On the Fourier Coefficients of Outer Functions

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Communicated by M. ROSENBLATT

Introduction. In this note we consider the problem raised by Helson [3, p. 22] of identifying those sequences which can arise as

$$\sum_{j=n}^{\infty} |a_j|^2, \quad n = 0, 1, 2, \dots,$$

where $\{a_j\}$ are the Fourier coefficients of an outer function in H^2 . We obtain a lower estimate for such sequences and use it to show that they can vanish arbitrarily slowly as $n \rightarrow \infty$. The proof is applied to answer a question about cyclic vectors of the backward shift on H^2 (see the question raised after Proposition 2). Our methods are elementary. The interested reader can rephrase Proposition 1 and Theorem 1 in terms of linear prediction theory; see, *e.g.*, [2, Chapter 10].

I wish to thank Alan Zame for supplying the important part of the proof of Lemma 1.

First we need some notation. C is the unit circle in the complex plane and σ is normalized Lebesgue measure on C (or, sometimes, on $[0, 2\pi)$). χ is the identity function on C : $\chi(e^{ix}) = e^{ix}$. The value of a function f at e^{ix} will be written $f(e^{ix})$ or $f(x)$.

For $1 \leq p \leq \infty$, L^p and H^p denote the Lebesgue and Hardy spaces with respect to σ ; the relevant information about these spaces can be found in [3] or [4]. If $0 \leq w \in L^1$, $G(w)$ is the geometric mean of w :

$$G(w) = \exp \int \log w \, d\sigma,$$

where the right side is taken to be zero if $\log w$ is not integrable. $G(w)$ never exceeds the arithmetic mean $\int w \, d\sigma$.

A function f in H^2 is *outer* provided $\{\chi^n f: n \geq 0\}$ spans H^2 ; $g \in H^2$ is *inner* if $|g| = 1$ a.e. It is well known that $\log |f| \in L^1$ (*i.e.*, $G(|f|) > 0$) whenever $f \in H^1$ and f is not the zero function. If $0 \leq w \in L^1$, then in order for $G(w)$ to be positive, it is necessary and sufficient that $w = |f|^2$ a.e. for some outer function f .

For any positive integer n , let

$$\rho_n(w) = \inf \int |1 - \chi^n p|^2 w \, d\sigma,$$

where the infimum is taken over the analytic trigonometric polynomials p . Szegő's Theorem [4, p. 49] identifies $\rho_1(w)$ as $G(w)$. $\{\rho_n(w)\}$ is a non-decreasing sequence and for all of the $\rho_n(w)$ to vanish it is necessary and sufficient that any one of them vanish. If $w = |f|^2$ a.e. where f is outer with Fourier coefficients $\{a_i\}$, it is known [3, p. 22] that

$$(1) \quad \rho_n(w) = \sum_{j=0}^{n-1} |a_j|^2.$$

Clearly $\rho_n(w) \rightarrow \int |f|^2 d\sigma = \int w d\sigma$ as $n \rightarrow \infty$.

For any real t , w_t denotes the translate $w_t(x) = w(x + t)$. It will be convenient to use the notation $G_n(w)$ for the geometric mean of the function

$$n^{-1} \sum_{j=0}^{n-1} w_{2\pi j/n}.$$

Clearly $G_1(w) = G(w)$.

1. Coefficients of outer functions.

Theorem 1. *Let $\{d_j\}$ be any sequence of numbers converging to zero with $0 < d_j < 1$ for each j . Then there is a measurable function w on C with $0 \leq w \leq 1$ satisfying*

$$(2) \quad \int \log w d\sigma > -\infty, \quad \text{and}$$

$$(3) \quad \rho_n(w) \leq \int w d\sigma - d_n \quad (n = 1, 2, \dots).$$

Two preliminary results are needed to prove the theorem, but the general idea is easily stated. In Proposition 1 it is proved that $G_n(w)$ lies between $\rho_n(w)$ and $\int w d\sigma$. Therefore the theorem will follow if it can be shown that $G_n(w)$ approaches $\int w d\sigma$ slowly enough. $G_n(w)$ will be small, relative to $\int w d\sigma$, if

