Plastic Deformations and Instability of Spherical Shells Under Internal Pressure

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1. Introduction. The basic theory of a discontinuity surface of strong instability was proposed by T. Y. Thomas [1] and then has been applied on the several fracture problems. The plastic flow and fracture of the round bars by tension and of the circular pipes by internal pressure etc. were investigated by Thomas [2]. Tsuan Wu Ting [3] studied the fracture of closed circular pipes under internal pressure and axial tension and his theoretical conclusions are in good agreement qualitatively with the experimental results, and other fracture problems were reported by several investigators [4, 5 and 6]. The basic theories and many applications are summarized by Thomas [7]. The present study is an analysis of plastic deformations, flows and instability of spherical shells under the internal pressure by means of the similar method with Ting.

If the internal pressure is continually increased from a value for which a spherical shell is entirely in a state of elastic equilibrium, the inner surface of the shell begins to yield, and then the entire region of the shell is subjected to a plastic equilibrium state, where the material of the shell, in this paper, is assumed to be incompressible and governed by Hencky's stress-strain relations with the Tresca yield condition. In Sect. 2 the deformation of the spherical shell in the plastic static equilibrium is analysed.

In the nonequilibrium plastic state the material is assumed to be incompressible and governed by von Mises stress-rate of strain relations with the Tresca yield condition. In Sects. 3 and 4 we consider the intersection Σ of the spherical shell with the surface of a cone whose vertex coincides with the center of the sphere. In the both sides of Σ the stress and the gradient of stress are assumed to be continuous while the velocity components are assumed to have the different values.

In Sect. 3 the surface Σ is regarded to separate a region of plastic equilibrium and a region of plastic flow and the following result is found: The surface Σ is not a slip surface and then no fracture occurs in this case. In Sect. 4 the surface

Σ is assumed to separate two regions of plastic flow and it is shown that the surface Σ is a surface of strong instability if, and only if, the normal stress acting on Σ is greater than the normal stress on the plane through the axis of cone.

In Sect. 5 the obtained results are compared with those of Ting, the definitions of the reference side of Σ and of the discontinuity quantity are examined and the instability condition of the stress is discussed.

2. Plastic deformation of the spherical shell at static equilibrium due to the internal pressure. To secure the utmost mathematical simplification in the following discussion we assume the body to be composed of rigid-plastic material, subject to the Tresca yield condition.

The basic equations are

$$(2.1) \quad g^{ij} u_{i,j} = 0 \quad (\text{equation of incompressibility}),$$

$$(2.2) \quad e_{ij} = \psi \sigma_{ij}^* \quad (\text{Hencky stress-strain equations}),$$

$$(2.3) \quad g^{ik} \sigma_{i,j,k} = 0 \quad (\text{equation of equilibrium}),$$

$$(2.4) \quad \tau_1 - \tau_3 = 2k \quad (\text{Tresca yield condition}),$$

where the spherical coordinates r, θ, ϕ are denoted by x^1, x^2, x^3 , respectively; the covariant and contravariant components of the metric tensor are given by the matrices

$$(2.5) \quad ||g_{ij}|| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{vmatrix}, \quad ||g^{ij}|| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/r^2 \sin^2 \theta \end{vmatrix};$$

u_i and e_{ij} are the components of displacement vector and strain tensor, respectively, and related with

$$(2.6) \quad e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

where and in (2.3) the comma denotes covariant differentiation; σ_{ij} are the components of the stress tensor and the stress deviator is defined by

$$(2.7) \quad \sigma_{ij}^* = \sigma_{ij} - \frac{1}{3}(g^{ab} \sigma_{ab}) g_{ij};$$

also τ_1, τ_2 and τ_3 are the principal stresses with the ordering $\tau_1 \geq \tau_2 \geq \tau_3$; the proportionality factor ψ in (2.2) is positive by hypothesis and k is a material constant.

The components of the affine connection of the spherical coordinate system are

$$(2.8) \quad \begin{cases} \Gamma_{22}^1 = -r, & \Gamma_{33}^1 = -r \sin^2 \theta, & \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r}, \\ \Gamma_{33}^2 = -\sin \theta \cos \theta, & \Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{r}, & \Gamma_{23}^3 = \Gamma_{32}^3 = \cot \theta, \\ \Gamma_{jk}^i = 0 & \text{otherwise,} \end{cases}$$

and the physical components of a vector and a second order tensor are defined (See e.g. Eringen [8]), in general, by

$$(2.9) \quad \begin{cases} u^{(i)} = u^i \sqrt{g_{ii}}, \\ \sigma_{(i)}^{(j)} = \sigma_i^j \sqrt{\frac{g_{jj}}{g_{ii}}}, \end{cases} \quad (\text{not summed on } i \text{ and } j).$$

