Characteristic Planes and Pointwise Solutions of the Heat Equation

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1. Introduction. We shall operate in N -dimensional Euclidean space, E_N , $N \geq 2$, and use the notation $x = (x_1, \dots, x_N)$ and $B(x, r)$ = the open N -ball with center x and radius r . We say the function $u(x)$ continuous in a neighborhood of the point x^0 is in $T_k^1(x^0)$, k a positive integer, provided there is a polynomial $P(x)$ of degree $\leq k - 1$ such that

$$(1.1) \quad r^{-N} \int_{B(0, r)} |u(x^0 + x) - P(x)| \, dx = O(r^k) \quad \text{as } r \rightarrow 0.$$

We say that $u(x)$ is in $t_k^1(x^0)$ provided there is a polynomial $P(x)$ of degree $\leq k$ such that

$$r^{-N} \int_{B(0, r)} |u(x^0 + x) - P(x)| \, dx = o(r^k) \quad \text{as } r \rightarrow 0.$$

If $u(x)$ is in $t_2^1(x^0)$ and

$$P(x) = \beta_0 + \sum_{i=1}^N \beta_i x_i + 2^{-1} \sum_{i=1}^N \sum_{k=1}^N \beta_{ik} x_i x_k,$$

with $\beta_{ik} = \beta_{ki}$ for $j, k = 1, \dots, N$, we shall set $D_i u(x^0) = \beta_i$ and $D_{ik} u(x^0) = \beta_{ik}$. It is clear that $D_i u(x^0)$, $D_{ik} u(x^0)$ represent respectively generalizations of the usual $\partial u(x^0)/\partial x_i$ and $\partial^2 u(x^0)/\partial x_i \partial x_k$.

(The classes $t_k^1(x^0)$ and $T_k^1(x^0)$ are special cases of the classes $t_\alpha^p(x^0)$ and $T_\alpha^p(x^0)$ defined in [1].)

The function $u(x)$ continuous in a neighborhood of the point x^0 will be said to be a pointwise solution of the heat equation at x^0 provided $u(x)$ is in $t_2^1(x^0)$ and

$$(1.2) \quad D_N u(x^0) = \sum_{i=1}^{N-1} D_{ii} u(x^0).$$

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Next, with $\alpha > 0$, we designate by $C[x^*, \alpha]$ the open N -cube

$$C[x^*, \alpha] = \{x: |x_i - x_i^*| < \alpha, \quad j = 1, \dots, N\},$$

and we let Q denote a Borel set of Lebesgue measure zero contained in the open interval $(x_N^* - \alpha, x_N^* + \alpha)$. We observe that to each q in Q , there corresponds a characteristic plane of the heat equation, namely, $\{x: x_N = q\}$. By Q^s , we designate the set theoretic union of all these planes. To be specific, $Q^s = \{x: x_N = q, q \text{ in } Q\}$.

Next, we introduce four conditions which will be used in defining the concept of a removable set for pointwise solutions of the heat equation in $C[x^*, \alpha]$; namely,

- (i) u is continuous in $C[x^*, \alpha]$;
- (ii) u is in $T_2^1(x)$ for x in $C[x^*, \alpha] - Q^s$;
- (iii) u is in $t_2^1(x)$ almost everywhere in $C[x^*, \alpha]$;
- (iv) u is a pointwise solution of the heat equation almost everywhere in $C[x^*, \alpha]$.

We say u is a classical solution of the heat equation in $C[x^*, \alpha]$ if u is in $C^2(C[x^*, \alpha])$ and if $\partial u(x)/\partial x_N = \sum_{i=1}^{N-1} \partial^2 u(x)/\partial x_i^2$ for x in $C[x^*, \alpha]$.

$Q^s \cap C[x^*, \alpha]$ will be called a removable set for pointwise solutions of the heat equation in $C[x^*, \alpha]$ provided the following holds:

If u meets conditions (i), (ii), (iii), and (iv) above, then u is a classical solution of the heat equation in $C[x^*, \alpha]$. We intend to establish the following result:

Theorem. *A necessary and sufficient condition that $Q^s \cap C[x^*, \alpha]$ be a removable set for pointwise solutions of the heat equation in $C[x^*, \alpha]$ is that Q be countable.*

The necessary condition for the above theorem is easy, and we establish it forthwith.

Suppose Q is an uncountable Borel set of Lebesgue measure zero contained in the open interval $(x_N^* - \alpha, x_N^* + \alpha)$. Then Q contains a non-empty perfect subset Q_1 , [5, p. 205]. We set $\alpha_0 = \inf \{q: q \text{ in } Q_1\}$ and $\alpha_1 = \sup \{q: q \text{ in } Q_1\}$ and construct on the interval $\alpha_0 \leq \alpha \leq \alpha_1$ a Lebesgue-Cantor function, $f(t)$, with respect to the set Q_1 , [7, p. 101]. Then $f(t)$ has the following properties: $f(t)$ is continuous and non-decreasing on the closed interval $[\alpha_0, \alpha_1]$; $f(t)$ is constant on the open contiguous intervals of Q_1 ; $f(\alpha_0) = 0$ and $f(\alpha_1) = 1$. We extend $f(t)$ to the rest of interval $(x_N^* - \alpha, x_N^* + \alpha)$ by setting $f(t) = 0$ for $x_N^* - \alpha < t \leq \alpha_0$ and $f(t) = 1$ for $\alpha_1 \leq t < x_N^* + \alpha$. Before proceeding, we observe in particular that

$$(1.3) \quad f(t) \text{ is not in class } C^1 \text{ on the interval } (x_N^* - \alpha, x_N^* + \alpha).$$

