Two Variational Problems in Hermitian Geometry

ISAAC CHAVEL

Communicated by S. S. CHERN

Let M be a compact connected $2n$ -dimensional complex analytic Hermitian manifold (n is the complex dimension), with associated 2-form Ω . (Ω uniquely determines the metric and we often refer to the metric as Ω . Also Ω^n is $n!$ times the volume element of the induced Riemannian metric.) We let R denote the Hermitian scalar curvature, and Δ the Hermitian-Laplace operator of the Hermitian metric (cf. definitions below). Then the integral

$$(1) \quad I(\Omega) = \int_M R^n \Omega^n,$$

is an invariant of the metric in the sense that if the metric is multiplied by a constant, the integral remains the same. In this note we prove

Theorem 1. *A Hermitian metric Ω_0 is a critical point of the functional $I(\Omega)$ —where Ω varies over all Hermitian metrics—if and only if either the Hermitian scalar curvature of Ω_0 vanishes identically on M , or Ω_0 is a Hermitian-Einstein metric. In the Hermitian-Einstein case, Ω_0 is Kählerian, i.e. $d\Omega_0 = 0$ (d denotes exterior differentiation) if and only if Δ_0 is self-adjoint elliptic.*

The Riemannian version of this theorem has been investigated originally by A. Einstein [8] and D. Hilbert [9], and more recently by M. Berger [3], and T. Nagano [11]. Indeed, one has that a Riemannian metric is Einstein if and only if its first variation of total scalar curvature vanishes.

In §1. we define the necessary concepts, and in §2 calculate the first variation of total scalar curvature, and prove Theorem 1. In §3 we restrict our attention to conformal deformations of a fixed Hermitian metric and prove

Theorem 2. *Assume $n > 1$, and let Ω_0 be a Hermitian metric such that Δ_0 is self-adjoint, elliptic. Then Ω_0 is a critical point of $I(\Omega)$ —where Ω varies over Hermitian metrics.*

* Partially supported by National Science Foundation Grant No. GP-8691.

tion metrics of the form $\Omega = e^{2u}\Omega_0$, where $u: M \rightarrow \mathbf{R}^+ - \{0\}$ is a C^∞ function—if and only if Ω_0 has constant Hermitian scalar curvature R_0 . In such a case, the second variation of $I(\Omega)$ is non-negative and is zero if and only if $R_0 \equiv 0$, or the infinitesimal deformation is a constant.

Note that for $n = 1$, Theorem 2 is false by the Gauss-Bonnet Theorem.

In the second part of this paper we consider the following: Let CP_n be n -(complex) dimensional complex projective space and CP_k a naturally imbedded subspace, $k = 1, \dots, n - 1$. For any Hermitian metric Φ on CP_n consider the number

$$\lambda(\Phi) = \frac{\inf_M \int_M \Phi^k}{\left\{ \int_{CP_n} \Phi^n \right\}^{k/n}}.$$

where M ranges over all compact real submanifolds homologous to CP_k . In §4 we prove

Theorem 3. *For any Kählerian metric Φ , $\lambda(\Phi) = 1$, [2, p. 311]. Also, any Kählerian metric is a local maximum of the function λ (of Hermitian metrics), and for any deformation of a Kählerian metric, the first variation of λ vanishes if and only if the infinitesimal deformation is Kählerian.*

This theorem is a weak generalization of the calculation of the extremal length of the family of curves in a given homology class of a Riemann surface. We refer the reader to [1; 2, pp. 303–311; 4; 12] for details of that theory. We also note that by Wirtinger's Inequality [13], replacing the numerator by k -dimensional volume (induced by the associated Riemannian metric) increases the integral, unless M is a complex submanifold. We finally note (cf. §4) that if the Kählerian metrics do not realize $\sup_\Phi \lambda(\Phi)$ then no Hermitian metric does.

§1. Review of formulae of Hermitian geometry. We calculate with the method of moving frames. For the necessary background the reader is referred to [6; 7].

