positive number α such that $T_\alpha = 0$. The smallest such number is called the index of T , and will be written $\text{ind } (T)$.

Definition 2. If T is a nilpotent semi-group on X , define

$$J(T) = \int_0^\infty T_t dt,$$

the integral converging in the strong operator topology.

We will usually employ some slightly abusive notation by writing $J^n(T)$ or $J^*(T)$ for $(J(T))^n$ and $(J(T))^*$. Where no confusion is likely to arise, we will often write merely J for $J(T)$.

Lemma 1. If T is a nilpotent semi-group on X , then

$$J^n(T) = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} T_t dt, \quad n = 1, 2, \dots$$

Proof. Assuming $\text{ind } (T) = \alpha$, the induction step is

$$\begin{aligned} J^{n+1}(x) &= J^n(Jx) \\ &= \frac{1}{(n-1)!} \int_0^\alpha t^{n-1} T_t Jx dt = \frac{1}{(n-1)!} \int_0^\alpha t^{n-1} \int_0^\alpha T_{t+u} x du dt \\ &= \frac{1}{(n-1)!} \int_0^\alpha t^{n-1} \int_t^\alpha T_v x dv dt = \frac{1}{n!} \int_0^\alpha t^n T_t x dt, \end{aligned}$$

the last step using integration by parts.

Corollary. $J(T)$ is quasi-nilpotent.

Lemma 2. If T is a nilpotent semi-group on H with $\text{ind } (T) = \alpha$, then the von Neumann algebra generated by $\{T_t\}_{t \geq 0}$ is generated by $J(T)$.

Proof. Since $n \int_0^{t+1/n} T_u du$ tends strongly to T_t , it more than suffices to construct a sequence of polynomials p_n such that $p_n(J)$ tends strongly to $\int_0^t T_u du$ (here, t is fixed and $0 < t < \alpha$). To do this, pick a sequence of polynomials q_n such that $\{q_n\}$ is uniformly bounded on $[0, \alpha]$, $q_n(u) \rightarrow 1$ if $0 \leq u \leq t$ and $q_n(u) \rightarrow 0$ if $t < u \leq \alpha$. Then $\int_0^\alpha q_n(u) T_u x du$ tends to $\int_0^t T_u x du$ for all $x \in H$. If $q_n(u) = \sum \alpha_i u^i$, set $p_n(u) = \sum j! \alpha_j u^{j+1}$ so that $p_n(J) = \int_0^\alpha q_n(u) T_u du$. It is useful to note that p_n has no constant term.

Lemma 3. *Let A be an operator on H satisfying $A + A^* \geq 0$. Let $\mathfrak{A}$ be the von Neumann algebra generated by A . Then the projection P on $\ker(A)$ is a central projection in $\mathfrak{A}$.*

Proof. If $x \in \ker(A)$, then $0 = \langle x, Ax \rangle = \langle A^*x, x \rangle = \langle (A + A^*)x, x \rangle$. Hence $x \in \ker(A + A^*)$ so $x \in \ker(A^*)$. Taking adjoints, it follows that $\ker(A) = \ker(A^*)$ and so $P \leftrightarrow A$, hence $P \in \mathfrak{A}'$ (the commutant of $\mathfrak{A}$). On the other hand, if $B \in \mathfrak{A}'$, then B leaves $\ker(A)$ invariant as does B^* , hence $P \leftrightarrow B$ so $P \in \mathfrak{A} \cap \mathfrak{A}'$ as stated.

Definition 3. For $\alpha > 0$, $S(\alpha)$ is the semi-group on $L_2(0, \alpha)$ defined by

$$S_t(\alpha) = 0 \quad \text{if } t \geq \alpha,$$

and for $0 < t < \alpha$

$$S_t(\alpha)f(x) = \begin{cases} 0 & \text{if } x \leq t \\ f(x - t) & \text{if } t < x \leq \alpha. \end{cases}$$

Definition 4. A semi-group of partial isometries, abbreviated hereafter as a s.p.i., is a truncated shift if it is unitarily equivalent to a direct sum of n copies of some $S(\alpha)$ (n may be infinite).

A word of caution: for each $t \geq 0$, $S_t(1)$ is unitarily equivalent to $S_t(1) \oplus S_t(1)$ but the semi-groups $S(1)$ and $S(1) \oplus S(1)$ are not equivalent. Reason: $J(S(1)) + J^*(S(1))$ is a projection of rank 1, but the corresponding expression for $S(1) \oplus S(1)$ is a projection of rank 2.

§3. Some theorems.

Theorem 0. *A nilpotent s.p.i. W is a truncated shift if and only if the von Neumann algebra $\mathfrak{A}(W)$ generated by the operators $\{W_i\}_{i \geq 0}$ is a factor.*

This is the theorem of [2]. The theorem we prove below gives an alternate characterization of truncated shifts. In the case of separable H , it will follow from the structure theory. However, the methods of [2] are adequate to prove it in general.

Theorem 1. *Let W be a s.p.i. of index α . Then W is a truncated shift if and only if $(1/\alpha)(J(W) + J^*(W))$ is a projection.*

We need a lemma which will be useful later also.