Next, we define $u(x) = f(x_N)$ for x in $C[x^*, \alpha]$ and observe that u meets condition (i). We set $Q_1^s = \{x: x_N = q, q \text{ in } Q_1\}$ and observe that $Q_1^s \subset Q^s$ and that both Q_1^s and Q^s are of n -dimensional Lebesgue measure zero. If x is in $C[x^*, \alpha] - Q^s$, then x is in $C[x^*, \alpha] - Q_1^s$ and consequently x_N is in $(x_N^* - \alpha, x_N^* + \alpha) - Q_1$. But then u is constant in a neighborhood of x , and we conclude that u also meets conditions (ii), (iii), and (iv) above. If $Q^s \cap C[x^*, \alpha]$ were a removable set, we would then have in particular that $u(x)$ would be in $C^1(C[x^*, \alpha])$. But this would imply that $u(x_1^*, \dots, x_{N-1}^*, t) = f(t)$ would be in class C^1 on the interval $(x_N^* - \alpha, x_N^* + \alpha)$. This is a contradiction to (1.3), and the necessary condition of the theorem is established.

2. Fundamental lemmas. In order to establish the sufficiency condition of the above theorem, we shall need some lemmas from the theory of multiple trigonometric series.

We shall use the following notation: $T_N = \{x: -\pi < x_j \leq \pi, j = 1, \dots, N\}$; m will designate an integral lattice point; for a function f in $L^1(T_N)$, we shall set

$$\hat{f}(m) = (2\pi)^{-N} \int_{T_N} f(x) e^{-i(m, x)} dx,$$

and call $\sum \hat{f}(m) e^{i(m, x)}$ the Fourier series of f .

Given f in $L^1(T_N)$, we shall say f is extended by periodicity to all of E_N if f is defined in all of E_N and is periodic of period 2π in each variable.

With $f_{x_i}(x, t) = \partial^1 f(x, t) / \partial x_i$ and $f_{x_i x_k}(x, t)$ designating in a similar manner the second partial derivative of $f(x, t)$ with respect to x_i and x_k , the first lemma we prove is the following:

Lemma 1. *Let f be in $L^1(T_N)$ and extended by periodicity to all of E_N . For $t > 0$, set $f(x, t) = \sum \hat{f}(m) e^{i(m, x) - |m|^2 t}$. Suppose f is in $T_2^1(x^0)$. Then for $j = 1, \dots, N$ and $k = 1, \dots, N$, $\limsup_{t \rightarrow 0} |f_{x_j x_k}(x^0, t)| < \infty$.*

To establish the lemma, with no loss in generality we can assume that $x^0 = 0$. Also, to establish the lemma it is clear that we can take the linear polynomial arising in the definition of $T_2^1(0)$ [see (1.1)] to be identically zero. Consequently with j and k fixed, the lemma will be established if we show that

$$(2.1) \quad \int_{B(0, r)} |f(x)| dx \leq M r^{N+2} \quad \text{for } 0 < r < r_0$$

implies that

$$(2.2) \quad \limsup_{t \rightarrow 0} |f_{x_j x_k}(0, t)| \leq \gamma_N M,$$

where γ_N is a constant that depends only on N .

Now using [8; pp. 56–57], we see that

$$(2.3) \quad f_{x_j}(x, t) = -(N+1)b_N t \int_{E_N} f(y) [t^2 + |x - y|^2]^{-(N+3)/2} (x_j - y_j) dy,$$

where b_N is a constant depending only on N . Consequently it follows from (2.3) and another computation that

$$(2.4) \quad |f_{x_j x_k}(0, t)| \leq b'_N t \int_0^{r_0} [t^2 + r^2]^{-(N+3)/2} dg(r) + o(1),$$

where $g(r)$ is equal to the expression on the left side of the inequality in (2.1) and b'_N is a constant depending only on N . Integrating by parts in (2.4) and using (2.1), it follows that

$$(2.5) \quad \limsup_{t \rightarrow 0} |f_{x_j x_k}(0, t)| \leq M(N+3)b'_N \int_0^\infty r^{N+3} [1 + r^2]^{-(N+5)/2} dr.$$

We see that (2.5) implies (2.2) and the proof of the lemma is complete.

A close look at the proof just given shows that we have established the second part of the next lemma; a similar procedure establishes the first part (where for u in $t_1^1(x^0)$, $D_i(x^0)$ is taken to be the coefficient of x_i in $P(x)$, $j = 1, \dots, n$).

Lemma 2. *Let f be in $L^1(T_N)$ and extended by periodicity to all of E_N . For $t > 0$, set $f(x, t) = \sum_m \hat{f}(m) e^{i(m, x) - |m|t}$.*

(i) *If f is in $t_1^1(x^0)$, then for $j = 1, \dots, N$,*

$$(2.6) \quad \lim_{t \rightarrow 0} f_{x_j}(x^0, t) = D_j f(x^0).$$

(ii) *If f is in $t_2^1(x^0)$, then for $j, k = 1, \dots, N$,*

$$(2.7) \quad \lim_{t \rightarrow 0} f_{x_j x_k}(x^0, t) = D_{jk} f(x^0).$$

The next lemma we establish is the following:

Lemma 3. *Let $f(x)$ be a continuous function in E_N of period 2π in each variable. For $t > 0$, set $f(x, t) = \sum_m \hat{f}(m) e^{i(m, x) - |m|t}$ and suppose that*

$$(2.8) \quad \lim_{t \rightarrow 0} \left[\sum_{i=1}^{n-1} \partial^2 f(x, t) / \partial x_i^2 - \partial f(x, t) / \partial x_N \right] = 0$$

uniformly for x in $B(x^0, r_0)$, $r_0 > 0$. Then $f(x)$ is in class $C^2[B(x^0, r_0)]$ and is a classical solution of the heat equation in $B(x^0, r_0)$.