Relative to the spherical system it is obvious, from symmetry, that the components u_2 and u_3 must vanish and the radial component u_1 depends only on r , hence we have

$$(2.10) \quad u_1 = u_1(r), \quad u_2 = u_3 = 0.$$

Similarly we see that the stress components must be independent of the θ and ϕ coordinates, and from symmetry and (2.9) we can write

$$(2.11) \quad \begin{cases} \sigma_{(1)}^{(1)} = \sigma_{11}(r), \quad \sigma_{(2)}^{(2)} = \frac{1}{r^2} \sigma_{22}(r), \quad \sigma_{(3)}^{(3)} = \frac{1}{r^2 \sin^2 \theta} \sigma_{33}(r), \\ \sigma_i^j = 0 \quad (i \neq j). \end{cases}$$

Making use of (2.10), (2.11), (2.8) and the formula for covariant differentiation we have

$$(2.12) \quad \begin{cases} u_{1,1} = \frac{du_1}{dr}, \quad u_{1,2} = 0, \quad u_{1,3} = 0, \\ u_{2,1} = 0, \quad u_{2,2} = ru_1, \quad u_{2,3} = 0, \\ u_{3,1} = 0, \quad u_{3,2} = 0, \quad u_{3,3} = r \sin^2 \theta u_1 \end{cases}$$

and

$$(2.13) \quad \begin{cases} \sigma_{11,1} = \frac{\partial \sigma_{11}}{\partial r}, \quad \sigma_{12,2} = r\sigma_{11} - \frac{1}{r} \sigma_{22}, \quad \sigma_{13,3} = r \sin^2 \theta \sigma_{11} - \frac{1}{r} \sigma_{33}, \\ \sigma_{21,1} = 0, \quad \sigma_{22,2} = 0, \quad \sigma_{23,3} = -\cot \theta \sigma_{33} + \sin \theta \cos \theta \sigma_{22}, \\ \sigma_{31,1} = 0, \quad \sigma_{32,2} = 0, \quad \sigma_{33,3} = 0. \end{cases}$$

Now consider the equation of incompressibility (2.1). Using (2.12) we find that

$$\frac{du_1}{dr} + \frac{2u_1}{r} = 0,$$

which produces the solution

$$(2.14) \quad u_1 = \frac{A}{r^2},$$

where A is a constant.

The components of strain tensor are specified by (2.6), (2.12) and (2.14) and then Hencky stress-strain equation (2.2) gives the two independent relations

$i, j = 1, 1$ and $2, 2$ such that

$$(2.15) \quad \begin{cases} -\frac{2A}{r^3} = \frac{\psi}{3} (2\sigma_{(1)}^{(1)} - \sigma_{(2)}^{(2)} - \sigma_{(3)}^{(3)}), \\ \frac{A}{r^3} = \frac{\psi}{3} (2\sigma_{(2)}^{(2)} - \sigma_{(1)}^{(1)} - \sigma_{(3)}^{(3)}), \end{cases}$$

by which we can obtain the relations:

$$(2.16) \quad \sigma_{(2)}^{(2)} = \sigma_{(3)}^{(3)}$$

and

$$(2.17) \quad -\frac{A}{r^3} = \frac{\psi}{3} (\sigma_{(1)}^{(1)} - \sigma_{(2)}^{(2)}).$$

In the equation of equilibrium, the third equation of (2.3) for $i = 3$ is satisfied identically from (2.13). For $i = 2$ we see immediately from (2.13) and (2.16) that (2.3) is satisfied and for $i = 1$ we can write it as

$$(2.18) \quad \frac{d\sigma_{(1)}^{(1)}}{dr} + \frac{2\sigma_{(1)}^{(1)}}{r} - \frac{2\sigma_{(2)}^{(2)}}{r} = 0.$$

Suppose $A = 0$ then (2.17) implies $\sigma_{(1)}^{(1)} = \sigma_{(2)}^{(2)}$ at any point of the shell. Hence, from (2.18), we have

$$\frac{d\sigma_{(1)}^{(1)}}{dr} = 0$$

in the entire plastic equilibrium region. Thus $\sigma_{(1)}^{(1)}$ is independent of r . But on the outer spherical surface $\sigma_{(1)}^{(1)} = 0$ and on the inner surface $\sigma_{(1)}^{(1)} = -p$ which is impossible for the positive value of p . Then we will assume $A \neq 0$. Suppose now $A < 0$; since $\psi > 0$ for loading we have from (2.17) and (2.18)

$$\frac{d\sigma_{(1)}^{(1)}}{dr} < 0,$$

which contradicts also the boundary conditions. Thus $A > 0$ and, consequently,

$$(2.19) \quad \sigma_{(1)}^{(1)} < \sigma_{(2)}^{(2)}$$

everywhere.