Lemma 4. *With the hypothesis of Theorem 1, $0 \leq J(W) + J^*(W) \leq \alpha I$.*

Proof. We may take $\alpha = 1$ by a trivial change of variable. Define

$$\sigma_k = 2^{-k} \sum_{j=0}^{2^k-1} W_{j/2^k},$$

$$\tau_k = 2^{-k} \sum_{j=1}^{2^k-1} W_{j/2^k}^*.$$

Since σ_k and τ_k are just Riemann sum approximations to $J(W)$ and $J^*(W)$, we

have strong $\lim_{k \rightarrow \infty} \sigma_k = J$ and strong $\lim_{k \rightarrow \infty} \tau_k = J^*$. Now, according to [2], $W_{2^{-k}}$ is unitarily equivalent to $\sum_{p=1}^{2^k} \oplus J_p$ ($\sum'$ means some of the summands may be missing), where J_p acts on a Hilbert space of p -tuples via the matrix

$$\left\{ \begin{pmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ I & 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \cdot & \cdot & \cdot & I & 0 \end{pmatrix} \right\} p.$$

Hence $\sigma_k + \tau_k$ is unitarily equivalent to

$$\sum_{p=1}^{2^k} \oplus \frac{1}{2^k} \left\{ \begin{pmatrix} I & \cdot & \cdot & \cdot & I \\ \vdots & & & & \vdots \\ I & \cdot & \cdot & \cdot & I \end{pmatrix} \right\} p.$$

But each of the above summands is $p/2^k$ times a projection, hence $0 \leq \sigma_k + \tau_k \leq I$, finishing the proof.

Proof of Theorem 1. We may again assume $\alpha = 1$. Since $\int_0^1 S_t(1) dt$ is the ordinary Volterra operator V defined by $Vf(x) = \int_0^x f(t) dt$ and since $V + V^*$ is a projection of rank 1, the "only if" part is immediate.

For the sufficiency, let E_ξ be the class of projections defined in [2], where $0 \leq \xi$, and denote the projection $J + J^*$ by P . Since E_ξ reduces W_t , then if $E_\xi \neq 0$, the restriction $W_t(\xi)$ of W_t to $\text{ran}(E_\xi)$ is a s.p.i. of index at most ξ ([2], Lemma 3). Call this semi-group $W(\xi)$ and denote $J(W(\xi))$ by $J(\xi)$. Now suppose $\xi < 1$. Then Lemma 4 above tells us that $J(\xi) + J^*(\xi) \leq \xi < 1$. Since $J(\xi) + J^*(\xi)$ is a direct summand of P , it must be zero. But then $J(\xi)$ is skew-Hermitian and since it is quasi-nilpotent by the corollary to Lemma 1, $J(\xi) = 0$. But then Lemma 2 implies $W_t(\xi) = 0$ for $t > 0$ which contradicts $W_0 = I$. Hence $E_\xi = 0$. But the vanishing of all E_ξ , $\xi < 1$, forces the conclusion of Lemma 4 of [2], and this implies that W is a truncated shift.

Theorem 2. *Every nilpotent s.p.i. on a separable H is a direct integral of truncated shifts. More precisely, there exists a finite Borel measure on the real line and a decomposition $H = \int \oplus K(\lambda) d\mu(\lambda)$ such that*

- (a) $K(\lambda) = K$ where K is a fixed separable Hilbert space,
- (b) for each λ , there is defined a s.p.i. $\tilde{W}(\lambda)$,
- (c) $\text{ind}(\tilde{W}(\lambda)) \leq \text{ind}(W) = \alpha$,
- (d) for each $t \geq 0$, $W_t = \int \oplus \tilde{W}_t(\lambda) d\mu(\lambda)$, and
- (e) each $\tilde{W}(\lambda)$ is a truncated shift.

Proof. Let $\mathfrak{A}(W)$ be the von Neumann algebra generated by $\{W_t\}_{t \geq 0}$. Then, according to [3], p. 53, we can find a finite Borel measure μ on the real line such that $H = \int \oplus K(\lambda) d\mu(\lambda)$ and $\mathfrak{A}(W) = \int \oplus \mathfrak{A}(\lambda) d\mu(\lambda)$ where each $\mathfrak{A}(\lambda)$ is a factor. If we write $J(W) = \int \oplus J(\lambda) d\mu(\lambda)$, so that $J^n =$

$\int \oplus J^n(\lambda) d\mu(\lambda)$, then $\mu - \text{a.e.}$, $\|J^n(\lambda)\| \leq \|J^n(W)\|$. In particular, $J(\lambda)$ is quasi-nilpotent except on a null set.