To prove the lemma, we observe first that with no loss in generality we can take $x^0 = 0$ and $0 < r_0 < 1$. Next, we take $0 < r_1 < r_0 < 1$ and observe that the lemma will be established if we can show

$$(2.9) \quad f(x) \text{ is in class } C^2[B(0, r_1)] \text{ and is a classical solution of the heat equation in } B(0, r_1).$$

In order to establish (2.9), we define $f_h(x)$, for $h > 0$, as follows:

$$f_h(x) = |B(0, h)|^{-1} \int_{B(0, h)} f(x + y) dy,$$

where $|B(0, h)|$ is the N -volume of $B(0, h)$. Also, we set $f_{hh}(x) = [f_h]_h(x)$ and observe that $f_{hh}(x)$ is in class $C^2(E_N)$ and periodic of period 2π in each variable. Since $\lim_{h \rightarrow 0} f_{hh}(x) = f(x)$ uniformly for x in $B(0, r_1)$, it follows from well-known facts concerning classical solutions of the heat equation that (2.9) will be established if we can show

$$(2.10) \quad f_{hh}(x) \text{ is a classical solution of the heat equation in } B(0, r_1) \text{ for } 0 < h < (r_0 - r_1)/2.$$

To establish (2.10), we first use Lemma 2 and obtain that for each $h > 0$, for x in E_N and $j = 1, \dots, N - 1$,

$$(2.11) \quad \lim_{t \rightarrow 0} \partial f_{hh}(x, t) / \partial x_N = \partial f_{hh}(x) / \partial x_N, \quad \text{and} \\ \lim_{t \rightarrow 0} \partial^2 f_{hh}(x, t) / \partial x_i^2 = \partial^2 f_{hh}(x) / \partial x_i^2,$$

where $f_{hh}(x, t) = \sum_m \hat{f}_{hh}(m) e^{i(m, x) - |m|t}$.

Observing that $f_{hh}(x, t) = |B(0, h)|^{-2} \int_{B(0, h)} \int_{B(0, h)} f(x + y + z, t) dy dz$, we infer from (2.8) in the hypothesis of the lemma that for each h such that $0 < h < (r_0 - r_1)/2$ the following fact obtains:

$$(2.12) \quad \lim_{t \rightarrow 0} \left[\sum_{i=1}^{N-1} \partial^2 f_{hh}(x, t) / \partial x_i^2 - \partial f_{hh}(x, t) / \partial x_N \right] = 0 \quad \text{uniformly}$$

for x in $B(0, r_1)$.

From (2.11) and (2.12), we obtain that $\sum_{i=1}^{N-1} \partial^2 f_{hh}(x) / \partial x_i^2 = \partial f_{hh}(x) / \partial x_N$ for x in $B(0, r_1)$ and $0 < h < (r_0 - r_1)/2$. Consequently (2.10) is established and the proof of the lemma is complete.

Using $\sum'$ to designate a summation over the integral lattice points in E_N with the 0-lattice point excluded, the next lemma that we deal with is the following:

Lemma 4. *Let $f(x)$ be a continuous function in E_N of period 2π in each variable. Let $\tilde{f}(x)$ be a function in $L^2(T_N)$ extended by periodicity to all of E_N having*

$$-\sum' \hat{f}(m) \left[\sum_{i=1}^{N-1} m_i^2 - (N-1)m_N^2 \right] |m|^{-2} e^{i(m, x)}$$

as its Fourier series. Then $\lim_{h \rightarrow 0} [\sup_{x \text{ in } T_N \text{ and } |p| \leq h} |\tilde{f}_h(x + p) - \tilde{f}_h(x)|] = 0$.

To establish the above lemma, we set for $x \neq 0$

$$(2.13) \quad K(x) = \pi^{N/2} 2^N \Gamma[(N+2)/2] \left(\sum_{i=1}^{N-1} x_i^2 - (N-1)x_N^2 \right) / |x|^{N+2},$$

and for $x/2\pi$ not equal to an integral lattice point

$$(2.14) \quad K^*(x) = K(x) + \lim_{R \rightarrow \infty} \sum_{1 \leq |m| \leq R} [K(x + 2\pi m) - K(2\pi m)],$$

where the series in (2.14) converges uniformly for x in T_N .

With no loss in generality, we can assume $(2\pi)^{-N} \int_{T_N} f(x) dx = 0$, and for $0 < \epsilon < 1$ we set for x in E_N

$$(2.15) \quad \epsilon \tilde{f}(x) = (2\pi)^{-N} \int_{T_N - B(0, \epsilon)} f(x - p) K^*(p) dp.$$