Relations (2.16) and (2.19) indicate that

$$(2.20) \quad \tau_1 = \tau_2 = \sigma_{(2)}^{(2)}, \quad \tau_3 = \sigma_{(1)}^{(1)}.$$

Therefore Tresca yield condition (2.4) is replaced by

$$(2.21) \quad \sigma_{(2)}^{(2)} - \sigma_{(1)}^{(1)} = 2k.$$

Substituting (2.21) into (2.18) and integrating it, we get

$$(2.22) \quad \sigma_{(1)}^{(1)} = 4k \log r + \text{constant}.$$

The constant and the required pressure are determined by the boundary conditions and we obtain

$$(2.23) \quad \begin{cases} \sigma_{(1)}^{(1)} = 4k \log \frac{r}{b}, \\ \sigma_{(2)}^{(2)} = \sigma_{(3)}^{(3)} = 4k \left(\frac{1}{2} + \log \frac{r}{b} \right), \end{cases}$$

and

$$(2.24) \quad p = 4k \log \frac{b}{a},$$

where a and b are inner and outer radii of the spherical shell, respectively.

From (2.17) and (2.21) the proportional factor ψ is specified by

$$(2.25) \quad \psi = \frac{3A}{2kr^3},$$

which gives positive values everywhere.

3. Stability criterion of a surface as a boundary of a region of plastic equilibrium and a region of plastic flow. In a region of plastic flow the following relations are assumed to hold:

$$(3.1) \quad g^{ij}v_{i,j} = 0 \quad (\text{equation of incompressibility}),$$

$$(3.2) \quad \epsilon_{ij} = \psi \sigma_{ij}^* \quad (\text{von Mises stress and rate of strain relations}),$$

$$(3.3) \quad g^{ik}\sigma_{ij,k} = \rho \left(\frac{\partial v_i}{\partial t} + v_{i,j}v^j \right) \quad (\text{equations of motion}),$$

$$(3.4) \quad \tau_1 - \tau_3 = 2k \quad (\text{Tresca yield condition}),$$

where v_i are the components of velocity vector and ϵ_{ij} are the components of the rate of strain tensor defined by

$$(3.5) \quad \epsilon_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i});$$

ρ is the density and the proportional factor ψ is positive by hypothesis.

There are two dynamical conditions on discontinuity across a surface [1]:

$$(3.6) \quad \begin{cases} \rho(G - v_n) = \bar{\rho}(G - \bar{v}_n), \\ [\sigma_{ij}]v^j = -\rho(G - v_n)[v_i], \end{cases}$$

where square bracket [] denote the discontinuity on the two sides of a discontinuity surface; G , v_n and v^i denote the coordinate velocity of the surface in the normal direction, the normal velocity of the material particles on the surface and the components of the unit normal of the surface, respectively; and the quantities without or with bar denote their two values on the two sides of the discontinuity surface.

In this problem we have $G = 0$ and $v_n = \bar{v}_n = 0$, so that the first condition (3.6) is satisfied identically and the second condition becomes

$$(3.7) \quad [\sigma_{ii}]v^i = 0.$$

The physical components of the velocity vector becomes from (2.9)

$$(3.8) \quad v^{(1)} = v_1, \quad v^{(2)} = \frac{v_2}{r}, \quad v^{(3)} = \frac{v_3}{r \sin \theta}.$$

Now we assume that the supposed discontinuity surface Σ is the intersection of the spherical shell with the surface of a cone, which is represented by

$$(3.9) \quad \theta = \xi, \quad (0 < \xi < \pi),$$

where we assume that the material flows in the region $0 \leq \theta < \xi$ and it is in the plastic equilibrium in the region $\xi < \theta \leq \pi$.

Here we assume that by the circular symmetry with respect to the azimuth ϕ the velocity is independent of it and the component v_3 must be zero. A further simplifying assumption will be introduced by the requirement that radial lines are displaced into radial lines. Hence the angular velocity $v^{(2)}/r = v_2/r^2$ is independent of r . In order to avoid separation of material, $v^{(2)}$ must vanish on the surface Σ . Hence we have the following components:

$$(3.10) \quad v^{(1)} = v_1(r, \theta, t), \quad v^{(2)} = rf(\theta, t), \quad v^{(3)} = 0$$

for $0 \leq \theta < \xi$ and

$$(3.11) \quad \bar{v}^{(2)} = 0,$$

where a bar over $v^{(2)}$ indicates the value of $v^{(2)}$ evaluated on Σ : $\theta = \xi$.