$$n^{-1} \sum_{j=0}^{n-1} w_{2\pi j/n}$$

is suitably small on some subset of C . This will be the case if w itself is small on a set which recurs with period $2\pi/n$ as one moves around the unit circle. We define w as an infinite product whose j^{th} factor has period $2\pi/j$. The factors are chosen in such a way that those corresponding to multiples of n are small enough in spots to keep $G_n(w)$ away from $\int w d\sigma$. Lemma 1 implies that this can be done while ensuring that $\int \log w d\sigma > -\infty$.

Proposition 1. *Let $0 \leq w \in L^1$. Then*

$$\rho_n(w) \leq G_n(w) \leq \int w d\sigma \quad (n = 1, 2, \dots).$$

Proof. The second inequality follows immediately from the inequality of the geometric and arithmetic means and the fact that σ is translation invariant. To prove the first inequality, note that

$$(4) \quad \rho_n(w) \leq \int_0^{2\pi} |1 - e^{inx} p(e^{inx})|^2 w(x) d\sigma(x),$$

for any analytic polynomial p . Making the change of variable $t = nx$ and keeping in mind that

$$n^{-1} \sum_{i=0}^{n-1} w_{2\pi i/n}(t/n)$$

has period 2π as a function of t , we find, after a straightforward computation, that the right side of (4) is

$$\int_0^{2\pi} |1 - e^{it} p(e^{it})|^2 \left[n^{-1} \sum_{i=0}^{n-1} w_{2\pi i/n}(t/n) \right] d\sigma(t).$$

Using this and taking the infimum of (4) over all analytic polynomials p , we have, by Szegő's Theorem,

$$(5) \quad \rho_n(w) \leq \exp \int_0^{2\pi} \log \left[n^{-1} \sum_{i=0}^{n-1} w_{2\pi i/n}(t/n) \right] d\sigma(t).$$

The reverse change of variables $x = t/n$ and another short computation show that the right side of (5) is exactly $G_n(w)$. This completes the proof.

Lemma 1. Let $r > 0$. Suppose that $\{c_n\}$ and $\{h_n\}$ are sequences of positive and negative numbers, respectively, both converging to zero. Then there are positive sequences $\{a_n\}$ and $\{\epsilon_n\}$ with $0 < \epsilon_n < 1$ and satisfying the following conditions:

$$(A) \quad \sum_i a_i < r,$$

$$(B) \quad \sum_i a_i \log \epsilon_i > -\infty,$$

$$(C) \quad \sum_i a_{ni} \leq c_n \quad (n = 1, 2, \dots),$$

$$(D) \quad \sum_i a_{ni} \log \epsilon_{ni} \leq h_n \quad (n = 1, 2, \dots).$$

Proof. Only two of the above conditions require any work. This is apparent when the problem is restated as follows: assume that $0 < \epsilon_i < 1$ and set $v_i = -(\log \epsilon_i)^{-1}$, $b_i = -a_i \log \epsilon_i$, so that $a_i = b_i v_i$. The problem then becomes to find positive sequences $\{v_i\}$ and $\{b_i\}$ satisfying

$$(A') \quad \sum_i b_i v_i < r,$$

$$(B') \quad \sum_i b_i < \infty,$$

$$(C') \quad \sum_i b_{ni} v_{ni} \leq c_n \quad (n = 1, 2, \dots),$$

$$(D') \quad \sum_i b_{ni} \geq |h_n| \quad (n = 1, 2, \dots).$$

Given $\{b_i\}$, the existence of $\{v_i\}$ is obvious. To find $\{b_n\}$, first note that the sequence $\{|h_n|\}$ may not be monotone, so replace it by $z_n = \sup \{|h_j| : j \geq n\}$ ($n = 1, 2, \dots$). Clearly $\{z_n\}$ is non-increasing, $z_n \geq |h_n|$ and $z_n \rightarrow 0$ as $n \rightarrow \infty$. The following construction of $\{b_n\}$ is due to Alan Zame.