Let $(\ , \)$ denote the Hermitian inner product, and let $e_1, \dots, e_n$ be a local unitary frame on M , i.e., $(e_i, e_k) = \delta_{ik}$ (δ_{ik} is the Kronecker delta. All indices range from 1 to n). Also we let $\omega_1, \dots, \omega_n$ be the co-frame of $(1, 0)$ -forms dual to $e_1, \dots, e_n$. Then the Hermitian inner product is locally given by

$$(2) \quad ds^2 = \sum_k \omega_k \bar{\omega}_k,$$

and the associated 2-form by

$$(3) \quad \Omega = \frac{(-1)^{1/2}}{2} \sum_k \omega_k \wedge \bar{\omega}_k.$$

The Hermitian connection is given locally by a unique skew-Hermitian matrix (ω_{jk}) of local 1-forms such that the local forms

$$(4) \quad \Omega_i = d\omega_i - \Sigma_k \omega_k \wedge \omega_{ki} ,$$

are of type (2, 0). One easily checks that the forms Ω_i vanish identically if and only if $d\Omega = 0$, i.e., Ω is Kählerian. The curvature matrix (Ω_{ik}) is given by

$$(5) \quad \Omega_{ik} = d\omega_{ik} - \Sigma_l \omega_{il} \wedge \omega_{lk} .$$

Each of the forms is of type (1, 1) and the matrix (Ω_{ik}) is skew-Hermitian and tensorial of adjoint type. cf. [6, p. 36]. Therefore, if we let $\omega = \Sigma_k \omega_{kk}$ then $\rho \stackrel{\text{def}}{=} d\omega$ is invariant under change of frames and hence defines a *global* closed 2-form of type (1, 1) on M , which is given locally (using (5)) by

$$\rho = \Sigma_k \Omega_{kk}$$

ρ is called the *Ricci form*, is pure imaginary, i.e., $\bar{\rho} = -\rho$, and is of the form

$$\rho = \frac{1}{2} \Sigma_{i,k} R_{ik} \omega_i \wedge \bar{\omega}_k ,$$

where $\bar{R}_{ik} = R_{ki}$. We denote the matrix (R_{ik}) by $\mathfrak{R}$ and let

$$R = \text{tr } \mathfrak{R} = \Sigma_k R_{kk} ,$$

be the *Hermitian-scalar curvature*. It is a real-valued function on M .

We now let $u: M \rightarrow \mathbb{R}$ be C^∞ . Then we can write $du = \Sigma_k (u_k \omega_k + \bar{u}_k \bar{\omega}_k) = \partial u + \bar{\partial} u$, where $\partial u = \Sigma_k u_k \omega_k$. Then using (4), one has

$$\begin{aligned} \bar{\partial} \partial u &= d(\partial u) \\ &= \Sigma_k ((du_k - \Sigma_l u_l \omega_{kl}) \wedge \omega_k + u_k \Omega_k). \end{aligned}$$

If we set

$$du_k - \Sigma_l u_l \omega_{kl} = \Sigma_s u'_{ks} \omega_s + u''_{ks} \bar{\omega}_s ,$$

then we have

$$\bar{\partial} \partial u = -\Sigma_{k,s} u''_{ks} \omega_k \wedge \bar{\omega}_s ,$$

and $u''_{ks} = \bar{u}'_{sk}$. Following S. S. Chern [7], except for a minus sign, we define

$$\Delta u = -4 \Sigma_k u''_{kk} .$$

To relate Δu to the Laplace-Beltrami operator (which we denote by) $\square u$, we first note that if the forms Ω_j , $j = 1, \dots, n$ are written as

$$\Omega_k = \frac{1}{2} \Sigma_{r,s} T_{krs} \omega_r \wedge \omega_s \quad T_{krs} + T_{ksr} = 0,$$

then

$$\varphi = \Sigma_{k,l} T_{kkl} \omega_l$$

defines a global differential form on M . Let $*$ denote the Hodge isomorphism, $\delta = -*d*$ the adjoint of d , $\square = d\delta + \delta d$, then (cf. [5, p. 530]) for a real valued function $u: M \rightarrow \mathbb{R}$,

$$(6) \quad \square u = \Delta u + [du, \varphi + \bar{\varphi}],$$

where $[,]$ is the inner product of covectors at any given point. In particular $\square = \Delta$ if and only if $\varphi = 0$.