As is well-known, [2, pp. 253-254], $\lim_{\epsilon \rightarrow 0} \int_{T_N} |\epsilon \tilde{f}(x) - \tilde{f}(x)|^2 dx = 0$. Consequently, for $h > 0$, $\lim_{\epsilon \rightarrow 0} [\epsilon \tilde{f}]_h(x) = \tilde{f}_h(x)$ for every x . Observing from (2.13) and (2.14) that $\lim_{\epsilon \rightarrow 0} \int_{T_N - B(0, \epsilon)} K^*(p) dp = 0$, we consequently conclude from (2.15) that for $h > 0$

$$(2.16) \quad \tilde{f}_h(x) = \lim_{\epsilon \rightarrow 0} (2\pi)^{-N} \int_{T_N - B(0, \epsilon)} [f_h(x - p) - f_h(x)] K^*(p) dp$$

for x in E_N . Since $f(x)$ is a continuous function in E_N of period 2π in each variable, it follows that for $h > 0$ there is constant c_h such that $|f_h(x - p) - f_h(x)| \leq c_h |p|$ for x in E_N and p in T_N . But then we conclude from (2.13), (2.14), and (2.16) that

$$(2.17) \quad \tilde{f}_h(x) = (2\pi)^{-N} \int_{T_N} [f_h(x - p) - f_h(x)] K^*(p) dp.$$

The formula in (2.17) is the same as the formula in [10; (19)] and the remainder of the proof of our current Lemma 4 is essentially the same as that of [10, Lemma 2] from (19) forward. We leave the details to the reader.

Next, we establish the following lemma which is to be compared with [13, Theorem 7.20, p. 103].

Lemma 5. *Let f and $\tilde{f}$ be as in Lemma 4. For $t > 0$, set $\tilde{f}(x, t) = \sum_m \hat{f}(m) e^{i(m, x) - |m|t}$ and*

$$(2.18) \quad V(x, t) = \tilde{f}(x, t) - \lim_{R \rightarrow \infty} (2\pi)^{-N} \int_{B(0, R) - B(0, t)} f(x - y) K(y) dy,$$

where $K(y)$ is defined in (2.13). Then $\lim_{t \rightarrow 0} V(x, t) = 0$ uniformly for x in T_N .

With no loss in generality we can suppose that $\tilde{f}(0) = 0$. With $\beta = (N - 2)/2$, it follows from [9, p. 67 and p. 69] that

$$(2.19) \quad \tilde{f}(x, t) = \lim_{R \rightarrow \infty} (2\pi)^{-N} \int_{B(0, R)} f(x - y) A_2^\beta(|y| t^{-1}) K(y) dy,$$

where $A_2^\beta(r)$ is defined on [9, p. 64] to be the following expression: $A_2^\beta(r) = \int_0^\infty e^{-s/r} J_{\beta+2}(s) s^{\beta+1} ds / 2^{N/2} \Gamma[(N + 2)/2]$. Now from [9, p. 64, (i)], $|A_2^\beta(r)| \leq \Gamma(N) r^N$ for $r > 0$. Consequently

$$(2.20) \quad \left| \int_{B(0, t)} f(x - y) A_2^\beta(|y| t^{-1}) K(y) dy \right| \\ = \left| \int_{B(0, t)} [f(x - y) - f(x)] A_2^\beta(|y| t^{-1}) K(y) dy \right| \\ \leq \Gamma(N) \sup_{|y| \leq t} |f(x - y) - f(x)| t^{-N} \int_{B(0, t)} |y|^N |K(y)| dy.$$

Since the last expression in (2.20) goes to zero uniformly for x in T_N , we conclude from (2.18) and (2.19) that the conclusion of the lemma will follow provided that we show

$$(2.21) \quad \lim_{t \rightarrow 0} V'(x, t) = 0 \text{ uniformly for } x \text{ in } T_N,$$

where

$$(2.22) \quad V'(x, t) = \lim_{R \rightarrow \infty} \int_{B(0, R) - B(0, t)} [f(x - y) - f(x)] [A_2^\beta(|y| t^{-1}) - 1] K(y) dy.$$

Now from [9, p. 55 and p. 64], we have that there is a positive constant γ depending on N such that

$$(2.23) \quad |A_2^\beta(r) - 1| \leq \gamma r^{-1/2} \quad \text{for } r > 0.$$

Given $\epsilon > 0$, we choose $\delta > 0$ such that $|f(x - y) - f(x)| \leq \epsilon/\gamma$ for $|y| < \delta$ and x in T_N . Then letting $\|f\|_\infty$ designate the L^∞ -norm of f on T_N , we have from (2.22) and (2.23) that

$$(2.24) \quad \sup_{x \text{ in } T_N} |V'(x, t)| \leq 2 \|f\|_\infty \gamma t^{1/2} \int_{B_N - B(0, \delta)} |y|^{-1/2} |K(y)| dy \\ + \epsilon t^{1/2} \int_{B(0, \delta) - B(0, t)} |y|^{-1/2} |K(y)| dy.$$

Since the first integral on the right-hand side of the inequality in (2.24) is finite, it follows that there is a t_0 such that

$$(2.25) \quad 2 \|f\|_\infty \gamma t^{1/2} \int_{B_N - B(0, \delta)} |y|^{-1/2} |K(y)| dy < \epsilon \quad \text{for } 0 < t < t_0.$$

On the other hand, it follows from (2.13) that

$$(2.26) \quad \int_{B(0, \delta) - B(0, t)} |y|^{-1/2} |K(y)| dy \\ \leq \pi^{N/2} 2^{N+1} N \Gamma[(N+2)/2] |S(0, 1)| \int_t^\infty r^{-3/2} dr,$$

where $|S(0, 1)|$ represent the $(N-1)$ -volume of the unit $(N-1)$ -sphere.

We conclude from (2.24), (2.25), and (2.26) that for

$$0 < t < t_0, \quad \sup_{x \text{ in } T_N} |V'(x, t)| \leq \epsilon [1 + \pi^{N/2} 2^{N+2} N \Gamma[(N+2)/2] |S(0, 1)|].$$

But this last fact establishes (2.21), and the proof of the lemma is therefore complete.

