Applications of the Sz.-Nagy and Foias Lifting Theorem

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1. Introduction. The Sz.-Nagy and Foias Lifting Theorem [15], Theorem 2.1 below, has allowed a new approach to a certain class of Hilbert space extremal problems. Recently Pousson [11] and Rabindranathan [13] have derived results concerning the invertibility of Toeplitz operators relative to non-simple shifts. Theirs is a follow-up to previous work of Devinatz [3] and Widom [16]. Since the appearance of the work of Pousson and Rabindranathan, Devinatz and Shinbrot [4] have abstracted the problem further still by eliminating the shift entirely. The weakness in this approach is that one cannot regain from it the information that Pousson and Rabindranathan obtain. In Section 2 with the aid of the Sz.-Nagy and Foias Theorem we shall consider an alternate abstract approach which has the advantage of including and in fact extending the results of Pousson and Rabindranathan.

In Section 3 we give a new generalization of the extremal problem arising in the study of Hankel matrices as first solved by Nehari [9]. The present author has previously generalized the Nehari result to non-simple shifts [10]. The treatment appearing in Section 3 is a further extension. The techniques used in this section are very similar to those which have already proved useful in the study of another closely related extremal problem [8].

In Section 4 the theory of Section 2 is applied to obtain criteria for the invertibility of Toeplitz operators on the space $H^2(d\mu)$ where μ is a finite Borel measure on the unit circle. Finally, a special case of the extremal problem studied in Section 3 is used to obtain necessary and sufficient conditions for the convergence of Fourier series in the space $L^2(W)$, where W is an operator-valued weight function. Equivalently one can interpret such results to determine whether the conjugation operator on $L^2(W)$ is bounded. In the scalar case this is a problem that has been studied by Helson and Szego [7].

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2. An invertibility theorem. We begin by stating the form of the Sz.-Nagy and Foias theorem which best fits our purposes.

2.1. Theorem. *For $i = 1, 2$, let T_i be a contraction on a Hilbert space $\mathcal{H}_i$, and let U_i acting on a Hilbert space $\mathcal{K}_i$ be a unitary power dilation of T_i ([5], problem 178). Let P_i be the orthogonal projection of $\mathcal{K}_i$ onto $\mathcal{H}_i$.*

Then a bounded operator X on $\mathcal{H}_1$ to $\mathcal{H}_2$ satisfies $T_2 X = X T_1$ only if there exists an operator Y on $\mathcal{K}_1$ to $\mathcal{K}_2$ such that $U_2 Y = Y U_1$, $\|Y\| = \|X\|$, and $P_2 Y P_1 = X P_1$.

The general setting which will be employed throughout this section is as follows. We let U denote a unitary operator on a Hilbert space $\mathcal{K}$, and we fix a closed subspace $\mathcal{H}$ of $\mathcal{K}$ which is invariant under U . Let $P_{\mathcal{H}}$ denote the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.

The question we ask is the following: Given an invertible operator A in the commutant of U , under what conditions is $P_{\mathcal{H}} A | \mathcal{H}$ invertible? If U is a bilateral shift and if $\mathcal{K}$ is separable, then this is essentially the question asked by Rabindranathan. The first case to be considered is that of a unitary operator in the commutant of U . The theorem below generalizes analogous results of Pousson, Rabindranathan, and Devinatz and Shinbrot.

2.2. Theorem. *Suppose W is a unitary operator on $\mathcal{K}$ which commutes with U . Then $P_{\mathcal{H}} W | \mathcal{H}$ is invertible if and only if there exists an invertible operator B in the commutant of U such that $B\mathcal{H} = \mathcal{H}$ and $\|W - B\| < 1$.*

Proof. If such an operator B exists, then the fact that $P_{\mathcal{H}} W | \mathcal{H}$ is invertible follows from Theorem 2 of Devinatz and Shinbrot [4].