From the relation (3.10), we obtain

$$(3.12) \quad ||v_{i,i}|| = \left\| \begin{array}{ccc} \frac{\partial v_1}{\partial r} & \frac{\partial v_1}{\partial \theta} - \frac{v_2}{r} & 0 \\ \frac{\partial v_2}{\partial r} - \frac{v_2}{r} & \frac{\partial v_2}{\partial \theta} + rv_1 & 0 \\ 0 & 0 & v_1 r \sin^2 \theta + v_2 \sin \theta \cos \theta \end{array} \right\|,$$

and

$$(3.13) \quad ||\epsilon_{i,i}|| = \left\| \begin{array}{ccc} \frac{\partial v_1}{\partial r} & \frac{1}{2} \left(\frac{\partial v_1}{\partial \theta} + \frac{\partial v_2}{\partial r} - \frac{2v_2}{r} \right) & 0 \\ \frac{1}{2} \left(\frac{\partial v_1}{\partial \theta} + \frac{\partial v_2}{\partial r} - \frac{2v_2}{r} \right) & \frac{\partial v_2}{\partial \theta} + rv_1 & 0 \\ 0 & 0 & v_1 r \sin^2 \theta + v_2 \sin \theta \cos \theta \end{array} \right\|.$$

In view of (3.12) the equation of incompressibility (3.1) can be written as

$$(3.14) \quad \frac{\partial v_1}{\partial r} + \frac{2}{r} v_1 + \frac{1}{r^2} \frac{\partial v_2}{\partial \theta} + \frac{v_2}{r^2} \cot \theta = 0.$$

Differentiating (3.14) with respect to r we have

$$\frac{\partial^2 v_1}{\partial r^2} + \frac{2}{r} \frac{\partial v_1}{\partial r} - \frac{2}{r^2} v_1 = 0.$$

Thus

$$(3.15) \quad v_1 = v^{(1)} = rQ(\theta, t) + \frac{1}{r^2} S(\theta, t).$$

Substituting (3.15) into (3.14) we have

$$\frac{\partial v_2}{\partial \theta} + v_2 \cot \theta = -3r^2 Q(\theta, t).$$

Hence

$$(3.16) \quad \frac{v_2}{r^2} = \frac{v^{(2)}}{r} = \frac{3}{\sin \theta} \int_{\theta}^{\xi} Q(\theta', t) \sin \theta' d\theta',$$

where we use the boundary condition (3.11). Substituting (3.15), (3.16) and (3.10) into (3.13) we have the rate of strain tensor:

$$(3.17) \quad ||\dot{\epsilon}_{ij}|| = \left\| \begin{array}{ccc} \bar{Q} - \frac{2}{r^3} \bar{S} & \frac{1}{2} \left(r \frac{\partial \bar{Q}}{\partial \theta} + \frac{1}{r^2} \frac{\partial \bar{S}}{\partial \theta} \right) & 0 \\ \frac{1}{2} \left(r \frac{\partial \bar{Q}}{\partial \theta} + \frac{1}{r^2} \frac{\partial \bar{S}}{\partial \theta} \right) & -2r^2 \bar{Q} + \frac{1}{r} \bar{S} & 0 \\ 0 & 0 & \left(r^2 \bar{Q} + \frac{1}{r} \bar{S} \right) \sin^2 \theta \end{array} \right\|.$$

Now we assume that the stress components are continuous over the surface Σ at all times, namely

$$(3.18) \quad [\sigma_{ij}] = \bar{\sigma}_{ij} - \sigma_{ij} = 0 \quad \text{at} \quad \theta = \xi,$$

where σ_{ij} are the stress components given in (2.23) and (2.11). Then we have

$$(3.19) \quad \begin{cases} \bar{\sigma}_{(1)}^{(1)*} = -\frac{4}{3}k, & \bar{\sigma}_{(2)}^{(2)*} = \bar{\sigma}_{(3)}^{(3)*} = \frac{2}{3}k, \\ \bar{\sigma}_{i,j}^* = 0 & (i \neq j), \end{cases}$$

which satisfy the Tresca yield condition (3.4).

The von Mises stress and rate of strain relations (3.2) and the values of the components (3.17) and (3.18) give the following three independent relations

$$(3.20) \quad r \frac{\partial \bar{Q}}{\partial \theta} + \frac{1}{r^2} \frac{\partial \bar{S}}{\partial \theta} = 0,$$

and

$$(3.21) \quad \begin{cases} \bar{Q} - \frac{2\bar{S}}{r^3} = -\frac{4}{3}k\bar{\psi}, \\ -2\bar{Q} + \frac{\bar{S}}{r^3} = \frac{2}{3}k\bar{\psi}. \end{cases}$$

The relation (3.20) indicates that v_1 is independent of θ and the relations (3.21) are equivalent to

$$(3.22) \quad \bar{Q} = 0 \quad \text{and} \quad \bar{\psi} = \frac{3\bar{S}}{2kr^3}.$$