Next, we establish the following lemma:

Lemma 6. *Let f and $\tilde{f}$ be as in Lemma 4. Suppose that there are positive constants M and t_0 such that for x in Z and $j = 1, \dots, N$*

$$(2.27) \quad \left| \sum_m \hat{f}(m) m_j^2 e^{i(m, x) - |m|t} \right| \leq M \quad \text{for } 0 < t \leq t_0.$$

Then uniformly for x in Z , $\lim_{h \rightarrow 0} \tilde{f}_h(x)$ exists and is finite.

Let $K(x)$ be as in (2.13) and let $0 < h < h_0$. Then we first establish the following equality:

$$(2.28) \quad (2\pi)^{-N} \int_{B(0, h_0) - B(0, h)} f(x - p) K(p) dp = -[\tilde{f}_{h_0}(x) - \tilde{f}_h(x)].$$

To establish (2.28), we need only show as in [10, Lemma 3] that the m -th Fourier coefficients of the functions on the right- and left-hand sides of the

equality in (2.28) are the same, for both functions are continuous in E_N and periodic of period 2π in each variable.

The 0-th coefficient of both sides is clearly zero. For $m \neq 0$, an easy computation using [12, p. 368] with $G(x) = \tilde{f}_h(x) - \tilde{f}_{h_0}(x)$ shows that

$$(2.29) \quad \hat{G}(m)/[N2^{(N-2)/2}\Gamma(N/2)] \\ = |m|^{-2} \left[\sum_{j=1}^{n-1} m_j^2 - (N-1)m_N^2 \right] \hat{f}(m) J_{N/2}(r) r^{-N/2} \Big|_{|m|h}^{|m|h_0}.$$

On the other hand using [12, p. 368] once again, we see that for $m \neq 0$,

$$\int_{B(0, h_0) - B(0, h)} e^{-i(m, p)} \left[\sum_{j=1}^{N-1} p_j^2 - (N-1)p_N^2 \right] |p|^{-(N+2)} dp \\ = (-1)(2\pi)^{N/2} \int_h^{h_0} J_{1+N/2}(r) |m| r (r|m|)^{-(1+N/2)} dr \cdot \left[\sum_{j=1}^{N-1} m_j^2 - (N-1)m_N^2 \right].$$

This fact, in conjunction with the well-known formula

$$(-1) \int_{|m|h}^{|m|h_0} J_{1+N/2}(r) r^{-N/2} dr = J_{N/2}(r) r^{-N/2} \Big|_{|m|h}^{|m|h_0},$$

enables us to conclude that the m -th Fourier coefficient of the function on the left-hand side of the equality in (2.28) is the same as the value of $\hat{G}(m)$ given in (2.29). The formula in (2.28) is therefore established.

Next, we observe from (2.27) and the Cauchy criterion that uniformly for x in Z ,

$$\lim_{t \rightarrow 0} \sum'_m \hat{f}(m) m_j^2 |m|^{-1} e^{i(m, x) - |m|t}$$

exists and is finite for $j = 1, \dots, N$. Applying this technique once again and setting $\tilde{f}(x, t) = \sum_m \hat{f}(m) e^{i(m, x) - |m|t}$ for $t > 0$, we see that

$$(2.30) \quad \text{uniformly for } x \text{ in } Z, \lim_{t \rightarrow 0} \tilde{f}(x, t) \text{ exists and is finite.}$$

From Lemma 5, we have that

$$(2.31) \quad \text{uniformly for } x \text{ in } Z,$$

$$\lim_{h \rightarrow 0} \left\{ \tilde{f}(x, h) - \lim_{R \rightarrow \infty} (2\pi)^{-N} \int_{B(0, R) - B(0, h)} f(x - p) K(p) dp \right\} = 0.$$

On the other hand, the periodicity and continuity of f and [9, p. 45] implies that

$$(2.32) \quad \text{uniformly for } x \text{ in } Z,$$

$$\lim_{R \rightarrow \infty} \int_{B(0, R) - B(0, h_0)} f(x - p) K(p) dp \text{ exists and is finite.}$$

The conclusion to the lemma follows immediately from (2.28), (2.30), (2.31), and (2.32).

Next, we establish the following lemma:

Lemma 7. *Let $f(x)$ be a continuous function in E_N of period 2π in each variable, and let $C[x^*, \alpha]$, $\alpha > 0$, be a given open N -cube. For $t > 0$, set $f(x, t) = \sum_m \hat{f}(m) e^{i(m, x) - |m|t}$. Suppose that the following facts hold:*

$$(2.33) \quad \limsup_{t \rightarrow 0} |\partial f(x, t)/\partial x_N| < \infty \quad \text{for } x \text{ in } C[x^*, \alpha];$$

$$(2.34) \quad \limsup_{t \rightarrow 0} |\partial^2 f(x, t)/\partial x_i^2| < \infty \quad \text{for } x \text{ in } C[x^*, \alpha] \text{ and } j = 1, \dots, N;$$

$$(2.35) \quad \lim_{t \rightarrow 0} \left[\sum_{i=1}^{N-1} \partial^2 f(x, t)/\partial x_i^2 - \partial f(x, t)/\partial x_N \right] = 0$$

for almost every x in $C[x^*, \alpha]$.

Then $f(x)$ is in class $C^2(C[x^*, \alpha])$ and is a classical solution of the heat equation in $C[x^*, \alpha]$.