Suppose then that $P_{\mathcal{H}} W | \mathcal{H}$ is invertible. It is convenient to represent operators on $\mathcal{K}$ as 2×2 matrices, the matrix representation being that induced by the decomposition of $\mathcal{K}$, $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$. Suppose then that

$$U = \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}.$$

From the fact that U and W commute we find by an elementary matrix computation that $U_{22} W_{21} = W_{21} U_{11}$. Furthermore it is easy to see that U is a unitary power dilation for both U_{11} and U_{22} . Thus by Theorem 2.1 there exists an operator

$$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ W_{21} & Y_{22} \end{bmatrix}$$

on $\mathcal{K}$ which satisfies $\|Y\| = \|W_{21}\|$ and which commutes with U . Let

$$B = W - Y = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}.$$

Then B commutes with U and $B\mathcal{H} \subseteq \mathcal{H}$. The fact that W_{11} is invertible implies that $\|W_{21}\| < 1$ (Devinatz and Shinbrot [4], Corollary 1). Thus $\|W - B\| =$

$\|Y\| < 1$. But then $\|I - W^*B\| < 1$ which implies that W^*B is invertible and hence B is invertible. From the inequality $\|I - W^*B\| < 1$ and a matrix multiplication we deduce $\|I_{\mathcal{H}} - W_{11}^*B_{11}\| < 1$ where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. The fact that W_{11}^* is invertible allows us to conclude that B_{11} is invertible, or $B\mathcal{H} = \mathcal{H}$. The proof is now complete.

The following lemmas will be needed to derive a form of a well-known factorization theorem of Beurling (Theorem 2.6).

2.3. Lemma. *An isometry that is similar to a shift is a shift.*

Proof. Suppose S_0 on $\mathcal{H}_0$ is a shift, S_1 on $\mathcal{H}_1$ is an isometry, and that T is an invertible map satisfying $S_0 = T^{-1}S_1T$.

Then $\bigcap_{n=0}^{\infty} S_1^n \mathcal{H}_1 = \bigcap_{n=0}^{\infty} S_1^n T \mathcal{H}_0 = \bigcap_{n=0}^{\infty} T S_0^n \mathcal{H}_0 = T(\bigcap_{n=0}^{\infty} S_0^n \mathcal{H}_0) = \{0\}$. This shows that S_1 has no unitary part and must be a shift.

2.4 Lemma. *Two shifts which are similar are unitarily equivalent.*

Proof. Let S_0 on $\mathcal{H}_0$ and S_1 on $\mathcal{H}_1$ be shifts, and let $T: \mathcal{H}_0 \rightarrow \mathcal{H}_1$ satisfy $S_0 = T^{-1}S_1T$. Let $C_0 = (S_0 \mathcal{H}_0)^\perp$ and $C_1 = (S_1 \mathcal{H}_1)^\perp$, and let P_1 be the orthogonal projection of $\mathcal{H}_1$ onto C_1 .

Let $Q: C_0 \rightarrow C_1$ be defined by $Qf = P_1 T f$. For any $c_1 \in C_1$, $c_1 = T(c_0 + S_0 g_0)$ for some $c_0 \in C_0$, $g_0 \in \mathcal{H}_0$. So $c_1 = P_1 T c_0 + P_1 T S_0 g_0 = P_1 T c_0 + P_1 S_1 T g_0 = P_1 T c_0 = Q c_0$. We have shown that Q maps C_0 onto C_1 .

Now suppose $c_0 \in C_0$ and $Q c_0 = 0$. We can write $T c_0 = c_1 + S_1 g_1$, where $c_1 \in C_1$ and $g_1 \in \mathcal{H}_1$. Then $0 = P_1 T c_0 = c_1$, so $T c_0 = S_1 g_1$, and $c_0 = T^{-1} S_1 g_1 = S_0 T^{-1} g_1$. But c_0 is perpendicular to the range of S_0 , so $c_0 = 0$. We have just shown that $\ker Q = \{0\}$, and by the closed graph theorem it now follows that Q is invertible. Thus C_0 and C_1 have the same Hilbert space dimension, and S_0 and S_1 , being shifts of the same multiplicity, are unitarily equivalent.

2.5 Lemma. *Two isometries which are similar are unitarily equivalent.*

Proof. Let S_0 and S_1 be isometries on Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ respectively, and let T be an invertible map from $\mathcal{H}_0$ to $\mathcal{H}_1$ such that $S_0 = T^{-1}S_1T$. By the Wold Decomposition Theorem for isometries, $\mathcal{H}_0 = M_0 \oplus N_0$ where $S_0|_{M_0}$ is a shift and $S_0|_{N_0}$ is unitary.