Let $\{n(k) : k \geq 1\}$ be a subsequence of the positive integers with $n(1) = 1$ and $\sum_k z_{n(k)} < \infty$. Define $b_{n(k)+1} = z_{n(k)-1}$ ($k = 2, 3, \dots$) and $b_n = 0$ if $n \notin \{n(k) : k = 2, 3, \dots\}$. Clearly $\sum_n b_n \leq \sum_k z_{n(k)} < \infty$, so (B') holds. For any positive integer p , there exists $k \geq 1$ with $n(k) \leq p < n(k+1)$. Since p divides $n(k+1)!$, $b_{n(k+1)+1}$ occurs in the sequence $\{b_{pi} : i \geq 1\}$. Therefore $\sum_i b_{pi} \geq b_{n(k+1)+1} = z_{n(k)} \geq z_p$ where the last inequality follows by the monotonicity of $\{z_n\}$. Since $z_p \geq |h_p|$, (D') is satisfied and the proof is complete.

Proof of Theorem 1. Let $\{a_i\}$ and $\{\epsilon_i\}$ be unspecified (for the moment) sequences contained in $(0, 1)$. Define $f_i(e^{ix})$ to be ϵ_i when $0 \leq x < 2\pi a_i \pmod{2\pi}$ and $f_i(e^{ix}) = 1$ otherwise. Define w on C by

$$w(x) = \prod_{j=1}^{\infty} f_j(e^{ix}), \quad 0 \leq x < 2\pi,$$

where the pointwise limits of the partial products are allowed to be zero. We will choose $\{a_i\}$ and $\{\epsilon_i\}$ so that (2) and (3) are satisfied.

The partial products of w are a non-increasing sequence and $0 \leq f_i \leq 1$ for each j , so

$$\log w(x) = \sum_i \log f_i(e^{ix}), \quad 0 \leq x < 2\pi.$$

The two sides of this equation are either both finite or both $-\infty$. The same is true of the next equation which follows now by monotone convergence:

$$(6) \quad \int \log w \, d\sigma = \sum_i \int_0^{2\pi} \log f_i(e^{ix}) \, d\sigma(x) = \sum_i a_i \log \epsilon_i.$$

Let n be a fixed positive integer and define

$$u(x) = \prod_{p=1}^{\infty} f_{np}(e^{inpx}), \quad v(x) = \prod_{i \neq np} f_i(e^{ix}),$$

where the last product is taken over the non-multiples of n . Clearly $w = uv$ and

$$n^{-1} \sum_{j=0}^{n-1} w_{2\pi j/n} = un^{-1} \sum_{j=0}^{n-1} v_{2\pi j/n}.$$

Consequently

$$(7) \quad G_n(w) = G(u)G\left(n^{-1} \sum_{j=0}^{n-1} v_{2\pi j/n}\right)$$

$$\begin{aligned}
&\leq G(u) \int \left(n^{-1} \sum_{j=0}^{n-1} v_{2^j \pi j/n} \right) d\sigma \\
&= G(u) \int v d\sigma.
\end{aligned}$$

It follows from the definition of u that

$$(8) \quad G(u) = \exp \left(\sum_i a_{ni} \log \epsilon_{ni} \right).$$

Also, if h is the characteristic function of the set

$$\{e^{ix} : u(x) < 1\} = \bigcup_{j=1}^{\infty} \{e^{ix} : f_{nj}(e^{inix}) < 1\},$$

we have

$$(9) \quad \int h d\sigma \leq \sum_{j=1}^{\infty} a_{nj}.$$

Moreover, $1 - h \leq u$, so $(1 - h)v \leq uv = w$ and therefore

$$v \leq w + hv \leq w + h.$$

Integrating and applying (9) gives

$$(10) \quad \int v d\sigma \leq \int w d\sigma + \sum_i a_{ni}.$$

From the fact that $G(u) \leq \int u d\sigma \leq 1$ and from (7), (8) and (10), it follows that

$$(11) \quad G_n(w) \leq \left(\int w d\sigma \right) \exp \left(\sum_i a_{ni} \log \epsilon_{ni} \right) + \sum_i a_{ni}.$$