With no loss in generality, we can assume that $x^* = 0$, that $0 < \alpha < 1$, and that

$$(2.36) \quad \hat{f}(0) = 0.$$

To establish the lemma, we first observe that

$$-\sum' \hat{f}(m) \left[\sum_{i=1}^{N-1} m_i^2 + i m_N \right] |m|^{-2} e^{i(m, x)}$$

is the Fourier series of a function in $L^2(T_N)$. Next, for $t > 0$, we define $g(x, t)$ by

$$(2.37) \quad g(x, t) = -\sum_m \hat{f}(m) \left[\sum_{i=1}^{N-1} m_i^2 + i m_N \right] |m|^{-2} e^{i(m, x)} e^{-|m|t},$$

and set

$$(2.38) \quad g(x) = \limsup_{t \rightarrow 0} g(x, t) \quad \text{for } x \text{ in } E_N.$$

We conclude from [8, pp. 55–56] that $g(x)$ is finite almost everywhere and in particular that

$$(2.39) \quad g(x) \text{ is in } L^2(T_N), \quad \hat{g}(0) = 0, \quad \text{and}$$

$$\hat{g}(m) = -\hat{f}(m) \left[\sum_{i=1}^{N-1} m_i^2 + i m_N \right] |m|^{-2} \quad \text{for } m \neq 0.$$

Also, we observe from (2.37) that for $t > 0$,

$$(2.40) \quad \partial^2 g(x, t)/\partial t^2 = \sum_{i=1}^{N-1} \partial^2 f(x, t)/\partial x_i^2 - \partial f(x, t)/\partial x_N.$$

Consequently, as in Lemma 6, we conclude from the Cauchy criterion, (2.33), and (2.34) that

$$(2.41) \quad \begin{aligned} &g(x) \text{ is finite for } x \text{ in } C[0, \alpha] \text{ and} \\ &\lim_{t \rightarrow 0} g(x, t) = g(x) \text{ for } x \text{ in } C[0, \alpha]. \end{aligned}$$

If we can establish the following fact:

$$(2.42) \quad g(x) \text{ is harmonic in } C[0, \alpha],$$

the proof to our current lemma will be complete. For it follows from (2.42) and [11, Lemma 5, p. 609] that $\lim_{t \rightarrow 0} \partial^2 g(x, t) / \partial t^2 = 0$ uniformly on compact subsets of $C[0, \alpha]$. But this fact in conjunction with (2.40) in turn implies that

$$\lim_{t \rightarrow 0} \left[\sum_{i=1}^{N-1} \partial^2 f(x, t) / \partial x_i^2 - \partial f(x, t) / \partial x_N \right] = 0$$

uniformly on compact subsets of $C[0, \alpha]$. The conclusion to our current lemma will then follow immediately from Lemma 3. Therefore the proof of our current lemma will be complete once we establish (2.42).

(In the above paragraph, the lemma given in the reference [11, Lemma 5, p. 609] was established only for $N = 2$. It is clear that a similar fact prevails for $N \geq 3$ with the proof being essentially the same.)

To establish (2.42), we first observe from (2.33), (2.34), (2.35), and (2.40) that

$$(2.43) \quad \limsup_{t \rightarrow 0} |\partial^2 g(x, t) / \partial t^2| < \infty \quad \text{for } x \text{ in } C[0, \alpha],$$

and

$$(2.44) \quad \lim_{t \rightarrow 0} \partial^2 g(x, t) / \partial t^2 = 0 \quad \text{almost everywhere in } C[0, \alpha].$$

Next, we define the upper generalized Laplacian of g at the point x , denoted by $\Delta^* g(x)$, as follows: $\Delta^* g(x) = 2(N + 2) \limsup_{h \rightarrow 0} [g_h(x) - g(x)] / h^2$. In a similar manner, using $\liminf$ we define the lower generalized Laplacian of g and denote it by $\Delta_* g(x)$. It follows from (2.39), (2.41), and [8, Lemma 7, p. 66] that

$$(2.45) \quad \begin{aligned} &\liminf_{t \rightarrow 0} -\partial^2 g(x, t) / \partial t^2 \leq \Delta^* g(x), \quad \text{and} \\ &\limsup_{t \rightarrow 0} -\partial^2 g(x, t) / \partial t^2 \geq \Delta_* g(x) \quad \text{for } x \text{ in } C[0, \alpha]. \end{aligned}$$

Next, we observe from [6] that if $g(x)$ is continuous in $B(x^0, r_0)$, if $\Delta^* g(x) \geq 0$ and $\Delta_* g(x) \leq 0$ almost everywhere in $B(x^0, r_0)$, and if $\Delta^* g(x) > -\infty$ and $\Delta^* g(x) < +\infty$ for x in $B(x^0, r_0)$, then $g(x)$ is harmonic in $B(x^0, r_0)$. We consequently conclude from (2.43), (2.44), (2.45), and [6] that

$$(2.46) \quad \begin{aligned} &\text{if } g(x) \text{ is continuous in } B(x^0, r_0) \subset C[0, \alpha], \text{ then} \\ &g(x) \text{ is harmonic in } B(x^0, r_0). \end{aligned}$$

We denote by Z the set of points in $C[0, \alpha]$ which are the discontinuity points of g , i.e.,

$$(2.47) \quad Z = \{x : x \text{ in } C[0, \alpha] \text{ and } g \text{ is not continuous at } x\}.$$

It follows from (2.46) that (2.42) will follow once we show that Z is the empty set. We now establish this fact.