Now $\mathcal{H}_1 = TM_0 + TN_0$ where the sum is direct but not necessarily orthogonal. Since $S_1 TN_0 = TS_0 N_0 = TN_0$, however, we do know that TN_0 reduces S_1 . Thus $(TN_0)^\perp$ also reduces S_1 , and we shall show that the restriction of S_1 to this subspace is similar to $S_0|_{M_0}$.

Let P denote the orthogonal projection of $\mathcal{H}_1$ onto $(TN_0)^\perp$. Let Q be the operator from M_0 to $(TN_0)^\perp$ defined by $Qf = PTf$. We show that Q maps M_0 onto $(TN_0)^\perp$, that Q is invertible, and that $S_0 = Q^{-1}S_1Q$ on M_0 .

Suppose $f \in (TN_0)^\perp$. Then $f = Th + Tk$ for some $h \in M_0$, $k \in N_0$, and $Pf = f = PTh + PTk = PTh = Qh$. Thus we see that the range of Q is $(TN_0)^\perp$. That Q is invertible will follow from the closed graph theorem if we show that

$\ker Q = \{0\}$. Suppose then that $h \in M_0$ and that $Qh = 0$. Then $PT h = 0$, implying that $Th \in TN_0$. Since $h \perp N_0$ and T is invertible, this implies that $h = 0$.

Finally, if $f \in M_0$, $S_1 Q f = S_1 P T f = P S_1 T f = P T S_0 f = Q S_0 f$, so $S_0 = Q^{-1} S_1 Q$ on M_0 .

Since $S_0 \mid M_0$ and $S_1 \mid (TN_0)^\perp$ are similar and the former is a shift, it now follows from the previous two lemmas that they are unitarily equivalent. Also $S_0 \mid N_0$ and $S_1 \mid TN_0$ are similar unitary operators, the similarity map being simply $T \mid N_0$. A well-known theorem of Putnam ([12], p. 12) guarantees their unitary equivalence. It is now evident that S_0 and S_1 are unitarily equivalent, and the proof of Lemma 2.5 is complete.

As a result of Lemma 2.5 we obtain the following factorization which is very similar to a result of Devinatz and Shinbrot [4]. It is important to our development that the factors below both lie in the commutant of U . The Devinatz and Shinbrot result is precisely the special case of the following theorem in which U is the identity operator.

2.6 Theorem. *Suppose A is invertible on $\mathcal{K}$ and $AU = UA$. Then A can be factored in the form*

$$A = WB$$

where W is unitary and B is invertible with $B\mathcal{K} = \mathcal{K}$, and where both W and B commute with U .

Proof. Since $\mathcal{K}$ is invariant under U and $AU = UA$, it follows that $A\mathcal{K}$ is invariant under U .

Let $S_0 = U \mid \mathcal{K}$ and $S_1 = U \mid A\mathcal{K}$. Then S_0 and S_1 are similar isometries, the similarity map being simply the restriction of A to $\mathcal{K}$. By Lemma 2.5, S_0 and S_1 are unitarily equivalent. Let U_0 be a unitary map of $\mathcal{K}$ onto $A\mathcal{K}$ satisfying $S_0 = U_0^* S_1 U_0$.

Now let J_0 and J_1 be the smallest reducing subspaces for U which contain $\mathcal{K}$ and $A\mathcal{K}$ respectively. Since S_0 and S_1 are unitarily equivalent it is clear that their minimal unitary extensions are unitarily equivalent, i.e. that U_0 can be extended to a unitary mapping, again called U_0 , of J_0 onto J_1 . Since A maps any reducing subspace of U onto a reducing subspace, it follows that $J_1 = AJ_0$.

We now let $Q: J_0^\perp \rightarrow J_1^\perp$ be defined by $Qf = PAf$ where P is the orthogonal projection of $\mathcal{K}$ onto $J_1^\perp$. We show that Q is invertible. Suppose first that $f \perp J_1$. Then f can be written in the form $f = Ah + Ak$ where $h \in J_0$, $k \perp J_0$. Thus $Pf = f = PAh + PAk = PAk = Qk$, and we have shown that the range of Q is $J_1^\perp$. Finally, suppose $f \in \ker Q$, i.e. that $PAf = 0$. Then $Af \in J_1 = AJ_0$, implying that $Af = Ah$ for some $h \in J_0$. Since A is invertible and $f \perp J_0$, this implies that $f = 0$. By the closed graph theorem we conclude that Q is invertible.