The existence of the surface Σ separating the region of equilibrium and plastic flow evidently permits a violation of the condition of independence of θ . Therefore the stress components σ_{ij} must be taken to be functions of the form

$$\sigma_{ij} = \sigma_{ij}(r, \theta, t),$$

in the region of plastic flow. The covariant derivatives of the stress components on Σ can be calculated by (2.8), (2.11), (2.23) and (3.18), then we have

$$(3.23) \quad \begin{cases} \bar{\sigma}_{11,1} = \frac{4k}{r}, & \bar{\sigma}_{12,2} = \frac{\partial \bar{\sigma}_{12}}{\partial \theta} - 2kr, & \bar{\sigma}_{13,3} = -2kr \sin^2 \theta, \\ \bar{\sigma}_{21,1} = 0, & \bar{\sigma}_{22,2} = \frac{\partial \bar{\sigma}_{22}}{\partial \theta}, & \bar{\sigma}_{23,3} = 0, \\ \bar{\sigma}_{31,1} = 0, & \bar{\sigma}_{32,2} = \frac{\partial \bar{\sigma}_{32}}{\partial \theta}, & \bar{\sigma}_{33,3} = 0. \end{cases}$$

The equations of motion (3.3) have then the explicit expressions at points on Σ by means of (3.23), (3.12), (3.15), (3.16) and (3.22). For $i = 1$ the equation (3.3) reduces to

$$(3.24) \quad \frac{\partial \bar{\sigma}_{12}}{\partial \theta} = \bar{\rho} \left(\frac{1}{r^2} \frac{\partial \bar{S}}{\partial t} - \frac{2\bar{S}^2}{r^6} \right),$$

and for $i = 2, 3$ we have

$$(3.25) \quad \frac{\partial \bar{\sigma}_{22}}{\partial \theta} = 0, \quad \frac{\partial \bar{\sigma}_{23}}{\partial \theta} = 0$$

after some manipulations.

To find the expression for $\partial \bar{\sigma}_{12} / \partial \theta$, the relation

$$\psi \sigma_{12} = \epsilon_{12},$$

is differentiated with respect to θ and evaluated on the surface Σ , then we have

$$(3.26) \quad \bar{\psi} \frac{\partial \bar{\sigma}_{12}}{\partial \theta} = \frac{\partial \bar{\epsilon}_{12}}{\partial \theta} = \frac{1}{2} \left(\frac{\partial^2 \bar{v}_1}{\partial \theta^2} + \frac{\partial^2 \bar{v}_2}{\partial r \partial \theta} - \frac{2}{r} \frac{\partial \bar{v}_2}{\partial \theta} \right).$$

Calculating the right-hand side of (3.26) by (3.15), (3.16) and (3.22) and replacing $\bar{\psi}$ by (3.22), we have

$$(3.27) \quad \frac{3\bar{S}}{2kr^3} \frac{\partial \bar{\sigma}_{12}}{\partial \theta} = \frac{1}{2r^2} \frac{\partial^2 \bar{S}}{\partial \theta^2}.$$

Substituting the expression (3.24) for $\partial \bar{\sigma}_{12}/\partial \theta$ into (3.27) we find

$$(3.28) \quad \frac{3\bar{p}}{2k} \left(\frac{1}{r^5} \bar{S} \frac{\partial \bar{S}}{\partial t} - \frac{2}{r^3} \bar{S}^3 \right) = \frac{1}{2r^2} \frac{\partial^2 \bar{S}}{\partial \theta^2}.$$

This equation holds for all values of the radial coordinate r , hence the coefficients of the different powers of r must vanish. Then we have

$$(3.29) \quad \bar{S} = 0.$$

The first relation of (3.22), and (3.29) demand

$$\bar{v}_1 = \bar{v}_2 = \bar{v}_3 = 0,$$

in view of the expression (3.15) and the boundary conditions (3.10) and (3.11). Hence such a surface Σ is not a slip surface. Consequently it is not a surface of strong instability.

4. Stability criterion of a surface in plastic flow region. Let Σ be the intersection of the spherical shell with the surface of a cone $\theta = \xi$ ($0 < \xi < \pi$). The material of the shell on both sides of Σ is assumed to be in the state of plastic flow such that the stress components and their partial derivatives are continuous, but the velocity components are discontinuous across Σ .

By the same restriction depicted in Sect. 3 the velocity components have the same forms as (3.10) and (3.11) and by the condition of incompressibility we can obtain the components (3.15) and (3.16). Therefore the matrix of the velocity gradient can be written as

$$(4.1) \quad ||v_{i,j}|| = \left\| \begin{array}{ccc} \bar{Q} - \frac{2\bar{S}}{r^3} & r \frac{\partial \bar{Q}}{\partial \theta} + \frac{1}{r^2} \frac{\partial \bar{S}}{\partial \theta} & 0 \\ 0 & -2r^2 \bar{Q} + \frac{1}{r} \bar{S} & 0 \\ 0 & 0 & \left(r^2 \bar{Q} + \frac{1}{r} \bar{S} \right) \sin^2 \theta \end{array} \right\|$$

when evaluated on the plane Σ .