In view of Proposition 1, the proof will be complete if we can choose $\{a_i\}$ and $\{\epsilon_i\}$ so that (6) is finite and

$$\left(\int w d\sigma \right) \exp \left(\sum_i a_{ni} \log \epsilon_{ni} \right) + \sum_i a_{ni} \leq \int w d\sigma - d_n \quad (n = 1, 2, \dots).$$

Rewriting this inequality gives

$$(12) \quad \sum_i a_{ni} \log \epsilon_{ni} \leq \log \left[1 - \left(\int w d\sigma \right)^{-1} (d_n + \sum_i a_{ni}) \right] \quad (n = 1, 2, \dots).$$

Now choose $t > 1$ small enough so that $1 - td_n > 0$ for all n . Let $\{a_i\}$ and $\{\epsilon_i\}$ be the sequences produced by Lemma 1 when $r = (t - 1)(t + 1)^{-1}$, $c_n = rd_n$ and $h_n = \log(1 - td_n)$. Condition (A) and the inequality (9) with $n = 1$ imply that $w(x) = 1$ on a set whose measure is no less than $1 - r$, so $\int w d\sigma \geq 1 - r$. It follows from this together with condition (C) that

$$\log(1 - td_n) \leq \log \left[1 - \left(\int w d\sigma \right)^{-1} (d_n + \sum_i a_{ni}) \right]$$

for each n . Since $h_n = \log(1 - td_n)$, this last inequality coupled with condition (D) implies that (12) is in fact true. Finally, the condition (B) is equivalent to the finiteness of (6). This completes the proof.

Corollary 1. *Let $\{d_n\}$ be a real sequence converging to zero with $0 < d_n < 1$ for each n . Then there exists an outer function f with Fourier coefficients $\{a_i\}$ satisfying*

$$\begin{aligned} i) \quad & 0 \leq |f| \leq 1 \text{ a.e., and} \\ ii) \quad & \sum_{i=n}^{\infty} |a_i|^2 \geq d_n \quad (n = 1, 2, \dots). \end{aligned}$$

Proof. Let w be related to $\{d_n\}$ as in Theorem 1 and let f be an outer function with $w = |f|^2$ a.e. Now (1) implies that

$$\sum_{i=n}^{\infty} |a_i|^2 = \int w \, d\sigma - \rho_n(w),$$

so f is the desired outer function and the proof is complete.

I do not know if anything more can be said about the relationship between $G_n(w)$ and $\rho_n(w)$ than is contained in Proposition 1. The following example shows that $\{G_n(w)\}$ can be far from monotone and can approach $\int w \, d\sigma$ more rapidly than $\{\rho_n(w)\}$.

Example. For fixed positive $t \neq 1$, let $w(e^{ix}) = t$ if $0 \leq x < \pi$ and $w(e^{ix}) = 1$ otherwise. I will show that $\int w \, d\sigma - \rho_n(w)$ is the same size as n^{-1} while $\int w \, d\sigma - G_n(w)$ is zero for even n and asymptotically like n^{-2} for odd n .