Since P is the orthogonal projection onto a reducing subspace for U , it follows that $UP = PU$ and hence that $UQ = QU$ on $J_0^\perp$. This simply says, since Q is invertible, that $U \mid J_0^\perp$ and $U \mid J_1^\perp$ are similar. By the Putnam result mentioned

in the proof of Lemma 2.5, $U \mid J_0^\perp$ and $U \mid J_1^\perp$ are unitarily equivalent. Let V_0 be a unitary map from $J_0^\perp$ onto $J_1^\perp$ satisfying $UV_0 = V_0U$ on $J_0^\perp$.

Now define W on $\mathcal{K}$ by $W \mid J_0 = U_0$, $W \mid J_0^\perp = V_0$. Then W is unitary and clearly W commutes with U . Finally we set $B = W^*A$. Then B is invertible since A is invertible, B commutes with U , and $B\mathcal{I} = W^*A\mathcal{I} = U_0^*A\mathcal{I} = \mathcal{I}$. This completes the proof.

The following corollary is a form of the inner-outer factorization originated by Beurling [1]. References to such factorizations are too numerous to give here. The formulation given below, except for the restriction that A be invertible, is sufficiently general to include all previous inner-outer factorizations for bounded functions or bounded operators.

2.7 Corollary. *Suppose A is invertible on $\mathcal{K}$, that $UA = AU$, and that $\mathcal{I}$ is invariant under A .*

Then A can be factored in the form

$$A = WB$$

where W is unitary and $W\mathcal{I} \subseteq \mathcal{I}$, B is invertible and $B\mathcal{I} = \mathcal{I}$, and both W and B commute with U .

Proof. Let $A = WB$ as in Theorem 2.6. We need only show that $W\mathcal{I} \subseteq \mathcal{I}$. But this is immediate since $W = AB^{-1}$.

The following simple result may be found in numerous places, for example in Halmos ([5], p. 220).

2.8 Lemma. *Suppose A is an invertible operator on $\mathcal{K}$ which maps $\mathcal{I}$ into $\mathcal{I}$. Then the restriction of A to $\mathcal{I}$ is an invertible operator on $\mathcal{I}$ if and only if A^{-1} maps $\mathcal{I}$ into $\mathcal{I}$.*

We now come to the main result of this section. The Toeplitz operator version of the theorem is due to Pousson, Rabindranathan, and Devinatz, and the case that U is the identity is due to Devinatz and Shinbrot.

2.9 Theorem. *Let A be an invertible operator on $\mathcal{K}$ which commutes with U . Then the projection of A onto $\mathcal{I}$ is invertible if and only if A can be factored in the form*

$$A = B_1^*WB_2$$

where (i) B_1, B_2 , and W commute with U ,
(ii) W is unitary and B_1 and B_2 are invertible and map $\mathcal{I}$ onto $\mathcal{I}$,
and (iii) *there exists an invertible operator B_3 in the commutant of U which maps $\mathcal{I}$ onto $\mathcal{I}$ and satisfies $\|W - B_3\| < 1$.*

Proof. Given the factorization provided by Theorem 2.6, the proof is now the same as that used by Rabindranathan. If $P_{\mathcal{I}}A \mid \mathcal{I}$ is invertible, then, when we write $A = WB$ as in 2.6, $P_{\mathcal{I}}W \mid \mathcal{I}$ must be invertible, and the remainder follows from Theorem 2.2.

Conversely, if $A = B_1^* W B_2$ where (i), (ii), and (iii) hold, then $B_1 \mid \mathfrak{H}$ and $B_2 \mid \mathfrak{H}$ are invertible operators on $\mathfrak{H}$. By Theorem 2.2 $P_{\mathfrak{H}} W \mid \mathfrak{H}$ is invertible on $\mathfrak{H}$. A simple matrix multiplication shows that $P_{\mathfrak{H}} A \mid \mathfrak{H}$ is the product of three invertible operators and is hence invertible.

The final result of this section will be stated without proof. The theorem below is a generalization of a result of Devinatz [3] concerning the invertibility of Toeplitz operators. A proof for the theorem is found in Devinatz and Shinbrot [4] in the special case that U is the identity. The same proof works for us if the previous results of this section are used.