In order to accomplish this, we first introduce the function $H_N(x)$ as follows:

$$H_N(x) = \lim_{t \rightarrow 0} \sum_{m \neq 0} i m_N e^{i(m, x) - |m|t} |m|^{-2} \quad \text{for } x/2\pi \neq \text{integral lattice point.}$$

It follows from [8, Lemma 8, p. 72] that $H_N(x)$ is finite-valued in $E_N - \bigcup_m \{2\pi m\}$, that $H_N(x)$ is in $L^1(T_N)$ and that

$$(2.48) \quad \hat{H}_N(0) = 0 \text{ and } \hat{H}_N(m) = i m_N |m|^{-2} \text{ for } m \neq 0.$$

Next, we define $g'(x)$ for x in E_N as follows:

$$(2.49) \quad g'(x) = -N(2\pi)^{-N} \int_{T_N} f(x - p) H_N(p) dp.$$

The continuity and periodicity of f imply immediately that

$$(2.50) \quad g'(x) \text{ is continuous in } E_N \text{ and of period } 2\pi \text{ in each variable.}$$

Also, we have from (2.48) and (2.49) that

$$(2.51) \quad \hat{g}'(0) = 0 \text{ and } \hat{g}'(m) = -iN \hat{f}(m) m_N |m|^{-2}.$$

Next, we define $\tilde{f}(x)$ in E_N as follows:

$$(2.52) \quad \tilde{f}(x) = \limsup_{t \rightarrow 0} \left(-\sum' \hat{f}(m) \left[\sum_{i=1}^{N-1} m_i^2 - (N-1)m_N^2 \right] |m|^{-2} e^{i(m, x) - |m|t} \right).$$

Since we are essentially dealing with a function in $L^2(T_N)$, it follows from [8, pp. 55–56] that $\tilde{f}(x)$ is finite almost everywhere in E_N and in particular from the Cauchy criterion and (2.34) that

$$(2.53) \quad \tilde{f}(x) \text{ is finite-valued in } C[0, \alpha].$$

Also, we have

$$(2.54) \quad \tilde{f}(x) \text{ is in } L^2(T_N) \text{ with } \hat{\tilde{f}}(0) = 0 \text{ and } \hat{\tilde{f}}(m) = -\hat{f}(m) \left[\sum_{i=1}^{N-1} m_i^2 - (N-1)m_N^2 \right] |m|^{-2} \text{ for } m \neq 0.$$

It consequently follows from (2.36), (2.39), (2.51), and (2.54) that

$$(2.55) \quad \hat{g}(m) = [\hat{\tilde{f}}(m) + \hat{g}'(m) - (N-1)\hat{f}(m)]N^{-1} \text{ for all } m.$$

From the hypothesis of the lemma and (2.50), we have that $f(x)$ and $g'(x)$ are continuous in E_N and periodic of period 2π in each variable. Consequently,

it follows from [8, Theorem 2, p. 55], (2.37), (2.38), (2.39), (2.52), (2.54), and (2.55) that

$$(2.56) \quad g(x) = [\tilde{f}(x) + g'(x) - (N-1)f(x)]/N \text{ for every } x \text{ in } E_N.$$

We are now ready to show that the set Z defined in (2.47) is empty. In order to do this we write $g_t(x, t) = \partial g(x, t)/\partial t$ and $g_{tt}(x, t) = \partial^2 g(x, t)/\partial t^2$.

Now it follows from (2.37) that we can choose a strictly decreasing sequence $\{t_k\}_{k=1}^\infty$ with $\lim_{k \rightarrow \infty} t_k = 0$ such that the following fact obtains

$$(2.57) \quad |g_{tt}(x, t_k) - g_{tt}(x, t)| \leq 1 \text{ for } t_{k+1} \leq t \leq t_k \text{ and for } x \text{ in } T_N.$$

With $\bar{B}(x^0, r_0)$ designating the closure of $B(x^0, r_0)$ suppose that $\bar{B}(x^0, r_0) \subset C[0, \alpha]$ where $r_0 > 0$. Then it follows from the Baire Category Theory (see [13, p. 29]) and (2.43) that there is a finite constant M and a sub-ball $B(y^0, r_1) \subset B(x^0, r_0)$ with $r_1 > 0$ such that

$$(2.58) \quad |g_{tt}(x, t_k)| \leq M \text{ for } x \text{ in } B(y^0, r_1) \text{ and } k = 1, 2, \dots$$

But then it follows from (2.57) and (2.58) that

$$(2.59) \quad |g_{tt}(x, t)| \leq M + 1 \text{ for } x \text{ in } B(y^0, r_1) \text{ and } 0 < t \leq t_1.$$

But (2.59) and an application of the Cauchy criterion in conjunction with (2.41) enables us to conclude that

$$(2.60) \quad \lim_{t \rightarrow 0} g(x, t) = g(x) \text{ uniformly for } x \text{ in } B(y^0, r_1).$$

Consequently, we have that

$$(2.61) \quad Z \text{ is nowhere dense in } C[0, \alpha].$$

If Z is not non-empty, then there exists α_1 with $0 < \alpha_1 < \alpha$ such that

$$(2.62) \quad Z_1 \text{ is non-empty where } Z_1 = Z \cap C[0, \alpha_1].$$

Next, we choose another strictly decreasing sequence $\{t'_k\}_{k=1}^\infty$ with $\lim_{k \rightarrow \infty} t'_k = 0$ such that

$$(2.63) \quad \sum_{i=1}^N |\partial^2 f(x, t'_k)/\partial x_i^2 - \partial^2 f(x, t)/\partial x_i^2| \leq 1 \text{ for } x \text{ in } E_N \text{ and } t'_{k+1} \leq t \leq t'_k.$$