§2. The first variation of $I(\Omega)$. We now fix $\Omega = ((-1)^{1/2}/2)\Sigma_k\omega_k \wedge \omega_k$ and set

$$(7) \quad \Omega_t = \Omega + t\xi$$

where

$$\begin{cases} \xi = \frac{(-1)^{1/2}}{2} \Sigma_{i,k} h_{ik} \omega_i \wedge \bar{\omega}_k \\ h_{ik} = \bar{h}_{ki} \end{cases}$$

Let $\mathcal{H}$ denote the matrix (h_{ik}) ; then $h = \text{tr } \mathcal{H} = \Sigma_k h_{kk}$ is a real-valued function on M . For each t , let ρ_t , R_t denote the Ricci form and Hermitian-scalar curvature, respectively, corresponding to the Hermitian metric Ω_t . Then by [10, p. 158] we have

$$\begin{aligned} \rho_t &= \rho + t \bar{\partial} \partial h + t^2(\cdots), \\ R_t &= R + t(\tfrac{1}{2}\Delta h - \text{tr}(\mathcal{R}\mathcal{H})) + t^2(\cdots) \end{aligned}$$

from which one obtains

$$R_t^n \Omega_t^n = (R^n + \text{tr}(\tfrac{1}{2}R^{n-1}\Delta h - R^{n-1}\text{tr}(\mathcal{R}\mathcal{H}) + R^n h/n) + t^2(\cdots))\Omega^n.$$

Therefore if we set $I(t) = I(\Omega_t)$ then using (6) we have

$$\begin{aligned} I'(0) &= n \int_M R^{n-1} \left(\tfrac{1}{2}\Delta h + \tfrac{1}{n} h R - \text{tr}(\mathcal{R}\mathcal{H}) \right) \Omega^n \\ &= n \int_M \langle \langle \mathcal{R}, \mathcal{H} \rangle \rangle \Omega^n, \end{aligned}$$

where

$$(8) \quad \mathcal{R} = \left(\tfrac{1}{2} \square R^{n-1} + \tfrac{1}{2} \delta(R^{n-1}(\varphi + \bar{\varphi})) + \tfrac{R^n}{n} \right) I - R^{n-1} \mathcal{R}$$

$$\langle \langle \mathcal{R}, \mathcal{H} \rangle \rangle = \text{tr } \mathcal{R}\mathcal{H}$$

where $\mathcal{R}, \mathcal{H}$ are Hermitian matrices. Therefore, $I'(0) = 0$ for all ξ , *i.e.*, for all $\mathcal{H}$, if and only if either $R \equiv 0$, or (since tr is positive definite on Hermitian matrices) $\mathcal{R} \equiv 0$, *i.e.*, Ω is Hermitian-Einstein. In particular, $\rho = -2(-1)^{1/2}(R/n)\Omega$.

We now turn to the last statement of Theorem 1. If Ω is Kählerian, then $\Omega_j = 0$, $j = 1, \dots, n$, which implies $\varphi = 0$ and $\square = \Delta$. Conversely, if Δ is self-adjoint elliptic then (8) can be written as

$$\mathcal{R} = (\tfrac{1}{2}\Delta R^{n-1} + R^n/n)I - R^{n-1}\mathcal{R}.$$

Then $\Delta R = 0$, $R \equiv \text{const}$, and $\rho = -2(-1)^{1/2}(R/n)\Omega$ implies $d\Omega = 0$.

§3. Conformal deformations of Hermitian metrics. As before, assume the original metric is given by

$$ds^2 = \Sigma_k \omega_k \bar{\omega}_k, \quad \Omega = \frac{(-1)^{1/2}}{2} \Sigma_k \omega_k \wedge \bar{\omega}_k,$$

and the new metric given by

$$dr^2 = \Sigma_k \varphi_k \bar{\varphi}_k, \quad \Phi = \frac{(-1)^{1/2}}{2} \Sigma_k \varphi_k \wedge \bar{\varphi}_k,$$

where

$$(9) \quad \varphi_k = e^u \omega_k.$$

Then the first structure equation is given by

$$d\varphi_i = \Sigma_k \varphi_k \wedge \varphi_{ki} + \Phi_i,$$

where

$$\begin{aligned} \varphi_{ki} &= (\partial - \bar{\partial})u \cdot \delta_{ki} + \omega_{ki}, \\ \Phi_i &= 2\partial u \wedge \varphi_i + e^u \Omega_i. \end{aligned}$$