2.10 Theorem. *Suppose that A is an invertible operator on $\mathfrak{K}$ and that $UA = AU$.*

Then the projection of A onto $\mathfrak{H}$ is invertible if and only if there exists an invertible operator G in the commutant of U such that $G\mathfrak{H} = \mathfrak{H}$ and such that AG has strongly positive real part, i.e. $\operatorname{Re} AG > \delta I$ for some $\delta > 0$.

3. An extremal problem of Nehari type. In this section we fix a separable complex Hilbert space C , and an operator-valued function W from the unit circle to the algebra $B(C)$ of bounded operators on C . The function W is assumed to be an invertible weight function, i.e. for almost all real t , $W(e^{it})$ is an invertible positive operator, and $\int \|W(e^{it})\| dm(t) < \infty$ where m is normalized Lebesgue measure on $[-\pi, \pi]$. The space $L^2(W)$ will be as described by Devinatz [2]; $L^2(W)$ is the space of equivalence classes of weakly measurable functions f from the unit circle to C which satisfy

$$\int \langle W(e^{it})f(e^{it}), f(e^{it}) \rangle dm(t) < \infty.$$

With the inner product $\langle f, g \rangle_W = \int \langle W(e^{it})f(e^{it}), g(e^{it}) \rangle dm(t)$, $L^2(W)$ is a Hilbert space.

Since W is norm integrable, it follows that every trigonometric polynomial with coefficients in C is in $L^2(W)$. The closure in $L^2(W)$ of the analytic trigonometric polynomials is denoted by $H^2(W)$. If the identity function on the unit circle is denoted by χ , then the multiplication operator $S: f \rightarrow \chi f$ on $H^2(W)$ is clearly isometric. In case W is the constant function $W(e^{it}) \equiv I$, then $L^2(W)$ is simply the space L^2_C used by Helson [6] and others. Finally, a weakly measurable function B from the unit circle to $B(C)$ will be called *integrable* if $\int \|B(e^{it})\| dm(t) < \infty$, and *analytic* if in addition

$$\int \langle B(e^{it})c_1, c_2 \rangle e^{int} dm(t) = 0,$$

whenever $c_1, c_2 \in C$ and n is a positive integer.

Suppose now that φ is a weakly measurable function from the unit circle to $B(C)$ such that $\int \|W(e^{it})\varphi(e^{it})\| dm(t) < \infty$. We can then define a sesquilinear

form b_φ on the analytic trigonometric polynomials in $H^2(W)$ by setting

$$b_\varphi(f, g) = \int \langle W(e^{it})\varphi(e^{it})f(e^{it}), e^{-it}g(e^{-it}) \rangle dm(t).$$

If the form b_φ is bounded, then the operator H_φ on $H^2(W)$ given by $\langle H_\varphi f, g \rangle_W = b_\varphi(f, g)$ is “ S -Hankel,” i.e. $S^*H_\varphi = H_\varphi S$.

In the case that C is one dimensional and $W(e^{it}) \equiv 1$, the theorem below reduces to the well-known Nehari theorem concerning bounded Hankel matrices [9]. In case $\dim C > 1$ and $W(e^{it}) \equiv I$ the theorem reduces to the generalized Nehari theorem previously obtained by this author [10]. The techniques used in the proof below have been used in a previous joint work (Moore and Page [8], Theorem 3.2).

3.1 Theorem. *Suppose φ is a weakly measurable $B(C)$ -valued function satisfying $\int \|W(e^{it})\varphi(e^{it})\| dm(t) < \infty$. Then $\|H_\varphi\|$ is the solution of the extremal problem*

$$\min_B \|W^{-1/2}(e^{-it})[W(e^{it})\varphi(e^{it}) - B(e^{it})]W^{-1/2}(e^{it})\|_\infty$$

where $\|\cdot\|_\infty$ denotes the ess-sup norm, and where the minimum is taken over all integrable analytic $B(C)$ -valued functions. (The form b_φ above is not bounded in case the minimum is infinite.)

Every bounded S -Hankel operator H on $H^2(W)$ has a representation

$$(*) \quad \langle Hf, g \rangle_W = \int \langle W^{1/2}(e^{-it})F(e^{it})W^{1/2}(e^{it})f(e^{it}), g(e^{-it}) \rangle dm(t)$$

for $f, g \in H^2(W)$, where F is an essentially bounded $B(C)$ -valued function. Furthermore F can be chosen so that $\|F\|_\infty = \|H\|$. (Any F satisfying $(*)$ must satisfy $\|H\| \leq \|F\|_\infty$.)