From the above assumption of the continuity of σ_{ij} and $\sigma_{ij,k}$ across Σ and from the equations of motion (3.3), we obtain

$$(4.2) \quad 0 = \frac{\partial [v_i]}{\partial t} + [v_{i,j} v^j],$$

which is satisfied identically for $i = 2$ and 3 since $v_2 = v_3 = 0$ on each side of Σ and reduces to

$$(4.3) \quad \left(\frac{d[Q]}{dt} + [Q^2] \right) r + \left(\frac{d[S]}{dt} - [QS] \right) \frac{1}{r^2} - 2[S^2] \frac{1}{r^5} = 0$$

for $i = 1$ from (3.15), where $Q(\theta, t)$ and $S(\theta, t)$ become functions of t along over the surface Σ . This equation holds for all values of r ; hence it is necessary that the coefficients of the different powers of r vanish since the coefficients are independent of r . Consequently we have

$$(4.4) \quad \frac{d[Q]}{dt} = -[Q^2], \quad \frac{d[S]}{dt} = [QS], \quad [S^2] = 0.$$

The last equation of (4.4) asserts that either

$$(4.5) \quad \bar{S}_{(1)} = \bar{S}_{(2)},$$

or

$$(4.6) \quad \bar{S}_{(1)} = -\bar{S}_{(2)}.$$

Now consider the von Mises stress and rate of strain relations. Evaluating these on Σ we have the five independent relations:

$$(4.7) \quad \begin{cases} \bar{\psi}_{(\alpha)} \bar{\sigma}_{(2)}^{(2)*} = -2\bar{Q}_{(\alpha)} + \frac{1}{r^3} \bar{S}_{(\alpha)}, \\ \bar{\psi}_{(\alpha)} \bar{\sigma}_{(3)}^{(3)*} = \bar{Q}_{(\alpha)} + \frac{1}{r^3} \bar{S}_{(\alpha)}, \\ \bar{\psi}_{(\alpha)} \bar{\sigma}_{12} = \frac{1}{2} \left(r \frac{\partial \bar{Q}_{(\alpha)}}{\partial \theta} + \frac{1}{r^2} \frac{\partial \bar{S}_{(\alpha)}}{\partial \theta} \right), \\ \bar{\psi}_{(\alpha)} \bar{\sigma}_{13} = 0, \\ \bar{\psi}_{(\alpha)} \bar{\sigma}_{23} = 0, \end{cases}$$

where $\alpha = 1$ and 2 mean the two sides of the surface Σ and we have made use of (3.17). By the first and second relations of (4.7), we have

$$(4.8) \quad \bar{\psi}_{(\alpha)} (\bar{\sigma}_{(2)}^{(2)} - \bar{\sigma}_{(3)}^{(3)}) = -3\bar{Q}_{(\alpha)} \quad (\alpha = 1, 2).$$

To determine the strong instability property of the slip surface Σ , we have the following three possible cases to be considered.

Case 1. $\bar{\sigma}_{(2)}^{(2)} = \bar{\sigma}_{(3)}^{(3)}$ at time $t = 0$ for some value of r with $b \geq r \geq a$. From (4.8) we see that $\bar{Q}_{(1)} = \bar{Q}_{(2)} = 0$. If (4.5) holds it follows from (3.15) that $[v_1] = 0$. If (4.6) holds the first relations of (4.7) implies $\bar{\psi}_{(1)} \bar{\sigma}_{(2)}^{(2)*} = -\bar{\psi}_{(2)} \bar{\sigma}_{(2)}^{(2)*}$. Hence $\bar{\sigma}_{(2)}^{(2)*} = 0$ for $\bar{\psi}_{(\alpha)} > 0$ and then $\bar{S}_{(1)} = \bar{S}_{(2)} = 0$. Thus we always have $[Q] = [S] = 0$ in this case. Hence from (3.10) and (3.11) the assumed surface Σ is not a slip surface.

Case 2. $\bar{\sigma}_{(2)}^{(2)} > \bar{\sigma}_{(3)}^{(3)}$ at time $t = 0$ for all values of r with $b \geq r \geq a$. From the first relations of (4.7) and (4.8) we have

$$\frac{-2\bar{Q}_{(1)} + \bar{S}_{(1)}/r^3}{-3\bar{Q}_{(1)}} = \frac{\bar{\sigma}_{(2)}^{(2)*}}{\bar{\sigma}_{(2)}^{(2)*} - \bar{\sigma}_{(3)}^{(3)*}} = \frac{-2\bar{Q}_{(2)} + \bar{S}_{(2)}/r^3}{-3\bar{Q}_{(2)}}$$