(In particular, if $n > 1$, and the original metric is Kählerian, then the new metric is Kählerian if and only if $u \equiv \text{const.}$) If (Φ_{ik}) is the curvature matrix of the new connection then

$$\Phi_{ik} = \Omega_{ik} + 2\bar{\partial}\partial u \cdot \delta_{ik},$$

from which the Ricci form ζ of the new metric is seen to be

$$\zeta = \rho + 2n\bar{\partial}\partial u,$$

and the scalar curvature, S , is given by

$$Se^{2u} = R + n\Delta u.$$

We therefore have

$$S\Phi = (R + n\Delta u)\Omega.$$

Now let $\mu : M \times (-\frac{1}{2}, \frac{1}{2}) \rightarrow \mathbb{R}$ be given by

$$\mu(x, t) = tu_1(x) + t^2u_2(x) + t^3(\cdots),$$

and $\mu_t : M \rightarrow \mathbb{R}$ be given by $\mu_t(x) = \mu(x, t)$. For each $t \in (-\frac{1}{2}, \frac{1}{2})$, let S_t , Φ_t denote the scalar curvature, and associated 2-form of the Hermitian metric corresponding to the function μ_t , as described above. Then

$$\begin{aligned} (S_t \Phi_t)^n &= (R + n\Delta \mu_t)^n \Omega^n \\ &= (R^n + t\alpha_1 + t^2\alpha_2 + t^3(\cdots))\Omega^n, \end{aligned}$$

where

$$\begin{aligned}\alpha_1 &= n^2 R^{n-1} \Delta u_1, \\ \alpha_2 &= n^2 R^{n-1} (n(n-1) \Delta u_2 + R(\Delta u_1)^2).\end{aligned}$$

Therefore if $I(t) = \int_M S_t^n \Phi_t^n$ then $I'(0) = 0$ for all u_1 if and only if

$$\int_M R^{n-1} \Delta u_1 \Omega^n = 0$$

for all u_1 . Δ is self-adjoint elliptic then implies $R \equiv \text{const}$. In such a case, for any u_1, u_2

$$\begin{aligned}I''(0) &= n^2 R^{n-1} \int_M (n(n-1) \Delta u_2 + R(\Delta u_1)^2) \Omega^n \\ &= n^2 R^n \int_M (\Delta u_1)^2 \Omega^n \\ &\geq 0,\end{aligned}$$

which is equal to zero if and only if $R \equiv 0$ or $u_1 \equiv \text{const}$.

§4. Proof of Theorem 3. For the rest of the paper we will let Ω be the canonical 2-form associated with the Fubini-Study metric on CP_n , normalized so that

$$\int_{CP_k} \Omega^k = 1,$$

for all naturally imbedded $CP_k \subseteq CP_n$, $k = 1, \dots, n$. Now $d\Omega = 0$, and for any other Kähler metric, Φ , we have $\Phi = \kappa\Omega + d\alpha$, where κ is a constant and α is a real 1-form. Since the exterior product of a closed form and an exact form is exact, we have

$$\begin{aligned}\frac{\int_M \Phi^k}{\left\{ \int_{CP_n} \Phi^n \right\}^{k/n}} &= \frac{\int_M \kappa^k \Omega^k}{\left\{ \int_{CP_n} \kappa^n \Omega^n \right\}^{k/n}} \\ &= \frac{\int_{CP_k} \Omega^k}{\left\{ \int_{CP_n} \Omega^n \right\}^{k/n}} \\ &= 1.\end{aligned}$$

We now let ξ be a $(1, 1)$ -form, $\xi = ((-1)^{1/2}/2) \sum_{i,k} h_{ik} dz_i \wedge d\bar{z}_k$, $h_{ik} = \bar{h}_{ki}$ with de Rham decomposition

$$(10) \quad \xi = \kappa\Omega + d\theta_1 + \delta\theta_3,$$

where if $*$ denotes the Hodge star isomorphism, then 1) $\delta = -*d^*$ is the adjoint of d relative to the inner product

$$(11) \quad (\omega, \eta) = \int_{CP_n} \omega \wedge *\eta,$$

2) θ_1, θ_3 are real 1- and 3-forms, respectively, κ is a constant, and 3) the decomposition (10) is orthogonal relative to the inner product (11). Finally set