Proof. Suppose that H is an arbitrary S -Hankel operator on $H^2(W)$, and let U denote the operator on $L^2(W)$ described by $U: f \rightarrow \chi f$. Since the restriction of U to $H^2(W)$ is S , it is easy to see that U is a unitary power dilation of S and that U^* is a unitary power dilation of S^* . Thus by the lifting theorem of Sz.-Nagy and Foias, Theorem 2.1, there exists an operator J on $L^2(W)$ such that

- (i) $\|J\| = \|H\|$,
- (ii) $U^*J = JU$,
- and (iii) $\langle Jf, g \rangle_W = \langle Hf, g \rangle_W$ whenever $f, g \in H^2(W)$.

It is easily seen that $L^2(W)$ and L^2_C are isomorphic under the unitary mapping $Z: f(e^{it}) \rightarrow W^{1/2}(e^{it})f(e^{it})$ of $L^2(W)$ onto L^2_C . If we now let U_0 be the usual bilateral shift on L^2_C , $U_0: f \rightarrow \chi f$, we obtain $U = Z^*U_0Z$. If we define J_0 on L^2_C by $J_0 = ZJZ^*$, then a simple computation yields $U_0^*J_0 = J_0U_0$.

Now let R be the involution on L^2_C given by $R: f(e^{it}) \rightarrow f(e^{-it})$, and set $K_0 = RJ_0$. Then $U_0K_0 = U_0RJ_0 = RU_0^*J_0 = RJ_0U_0 = K_0U_0$, and we have shown that K_0 is in the commutant of U_0 . The commutant of U_0 is well known; the

operator K_0 must be left multiplication by an essentially bounded weakly measurable $B(C)$ -valued function F , with $\|F\|_\infty = \|H\|$. For any two analytic trigonometric polynomials f and g we then have

$$\begin{aligned}\langle Hf, g \rangle_W &= \langle Jf, g \rangle_W = \langle ZJf, Zg \rangle_{L^2 C} = \langle J_0 Zf, Zg \rangle_{L^2 C} = \langle RJ_0 Zf, RZg \rangle_{L^2 C} \\ &= \langle K_0 Zf, RZg \rangle_{L^2 C} = \int \langle F(e^{it})W^{1/2}(e^{it})f(e^{it}), W^{1/2}(e^{-it})g(e^{-it}) \rangle dm(t).\end{aligned}$$

It is now easy to see that this representation holds for all $f, g \in H^2(W)$.

Now if $H = H_\varphi$ where φ is as in the statement of the theorem, then

$$\langle H_\varphi f, g \rangle_W = \int \langle W(e^{it})\varphi(e^{it})f(e^{it}), e^{-it}g(e^{-it}) \rangle dm(t)$$

for all analytic trigonometric polynomials f and g . In particular, by taking $f(e^{it}) = e^{int}c_1$ and $g(e^{it}) = c_2$ where $c_1, c_2 \in C$ and $n \geq 0$ and by equating the two expressions above we obtain

$$\int \langle [W^{1/2}(e^{-it})F(e^{it})W^{1/2}(e^{it}) - e^{it}W(e^{it})\varphi(e^{it})]c_1, c_2 \rangle e^{int} dm(t) = 0,$$

implying that the function B_0 defined by

$$B_0(e^{it}) = e^{-it}W^{1/2}(e^{-it})F(e^{it})W^{1/2}(e^{it}) - W(e^{it})\varphi(e^{it}),$$

is an integrable analytic $B(C)$ -valued function. Also

$$\|H_\varphi\| = \|e^{-it}F(e^{it})\|_\infty = \|W^{-1/2}(e^{-it})[B_0(e^{it}) + W(e^{it})\varphi(e^{it})]W^{-1/2}(e^{it})\|_\infty.$$