or

$$(4.9) \quad \frac{\bar{S}_{(1)}}{\bar{Q}_{(1)}} = \frac{\bar{S}_{(2)}}{\bar{Q}_{(2)}}$$

for all time. First, if $\bar{S}_{(1)} = \bar{S}_{(2)} \neq 0$ then $\bar{Q}_{(1)} = \bar{Q}_{(2)}$ holds by means of (4.9). Consequently the surface Σ is not a slip surface. Secondary, if $\bar{S}_{(1)} = -\bar{S}_{(2)} \neq 0$ then $\bar{Q}_{(1)} = -\bar{Q}_{(2)}$; but under the assumption $\bar{\sigma}_{(2)}^{(2)} - \bar{\sigma}_{(3)}^{(3)} > 0$ this is impossible, in view of (4.8). Hence in either case we must have $\bar{S}_{(1)} = \bar{S}_{(2)} = 0$ for all time. Then the second and third equations of (4.4) are satisfied identically.

Without loss of generality, but for definiteness, we may assume

$$(4.10) \quad \bar{Q}_{(2)}(0) < \bar{Q}_{(1)}(0) < 0$$

in accordance with (4.8), and we define that

$$(4.11) \quad [Q(t)] \equiv \bar{Q}_{(2)}(t) - \bar{Q}_{(1)}(t).$$

If we adopt a side $\alpha = 1$ of Σ as the reference side, we can write

$$[Q(t)^2] = [Q(t)]^2 + 2\bar{Q}_{(1)}(t)[Q(t)].$$

Hence the first equation of (4.4) becomes

$$(4.12) \quad \frac{d[Q(t)]}{dt} = -[Q(t)]^2 - 2\bar{Q}_{(1)}(t)[Q(t)].$$

By the similar logical considerations with Ting [2] we may conclude that if $[Q(0)] < 0$ then $\bar{Q}_{(\alpha)}(t) < 0$ and $[Q(t)] < 0$ for all time t .

From (4.10) and (4.11) we have $[Q(0)] < 0$, and then $\bar{Q}_{(1)}(t)[Q(t)] > 0$ for all time t . Hence the equation (4.12) implies

$$\frac{d[Q(t)]}{dt} < -[Q(t)]^2,$$

which has the solution

$$(4.13) \quad [Q(t)] < \frac{1}{t + \frac{1}{[Q(0)]}} \quad \text{for } 0 < t < -\frac{1}{[Q(0)]}.$$

The right hand side of (4.13) approaches $-\infty$ as t approaches $-1/[Q(0)]$. Hence the left hand side $[Q(t)] \rightarrow -\infty$ as t approaches $-1/[Q(0)]$. Thus in this case the surface Σ is a surface of strong instability.

Case 3. $\bar{\sigma}_{(2)}^{(2)} < \bar{\sigma}_{(3)}^{(3)}$ at time $t = 0$ for all values of r with $b \geq r \geq a$. Using the same arguments as used in Case 2 we will also have $\bar{S}_{(1)} = \bar{S}_{(2)} = 0$ at $t = 0$.

Without loss of generality, we may assume

$$(4.14) \quad Q_{(2)}(0) > Q_{(1)}(0) > 0$$

and define (4.11).

If the reference side of Σ is adopted as $\alpha = 1$ we have (4.12). Assume $t = \tau$ is the first time for which $[Q(\tau)] = 0$. Hence we have $[Q(t)] > 0$ and $\bar{Q}_{(1)}(t) > 0$. Therefore the right hand side of (4.12) is negative for $t < \tau$ and it follows that $[Q(t)]$ decreases continuously to zero as $t \rightarrow \tau$. If there is no such first time τ , the similar arguments assert that $[Q(t)]$ is a continuously decreasing function of t . Hence Σ is not a surface of strong instability.

5. Discussions.

1. We consider the intersection Σ of the spherical shell with the surface of a cone whose vertex coincides with the center of the sphere. Then when Σ separates a region of plastic equilibrium and a region of plastic flow, Σ is not a slip surface. But when Σ has different values of velocity component in the plastic flow, Σ is a surface of strong instability if, and only if, the normal stress acting on Σ is greater than the normal stress acting on the plane through the axis of cone. These conclusions correspond with Ting's investigations [2] with respect to the stability criterion of a discontinuity surface which is the intersection of a pipe with a plane perpendicular to its axis, where the pipe is loaded by the internal pressure and the axial tension. On the other hand the pipe has the strong instability surface which is the intersection of the pipe with a plane through its axis. But no counterpart exists with respect to the spherical shell.