Let f be the outer function defined by

$$(13) \quad f(z) = \exp \left\{ \frac{1}{2} \int_0^\pi \frac{e^{ix} + z}{e^{ix} - z} \log t \, d\sigma(x) \right\}, \quad |z| < 1.$$

Clearly $|f|^2 = w$ a.e. on C . Evaluating the integral in (13) [6, Lemma 2] gives

$$f(z) = t^{1/4} (1 - z)^{(i \log t)/2\pi} (1 + z)^{(-i \log t)/2\pi}, \quad |z| < 1.$$

Let

$$g(z) = (1 - iz)^{(i \log t)/2\pi} (1 + iz)^{(-i \log t)/2\pi}, \quad |z| < 1.$$

If $\{a_n\}$ and $\{b_n\}$ are the Fourier (= Taylor) coefficients of f and g respectively, clearly $t^{1/4} |b_n| = |a_n|$ ($n = 0, 1, \dots$). It follows immediately from Darboux's method of estimating Taylor coefficients [7, p. 200] that

$$(14) \quad b_n = 2 \operatorname{Re} \left\{ 2^{-ia} \binom{ia}{n} i^n \right\} + O(n^{-2}),$$

where $a = (\log t)/2\pi$ and $\binom{ia}{n}$ is the generalized binomial coefficient. Therefore

$$(15) \quad \frac{2|a|}{n} \left| \cos \left(-a \log 2 + \arg \binom{ia}{n} + \arg i^n \right) \right| + O(n^{-2}) \leq |b_n| = O(n^{-1}).$$

A little calculation, which I omit, shows that the inequality in (15) can be replaced by the inequalities

$$(16) \quad |b_{2n+1}| \geq 2 |a| (2n+1)^{-1} |\sin(a \log(2n+1) + \theta)| + O(n^{-2}),$$

$$|b_{2n}| \geq 2 |a| (2n)^{-1} |\cos(a \log(2n) + \theta)| + O(n^{-2}),$$

for some fixed real θ .

There exists $B > 0$ so that

$$\sum_{j=n}^{\infty} [j^{-1} \sin(a \log j + \theta)]^2 \geq Bn^{-1} \quad (n = 1, 2, \dots);$$

to see this, approximate the sums by integrals. Moreover, B can be chosen so that the inequality remains valid when "sin" is replaced by "cos". This, together with (16) implies that

$$\sum_{j=n}^{\infty} |b_j|^2 \geq Cn^{-1} \quad (n = 1, 2, \dots)$$

for some $C > 0$. It now follows from the equality in (15) and the relationship between b_n and a_n that

$$Dn^{-1} \leq \sum_{i=n}^{\infty} |a_i|^2 \leq En^{-1} \quad (n = 1, 2, \dots),$$

for some $D, E > 0$. Therefore the sequences $\{\int w d\sigma - \rho_n(w)\}$ and $\{n^{-1}\}$ dominate each other.

On the other hand, it is not hard to show that

$$(17) \quad G_n(w) = 2^{-1}(1+t)$$

if n is even and

$$(18) \quad G_n(w) = 2^{-1}[(1+t)^2 - n^{-2}(1-t)^2]^{1/2}$$

if n is odd. To see this, divide C into n equal arcs with $1 = e^{i0}$ as a point of division. If n is even, -1 is also a division point, so on any arc of the form $\{e^{ix}: 2\pi k/n \leq x < 2\pi(k+1)/n\}$ half of the functions $w_{2\pi i/n}$ ($j = 0, \dots, n-1$) have constant value t and half have the value 1. Therefore, the average of these functions has constant value $2^{-1}(1+t)$, and (17) follows.

If n is odd, -1 bisects one of the arcs; bisect the remaining arcs to get $2n$ new arcs. On half of the new arcs (the 1st, 3rd, etc.), $(n+1)/2$ of the functions $w_{2\pi i/n}$ have constant value t and the remaining $(n-1)/2$ functions have constant value 1. The situation is exactly reversed on the other (i.e., the 2nd, 4th, etc.) new arcs. It follows easily from these facts that (18) gives the right value for $G_n(w)$. (18) implies that $n^2[\int w d\sigma - G_{2n+1}(w)]$ has a finite, non-zero limit as $n \rightarrow \infty$.

2. Cyclic vectors of the backward shift. The shift operator on H^2 is the isometry defined by $Sf = \chi f$. An H^2 function f is *cyclic* if it is a cyclic vector of

the backward shift S^* , that is if $\{S^{*n}f: n \geq 0\}$ spans H^2 . A nice summary of results about cyclic vectors of S^* is contained in [1].