Now suppose that B is an arbitrary integrable analytic $B(C)$ -valued function. Let f and g be analytic trigonometric polynomials in the unit ball of $H^2(W)$, and set $h = W^{1/2}f$, $k = W^{1/2}g$ so that h and k are in the unit ball of L^2_C . Then

$$\begin{aligned}|\langle H_\varphi f, g \rangle_W| &= \left| \int \langle W(e^{it})\varphi(e^{it})f(e^{it}), e^{-it}g(e^{-it}) \rangle dm(t) \right| \\ &= \left| \int \langle [W(e^{it})\varphi(e^{it}) - B(e^{it})]f(e^{it}), e^{-it}g(e^{-it}) \rangle dm(t) \right| \\ &= \left| \int \langle e^{it}[W(e^{it})\varphi(e^{it}) - B(e^{it})]W^{-1/2}(e^{it})h(e^{it}), W^{-1/2}(e^{-it})k(e^{-it}) \rangle dm(t) \right| \\ &= \left| \int \langle \{W^{-1/2}(e^{-it})[W(e^{it})\varphi(e^{it}) - B(e^{it})]W^{-1/2}(e^{it})\}h(e^{it}), k(e^{-it}) \rangle e^{it} dm(t) \right| \\ &\leq \|W^{-1/2}(e^{-it})[W(e^{it})\varphi(e^{it}) - B(e^{it})]W^{-1/2}(e^{it})\|_\infty.\end{aligned}$$

The proof is now complete.

4. An application to Toeplitz operators on $H^2(d\mu)$. Let μ be a finite Borel measure on the unit circle. The spaces $L^2(d\mu)$ and $L^\infty(d\mu)$ are then the familiar spaces of complex valued functions, and $H^2(d\mu)$ is the closure in $L^2(d\mu)$ of the analytic trigonometric polynomials. The intersection of $H^2(d\mu)$ and $L^\infty(d\mu)$ is $H^\infty(d\mu)$.

If $\psi \in L^\infty(d\mu)$, the multiplication operator L_ψ is defined on $L^2(d\mu)$ by

$$L_\psi : f \rightarrow \psi f.$$

If P_+ denotes the orthogonal projection of $L^2(d\mu)$ onto $H^2(d\mu)$ then a Toeplitz operator on $H^2(d\mu)$ is an operator of the form $T_\psi = P_+ L_\psi | H^2(d\mu)$.

4.1 Corollary to Theorem 2.9. *Suppose that ψ and $1/\psi$ are in $L^\infty(d\mu)$. Then T_ψ is invertible if and only if ψ can be factored in the form*

$$\psi = \bar{h}_1 u h_2,$$

where (i) $h_1, h_2, 1/h_1$, and $1/h_2$ are in $H^\infty(d\mu)$,

(ii) $|u(e^{it})| = 1$ a.e. $-\mu$,

and (iii) there exists a function h_3 in $H^\infty(d\mu)$ such that

$$1/h_3 \in H^\infty(d\mu) \text{ and } \|u(e^{it}) - h_3(e^{it})\|_\infty < 1.$$

Proof. In Theorem 2.9 let U be multiplication by χ on $L^2(d\mu)$ and let $A = L_\psi$. Then T_ψ is invertible if and only if $A = B_1^* W B_2$ as in Theorem 2.9 with $\mathcal{H} = H^2(d\mu)$. The commutant of U is the set of all multiplications by $L^\infty(d\mu)$ functions. The factorization of A corresponds to a factorization of the function ψ into a product

$$\psi = \bar{h}_1 u h_2.$$

In order for multiplication by h_1 and h_2 to be invertible and to map $H^2(d\mu)$ onto itself it is necessary and sufficient that $h_1, h_2, 1/h_1$, and $1/h_2$ be in $H^\infty(d\mu)$. In order for multiplication by u to be unitary it is necessary and sufficient that $|u(e^{it})| = 1$ a.e. $-\mu$.

Remark. It is possible to carry out the above discussion in the spaces $L^2(W)$ and $H^2(W)$ of Section 3. In this setting, however, the details would become much more cumbersome and the statement of the analogous corollary somewhat more complicated.

5. Fourier series in $L^2(W)$. The application to Fourier series was suggested by Marvin Rosenblum, and the direction of the remarks below is influenced by his work on the subject [14].

We return to the setting of Section 3 so that W is again an invertible operator valued weight function. The general problem to be considered is the following: Give necessary and sufficient conditions on W in order that the Fourier series of functions in $L^2(W)$ converge.