Since $J(W) + J^*(W) \geq 0$ by Lemma 4, $J(\lambda) + J^*(\lambda) \geq 0$ $\mu - \text{a.e.}$ Moreover, since $J(W)$ generates $\mathfrak{A}(W)$ by Lemma 2, then, except for a null set, $J(\lambda)$ generates $\mathfrak{A}(\lambda)$ (see [3], p. 46). But $\mathfrak{A}(\lambda)$ is a factor, and so, according to Lemma 3, for each λ outside a null set, $\ker(J(\lambda)) = 0$ or $\ker(J(\lambda)) = K(\lambda)$. Let F be the set on which $J(\lambda) = 0$. If $\mu(F) > 0$, then the operator $P = \int_F \oplus I(\lambda) d\mu(\lambda)$ is a non-zero projection in the center of $\mathfrak{A}(W)$, and $PJ(W) = 0$. In fact, if q is any polynomial without constant term, $Pq(J) = 0$. But then, by the remark at the end of the proof of Lemma 2, we can conclude that $PW_t = 0$ for $t > 0$. Letting t tend to 0, we get the contradiction, $P = 0$.

This argument may be applied verbatim to the semi-group W^* since $J(W^*) = J^*(W)$ and $\mathfrak{A}(W^*) = \mathfrak{A}(W)$. The conclusion is then that $J^*(\lambda)$ is one-one, $\mu - \text{a.e.}$, or $J(\lambda)$ has dense range $\mu - \text{a.e.}$ Thus, except on a null set, $-J^{-1}(\lambda)$ is a closed, densely defined operator on $K(\lambda)$. It is a routine verification that the resolvent $R(\lambda, z)$ of $-J^{-1}(\lambda)$ at z is given by the entire function $R(\lambda, z) = J(\lambda)(I(\lambda) + zJ(\lambda))^{-1}$. The resolvent for the semi-group W is given by $R(z) = \int_0^\infty e^{-zt} W_t dt = \sum_{n=0}^\infty (-1)^n z^n J^{n+1}(W) = J(W)(I + zJ(W))^{-1}$ and $\|R(z)\| \leq 1/z$ for positive z . For all z , $R(z) = \int \oplus R(\lambda, z) d\mu(\lambda)$ and so, except for a μ -null set, $\|R(\lambda, z)\| \leq 1/z$ holds for all positive rational z . But then, continuity of the resolvent shows that outside this null set $\|R(\lambda, z)\| \leq 1/z$ holds for all positive z .

We now use the Hille-Yosida Theorem (see *e.g.* [4], p. 626) to conclude that outside a μ -null set, there exist strongly continuous semi-groups of contractions, $\tilde{W}(\lambda)$, whose infinitesimal generators are $-J^{-1}(\lambda)$. Also we discard the bad null set we have amassed and assume that the "differential semi-groups" $\tilde{W}(\lambda)$ are defined for all λ . What we must show is that for fixed $t > 0$, $\lambda \rightarrow \tilde{W}_t(\lambda)$ is measurable and that $W_t = \int \oplus \tilde{W}_t(\lambda) d\mu$.

Having fixed $t > 0$, define $A(t, k) = [(k/t)R(k/t)]^k$ and $A(\lambda, t, k) = [(k/t)R(\lambda, k/t)]^k$. Hence $A(t, k) = \int \oplus A(\lambda, t, k) d\mu(\lambda)$ for $k = 1, 2, \dots$. According to [5] (p. 354, formula (E_9)), $W_t = \lim_{k \rightarrow \infty} A(t, k)$, and $\tilde{W}_t(\lambda) = \lim_{k \rightarrow \infty} A(\lambda, t, k)$, these limits being in the strong topology. If $W_t = \int \oplus W(\lambda, t) d\mu(\lambda)$ and $x = \int \oplus x(\lambda) d\mu(\lambda)$, then

$$\|W_t x - A(t, k)x\|^2 = \int \|W(\lambda, t)x(\lambda) - A(\lambda, t, k)x(\lambda)\|^2 d\mu(\lambda)$$

and this expression tends to zero as k tends to ∞ . On the other hand, $A(\lambda, t, k)x(\lambda)$ tends to $\tilde{W}_t(\lambda)x(\lambda)$ *a.e.* and so $\tilde{W}_t(\lambda)x(\lambda) = W(\lambda, t)x(\lambda)$ *a.e.* This shows that $\lambda \rightarrow \tilde{W}_t(\lambda)$ is a measurable operator function and $W_t = \int \oplus \tilde{W}_t(\lambda) d\mu(\lambda)$ for each $t > 0$.

We are almost done. If r is a positive rational, then

$$W_r^* W_r = \int \oplus \tilde{W}_r^*(\lambda) \tilde{W}_r(\lambda) d\mu(\lambda)$$

is a projection which is 0 if $r > \alpha$. Hence outside a μ -null set, $\tilde{W}_r(\lambda)$ is a partial isometry, and continuity implies $\tilde{W}_t(\lambda)$ is a partial isometry outside this null

set. Clearly $\text{ind } (\tilde{W}(\lambda)) \leq \alpha$ a.e. Also, the algebra generated by $\{\tilde{W}_t(\lambda)\}_{t \geq 0}$ is $\mathfrak{A}(\lambda)$ a.e. and so, by Theorem 0, $\tilde{W}(\lambda)$ is a truncated shift a.e. Finally, since no finite-dimensional space can support a nilpotent semi-group, (a) is apparent.

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