$$\Omega_t = \Omega + t\xi.$$

Then for sufficiently small t , Ω_t determines a Hermitian metric on CP_n . Direct calculation shows (since $*\Omega^k = \Omega^{n-k} k = 1, \dots, n$) that for any real M homologous to CP_k (henceforth written as $M \sim CP_k$)

$$\int_M \Omega_t^k = 1 + kt \left(\kappa + \int_M \delta\theta_3 \wedge \Omega^{k-1} \right) + t^2(\dots)$$

for $k = 1, \dots, n-1$, and

$$\int_{CP_n} \Omega_t^n = 1 + kt\kappa + t^2(\dots),$$

which implies that for $\sigma_M(t) = \int_M \Omega_t^k / \{ \int_{CP_n} \Omega_t^n \}^{k/n}$,

$$(12) \quad \sigma_M(t) = 1 + kt \int_M \delta\theta_3 \wedge \Omega^{k-1} + t^2(\dots).$$

Lemma 4. Let $G = SU(n+1)$ —the special unitary group on $n+1$ complex variables, acting as the group of isometries of CP_n — M a compact $2k$ -submanifold of CP_n and define $\beta_M : G \rightarrow \mathbb{R}$ by

$$\beta_M(g) = \int_{g \cdot M} \delta\theta_3 \wedge \Omega^{k-1}$$

where $g \cdot M$ is the image of M under the action of $g \in G$. Then

$$\int_G \beta_M d\mu = 0$$

where $d\mu$ is invariant measure on G .

Proof. It is known that the only non-zero invariant differential forms on CP_n are linear combinations (over the real numbers) of $1, \Omega, \Omega^2, \dots, \Omega^n$. For $g \in G$ and any form ω on CP_n , let $g \cdot \omega$ denote the pullback of ω under the action of g . Then $\int_G g \cdot \theta_3 d\mu$ (i.e., the integral of the form-valued function $g \cdot \theta_3$ on G) is an invariant 3-form on CP_n and hence zero. Also since g leaves the $*$ operator invariant, we have $g \cdot \delta\theta_3 = \delta g \cdot \theta_3$. Therefore

$$\int_G \beta_M d\mu = \int_G \left\{ \int_{g \cdot M} \delta\theta_3 \wedge \Omega^{k-1} \right\} d\mu$$

$$\begin{aligned}
&= \int_G \left\{ \int_M g \cdot (\delta \theta_3 \wedge \Omega^{k-1}) \right\} d\mu \\
&= \int_G \left\{ \int_M g \cdot \delta \theta_3 \wedge g \cdot \Omega^{k-1} \right\} d\mu \\
&= \int_G \left\{ \int_M \delta(g \cdot \theta_3) \wedge \Omega^{k-1} \right\} d\mu \\
&= \int_M \delta \left\{ \int_G g \cdot \theta_3 d\mu \right\} \wedge \Omega^{k-1} \\
&= 0,
\end{aligned}$$

which implies the lemma.

Note that since G is connected, M and $g \cdot M$ are homologous, for all $g \in G$. Therefore, if there exists $M_0 \sim CP_k$ such that $\int_{M_0} \delta \theta_3 \wedge \Omega^{k-1} \neq 0$, then

$$a \equiv \inf_{M \sim CP_k} \int_M \delta \theta_3 \wedge \Omega^{k-1} < 0 < \sup_{M \sim CP_k} \int_M \delta \theta_3 \wedge \Omega^{k-1} \equiv b,$$

which implies by (12)

$$\begin{aligned}
\sigma(t) &= \inf_{M \sim CP_k} \sigma_M(t) \\
&= 1 + \begin{cases} tak & t > 0 \\ t bk & t < 0 \end{cases} + t^2(\dots) \\
&< 1,
\end{aligned}$$

for sufficiently small $|t| > 0$. If $\int_M \delta \theta_3 \wedge \Omega^{k-1} = 0$ for all $M \sim CP_k$ then an elementary argument using Stokes Theorem implies $\delta \theta_3 = 0$, i.e., ξ is Kählerian. And the theorem is proven.