Specifically we proceed as follows. Let $\mathcal{O}$ be the class of trigonometric polynomials with coefficients in C . For each integer $n \geq 0$, the Dirichlet operator D_n is defined on $\mathcal{O}$ in the following way: If $g(e^{it}) = \sum_{j=-m}^m c_j e^{ijt}$ is in $\mathcal{O}$, then $(D_n g)(e^{it}) = g(e^{it})$ if $m \leq n$ while $(D_n g)(e^{it}) = \sum_{j=-n}^n c_j e^{ijt}$ if $n < m$.

Suppose for the moment that D_0 is bounded on $\mathcal{O}$ in $L^2(W)$ norm. Since $\mathcal{O}$ is dense in $L^2(W)$ (Devinatz, [2]), it follows that D_0 has a unique continuous extension to $L^2(W)$. In this case the n^{th} Fourier coefficient of a function f in $L^2(W)$ is defined to be $D_0(\chi^{-n} f)$. Since for $g \in \mathcal{O}$, $D_n g = \sum_{j=-n}^n D_0(\chi^{-n} g) \chi^n$, it follows that D_0 is bounded if and only if every operator D_n is bounded. In this case the n^{th} partial sum of the Fourier series of a function f in $L^2(W)$ is $D_n f$. The question then becomes, "When does $D_n f$ converge to f in $L^2(W)$ norm for all f in $L^2(W)$?" Since $D_n g \rightarrow g$ as $n \rightarrow \infty$ for all $g \in \mathcal{O}$, it is evident that $D_n \rightarrow I$ strongly on $L^2(W)$ if $\{D_n\}_{n=0}^\infty$ is totally bounded. The converse follows from the uniform boundedness principle.

Finally we let P be the projection (not necessarily orthogonal) defined on $\mathcal{O}$ by

$$P : \sum_{j=-m}^m c_j e^{ijt} \rightarrow \sum_{j=0}^m c_j e^{ijt}.$$

The following lemma was suggested to us by Rosenblum.

5.1. Lemma. *The projection P is bounded if and only if $D_n \rightarrow I$ strongly as $n \rightarrow \infty$.*

Proof. By the above remarks it suffices to show that P is bounded if and only if $\{D_n\}_{n=0}^\infty$ is totally bounded.

An elementary computation shows that $D_n g = \chi^{n+1}(I - P)\chi^{-2n-1}P\chi^n g$, $g \in \mathcal{O}$. Thus $\|D_n\| \leq \|P\| \|I - P\|$, and $\{D_n\}_{n=0}^\infty$ is totally bounded if P is bounded.

Conversely suppose $\{D_n\}_{n=0}^\infty$ is totally bounded. Let $g \in \mathcal{O}$, and write

$$g = \sum_{j=-2m}^{2m} c_j \chi^j.$$

Then an elementary calculation shows that $Pg = \chi^m D_m \chi^{-m} g$. Thus $\|Pg\| = \|\chi^m D_m \chi^{-m} g\| \leq \|D_m\| \|g\|$. So P is bounded, and the proof is complete.

Recalling that $\mathcal{O}$ is the class of analytic trigonometric polynomials with coefficients in C , we now let $\bar{\mathcal{O}}_0$ be the class of conjugate analytic trigonometric polynomials with coefficients in C and with mean-value zero. It is not difficult to show that if M and N are linear manifolds in a Hilbert space with $M \cap N = \{0\}$, then the unique idempotent operator on $M + N$ with range M and kernel N is bounded if and only if

$$\sup \{ |\langle f, g \rangle| : f \in M, g \in N, \|f\| = \|g\| = 1 \} < 1.$$

5.2 Corollary to Theorem 3.1. *The Fourier series of every function f in $L^2(W)$*

converges to f in $L^2(W)$ norm if and only if

$$\min_B \|W^{-1/2}(e^{-it})[W(e^{it}) - B(e^{it})]W^{-1/2}(e^{it})\|_\infty < 1$$

where the minimum is taken over the set of integrable analytic $B(C)$ -valued functions.

Proof. In Theorem 3.1 take $\varphi(e^{it}) \equiv 1$. Then

$$\langle H_\varphi f, g \rangle_W = \int \langle W(e^{it})f(e^{it}), e^{-it}g(e^{-it}) \rangle dm(t),$$

and

$$\|H_\varphi\| = \sup \{ |\langle h, k \rangle_W| : h \in \mathcal{O}, k \in \bar{\mathcal{O}}_0, \|h\|_W = \|k\|_W = 1 \}.$$

The result then follows from Theorem 3.1 and the remarks above.

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