2. In Sect. 4 we adopt the side $\alpha = 1$ as the reference side in view of (4.10) and (4.14). If we adopt the side $\alpha = 2$ as the reference side, we have the equation

$$(5.1) \quad \frac{d[Q(t)]}{dt} = [Q(t)]^2 - 2\bar{Q}_{(2)}(t)[Q(t)],$$

in place of (4.12). In the Case 2 in Sect. 4 we have

$$(5.2) \quad \frac{d[Q(t)]}{dt} < [Q(t)]^2$$

and then

$$(5.3) \quad [Q(t)] < \frac{-1}{t - \frac{1}{[Q(0)]}} \quad \text{for } t > 0.$$

This inequality gives no criterion with respect to the instability of Σ . In the Case 3 we have the same conclusion in Sect. 4.

Now consider the meaning of the reference side. In the Cases 2 and 3 in Sect. 4 $\bar{S}_{(1)} = \bar{S}_{(2)} = 0$ must hold. Hence (3.15), (4.10) and (4.14) results the

following inequalities

$$(5.4) \quad \begin{cases} \bar{v}_{(2)1} < \bar{v}_{(1)1} < 0 & \text{in Case 2,} \\ \bar{v}_{(2)1} > \bar{v}_{(1)1} > 0 & \text{in Case 3.} \end{cases}$$

Thus the side $\alpha = 1$ means the radial velocity of the material particles on this side of the discontinuity surface is slower than that of the other side. In the creation of a discontinuity surface in a plastic flow region we may consider that the deviated side has the greater plastic flow than the undeformed side. And we may adopt as a reference side the undeformed side with slow plastic flow. Hence it may be reasonable that the side $\alpha = 1$ is defined as a reference side.

3. If the definition (4.11) is altered such as

$$(5.5) \quad [Q(t)] \equiv \bar{Q}_{(1)}(t) - \bar{Q}_{(2)}(t),$$

we have the equation

$$(5.6) \quad \frac{d[Q(t)]}{dt} = [Q(t)]^2 - 2\bar{Q}_{(1)}(t)[Q(t)]$$

in place of (4.12). By the similar method of Ting [2] it results that if $[Q(0)] > 0$ then $\bar{Q}_{(1)}(t) > 0$ and $[Q(t)] > 0$ for all time t . Then in the Case 2 in Sect. 4 we have

$$(5.7) \quad \frac{d[Q(t)]}{dt} > [Q(t)]^2$$

and hence

$$(5.8) \quad [Q(t)] > \frac{1}{\frac{1}{[Q(0)]} - t} \quad \text{for } 0 < t < \frac{1}{[Q(0)]}.$$

The inequality indicates that $[Q(t)]$ approaches ∞ as t approaches $1/[Q(0)]$. Thus the surface Σ is a surface of strong instability. In the Case 3 in Sect. 4 we have the same conclusions in Sect. 4 from the definition (5.5).

If the side $\alpha = 2$ is adopted as the reference side, we can show that the same conclusions mentioned in the above Subsect. 5. 2 are deduced.

Then we may conclude that the same results are deduced from the definitions (4.11) and (5.5).

4. When the entire region of the shell is in plastic flow and when the deformation is subjected to be in spherical symmetry, the similar manipulations in Sect. 2 can hold if the components of the displacement vector and of the strain tensor are replaced by those of the velocity vector and of the rate of strain tensor, respectively. Hence from (2.23) we have $\sigma_{(2)}^{(2)} = \sigma_{(3)}^{(3)}$ in this restricted deformation state. Then the Case 1 in Sect. 4 is applied and no slip surface exists. Therefore in order to occur the plastic fracture on Σ in plastic flow, some spherical symmetry must be broken at any time *e.g.* $t = 0$.

REFERENCES

- [1] T. Y. THOMAS, Plastic flow and fracture in Solids, *J. Math. Mech.*, **7** (1958) 291–322.
- [2] T. Y. THOMAS, Rupture of round bars by tension, *J. Math. Mech.*, **8** (1959) 1–25,
Fracture of circular pipes by internal pressure, *J. Math. Mech.*, **8** (1959) 623–641.
- [3] TSUAN WU TING, Fracture of closed pipes under internal pressure and axial tension,
J. Math. Mech., **9** (1960) 821–867.
- [4] SADIA MURAD MAKKY, Plastic flow and fracture in round bars under pure torsion, *J. Math. Mech.*, **10** (1961) 199–221.
- [5] DALJIT S. AHLUWALIA, Plastic flow and fracture of a thick plate under uniform stress,
J. Math. Mech., **15** (1966) 713–734.
- [6] HUGH RANCE, Plastic flow and fracture in a torsionally stressed planetary sphere, *J. Math. Mech.*, **17** (1968) 953–974.
- [7] T. Y. THOMAS, *Plastic flow and fracture in solids*, (Mathematics in Science and Engineering Vol. 2), Academic Press, New York, 1961.
- [8] A. C. ERINGEN, *Nonlinear Theory of Continuous Media*, McGraw-Hill, New York, 1962.

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