If $\{S^{*n}f\}$ does not span H^2 , its orthogonal complement is invariant under S and therefore has the form bH^2 for some inner function b [4, p. 99]. Therefore f is non-cyclic if and only if $f \in (bH^2)^\perp$ for some inner b . In what follows, $q(0)$ denotes the value at 0 of the analytic extension of q to the open unit disk.

Proposition 2. *Let f be non-cyclic, not a polynomial, and suppose $\{a_i\}$ are the Fourier coefficients of f . There exists $r > 0$ and a non-negative integer k such that*

$$(19) \quad \sum_{i=k+n}^{\infty} |a_i|^2 \geq r^n \sum_{i=k}^{\infty} |a_i|^2 \quad (n = 0, 1, \dots).$$

Proof. Suppose b is inner and $(bH^2)^\perp$ contains f . Then for some $k \geq 0$, $b = \chi^k q$, where q is inner and $q(0) \neq 0$. It is easy to check that $S^{*k}f \in (qH^2)^\perp$ and

$$S^{*k}f \sim \sum_{i=k}^{\infty} a_i \chi^{i-k}.$$

It follows from a theorem of Moeller [5] that S^* is invertible (and thus bounded below) on $(qH^2)^\perp$. Therefore $\|S^{*n}S^{*k}f\|^2 \geq r^n \|S^{*k}f\|^2$ for some $r > 0$ and all $n = 0, 1, \dots$. This completes the proof.

Proposition 2 raises the question of whether the condition (19) or a stronger one of its type is sufficient for f to be non-cyclic. The answer, which appears in Proposition 4, is no.

Proposition 3. *Let f be non-cyclic and $\alpha > 0$. Then either $\log |\alpha - |f|| \in L^1$ or $|f| = \alpha$ a.e.*

Proof. $f \in (bH^2)^\perp$ for some inner function b and $\alpha \in (\chi H^2)^\perp$, so $(\chi b H^2)^\perp$ contains both f and α . Therefore

$$\int (\chi^n \chi b f) \bar{f} d\sigma = 0 = \int (\chi^n \chi b \alpha) \bar{\alpha} d\sigma \quad \text{for } n = 0, 1, \dots$$

It follows that

$$\int \chi^n \chi b (\alpha^2 - |f|^2) d\sigma = 0 \quad (n = 0, 1, \dots),$$

so $\chi b (\alpha^2 - |f|^2) \in H^1$. This implies that either $\chi b (\alpha^2 - |f|^2) = 0$ a.e. or

$$\int \log |\alpha^2 - |f|^2| d\sigma = \int \log |\chi b (\alpha^2 - |f|^2)| d\sigma > -\infty,$$

and this is clearly equivalent to the desired conclusion.

R. G. Douglas has found the following interesting generalization of this result: in the statement, replace α by $|g|$, where g is any non-cyclic vector; in the proof, replace α by g , χH^2 by qH^2 and χb by qb , where q is inner and $(qH^2)^\perp$ contains g .

Proposition 4. Suppose $0 < d_n < 1$ and $d_n \rightarrow 0$ as $n \rightarrow \infty$. Then there is a cyclic, outer, essentially bounded function f with Fourier coefficients $\{a_i\}$, satisfying

$$\sum_{i=n}^{\infty} |a_i|^2 \geq d_n \|f\|_{\infty}^2 \quad (n = 1, 2, \dots).$$

Proof. Let f be the outer function of Corollary 1. The proof of Theorem 1 shows that $|f| = 1$ on a set of positive measure, so $\log |1 - f|$ is not integrable. Since $|f|$ is not a.e. constant, it follows from Proposition 3 that f is cyclic. This completes the proof.

The fact that f is outer is not, of course, relevant to the question raised after Proposition 2.

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Date communicated: OCTOBER 2, 1969