We also note that for any non-Kählerian metric on CP_n if $\lambda(\Phi) > -\infty$, then by suitable perturbation $\lambda(\Phi)$ can be increased—we look at the case $CP_1 \subseteq CP_2$. Let $\Phi = \kappa\Omega + d\alpha + \delta\beta$, $\delta\beta \neq 0$. Then

$$\lambda(\Phi) = \frac{\kappa + \inf_M \int_M \delta\beta}{\left\{ \kappa^2 + 2 \int_{CP_2} d\alpha \wedge \delta\beta \right\}^{1/2}}.$$

Since $d\alpha$ is of the form $d\alpha = [(-1)^{1/2}/2] \bar{\partial} \partial f$ where $f: CP_2 \rightarrow \mathbf{R}$ we can always deform f to decrease the denominator, and still maintain the definiteness of the resulting Hermitian form.

The idea of using the averaging over G in such a problem goes back to [12]. The direct generalization of Pu's original result, restricted to conformal deformations of Riemannian homogeneous metrics is as follows:

Theorem 5. (P. M. Pu). *Let G be a compact Lie group, H a closed subgroup, G/H a Riemannian homogeneous space with Riemannian metric τ invariant under the action of G , $\dim G/H = n$. For any $k = 1, \dots, n-1$ let τ_k denote the k -dimensional volume element, ν_k be a homology class of $H_k(G/H)$, and*

$$\lambda(\nu_k) = \inf_{M \in \nu_k} \int_M \tau_k / \left\{ \int_{G/H} \tau_n \right\}^{k/n}.$$

Then for any $f: G/H \rightarrow \mathbb{R}^+ - \{0\}$ $\in C^\infty$, if we let $\lambda_f(\nu_k)$ denote the number corresponding to the metric $f \cdot \tau$, then

$$\lambda(\nu_k) \geq \lambda_f(\nu_k)$$

with equality if and only if $f \equiv \text{constant}$.

Proof. Let h be the function given by

$$h = \left\{ \int_G g \cdot f^k d\mu \right\}^{1/k}.$$

Then by the transitivity of G , h is a constant function on G/H . Now use Holder's Inequality and Fubini's Theorem.

REFERENCES

- [1] R. D. M. ACCOLA, Differentials and extremal length on Riemann surfaces, *Proc. Nat. Acad. Sci.*, **46** (1960) 540–543.
- [2] M. BERGER, *Lectures on geodesics in Riemannian Geometry*, Tata Institute on Fundamental Research, Bombay, 1965.
- [3] ———, Sur les variétés d' Einstein compactes, *C. R. de III^e Réunion du Groupement des Mathématiciens d' Expression Latine*, Centre Belge de Recherches Mathématiques, 1966.
- [4] C. BLATTER, Über Extremallängen auf geschlossenen Flächen, *Comment Math. Helv.*, **35** (1961) 153–168.
- [5] W. M. BOOTHBY, Some fundamental formulas for Hermitian manifolds with non-vanishing torsion, *Amer. J. Math.*, **76** (1954) 509–534.
- [6] S. S. CHERN, *Complex Manifolds without Potential Theory*, Van Nostrand Math. Studies #15, D. Van Nostrand Co., Princeton, 1967.
- [7] ———, On holomorphic mappings of Hermitian manifolds of the same dimension, *Proc. of Symposia in Pure Mathematics*, Vol. 11, AMS, 1968, 157–170.
- [8] A. EINSTEIN, Hamiltonisches Prinzip und allgemeine Relativitäts-theorie, *Preuss. Akad. Wiss. Sitz.*, 1916.
- [9] D. HILBERT, Die Grundlagen der physik, *Nach. Geo. Wiss. Göttingen*, 1915, 395.
- [10] S. KOBAYASHI & K. NOMIZU, *Foundations of Differential Geometry*, Vol. II, John Wiley and Sons, New York, 1969.
- [11] T. NAGANO, A problem on the existence of an Einstein metric, *J. Math. Soc. Japan*, **19** (1967) 30–31.
- [12] P. M. PU, Some inequalities on certain non-orientable manifolds, *Pacific J. Math.*, **11** (1952) 55–71.
- [13] W. WIRTINGER, Eine Determinantenidentität und ihre Anwendung auf Analytische Gebilde in euclidischer und hermitischer Massbestimmung, *Monatshefte Math. u. Physik*, **44** (1936) 343–365.

University of Minnesota

Date communicated: OCTOBER 3, 1969