Using the Baire Category Theory once again applied to the set $\bar{Z}_1$, the closure of Z_1 , it follows from (2.34), (2.63), and [13, p. 29] that there is a finite constant M' , an $h_0 > 0$, and a z^0 in Z_1 such that

$$(2.64) \quad \sum_{i=1}^N |\partial^2 f(x, t)/\partial x_i^2| \leq M' + 1 \text{ for } x \text{ in } \bar{Z}_1 \cap \bar{B}(z^0, h_0) \text{ and } 0 < t \leq t'_1.$$

For emphasis, we state explicitly that

$$(2.65) \quad z^0 \text{ is in } Z_1.$$

With $\tilde{f}(x, t)$ designating the expression in braces on the right-hand side of (2.52), we see from the Cauchy criterion and (2.64) that $\tilde{f}(x, t) \rightarrow \tilde{f}(x)$ uniformly for x in $\bar{Z}_1 \cap \bar{B}(z^0, h_0)$ as $t \rightarrow 0$. Since $g'(x)$ and $f(x)$ are continuous in E_N , we conclude from (2.56) that

$$(2.66) \quad g(x) \text{ is continuous for } x \text{ restricted to the closed set } \bar{Z}_1 \cap \bar{B}(z^0, h_0).$$

Also from Lemma 6 and (2.64), we infer that $\tilde{f}_h(x)$ tends to a finite limit uniformly for x in $\bar{Z}_1 \cap \bar{B}(z^0, h_0)$. But $\tilde{f}(x, t)$ must tend to this same limit. We consequently conclude from (2.52) that

$$(2.67) \quad \lim_{h \rightarrow 0} \tilde{f}_h(x) = \tilde{f}(x) \text{ uniformly for } x \text{ in } \bar{Z}_1 \cap \bar{B}(z^0, h_0).$$

But both $g'_h(x) \rightarrow g'(x)$ and $f_h(x) \rightarrow f(x)$ as $h \rightarrow 0$ uniformly for x in E_N ; so we conclude from (2.56) and (2.67) that

$$(2.68) \quad \lim_{h \rightarrow 0} g_h(x) = g(x) \text{ uniformly for } x \text{ in } \bar{Z}_1 \cap \bar{B}(z^0, h_0).$$

Also, from (2.56), Lemma 4, and the continuity of $g'(x)$ and $f(x)$, we have that

$$(2.69) \quad \lim_{h \rightarrow 0} \left[\sup_{x \text{ in } T_N \text{ and } |p| \leq h} |g_h(x+p) - g_h(x)| \right] = 0.$$

Next, given $\epsilon > 0$, we see from (2.66) and (2.68) that we can choose h_1 with $0 < h_1 < \min(1/2, h_0)$ such that

$$(2.70) \quad |g(x) - g(z^0)| < \epsilon \quad \text{and} \quad |g_h(x) - g(x)| < \epsilon \\ \text{for } x \text{ in } \bar{Z}_1 \cap \bar{B}(z^0, h_1) \quad \text{and} \quad 0 < h < h_1.$$

Also, from (2.69) we can choose h_2 with $0 < h_2 < h_1$ such that

$$(2.71) \quad |g_h(x) - g_h(x+p)| < \epsilon \text{ for } x \text{ in } T_N, |p| \leq h, \text{ and } 0 < h < h_2.$$

To obtain a contradiction to (2.62), we need only obtain a contradiction to (2.65). We now do this by showing

$$(2.72) \quad |g(x) - g(z^0)| < 3\epsilon \text{ for } x \text{ in } B(z^0, h_3)$$

where h_3 is chosen such that

$$(2.73) \quad \bar{B}(z^0, 2h_3) \subset C[0, \alpha_1] \text{ and } 0 < h_3 < h_2/2.$$

Since $h_2 < h_1$, (2.72) follows from (2.73) and (2.70) for x in $\bar{Z} \cap B(z^0, h_3)$. If on the other hand x is in $B(z^0, h_3) - \bar{Z}$, then g is continuous in a neighborhood of x , and we consequently infer from (2.73) and (2.46) that g is harmonic in a neighborhood of x . Let h_4 designate the distance of x to $\bar{Z}$. Then $0 < h_4 < h_3$ and

$$(2.74) \quad g_{h_*}(x) = g(x).$$

Let z^* be in $\bar{Z}$ and be such that $|z^* - x| = h_4$. Then z^* is in $\bar{Z}_1$, $|z^* - z^0| < h_2$, and we infer from (2.70), (2.71), and (2.74) that $|g(x) - g(z^0)| \leq |g_{h_*}(x) -$

$g_{h_*}(z^*)| + |g_{h_*}(z^*) - g(z^*)| + |g(z^*) - g(z^0)| \leq 3\epsilon$. Consequently, (2.72) is established. Therefore the set Z in (2.47) is empty and the proof to the lemma is complete.

Next, we concern ourselves with a lemma from the theory of the heat equation. But first we define $C'[x^*, \alpha]$, $\alpha > 0$, to be the open set

$$(2.75) \quad C'[x^*, \alpha] = \{x : x \text{ in } C[x^*, \alpha] \text{ and } x_N \neq x_N^*\}.$$

Lemma 8. *Let $C[x^*, \alpha]$, $\alpha > 0$, be a given open N -cube. Suppose $u(x)$ is continuous in $C[x^*, \alpha]$ and in class $C^2(C'[x^*, \alpha])$. Suppose, furthermore, that $u(x)$ is a classical solution of the heat equation in $C'[x^*, \alpha]$. Then $u(x)$ is in class $C^2(C[x^*, \alpha])$ and is a classical solution of the heat equation in $C[x^*, \alpha]$.*

To establish Lemma 8, we let $0 < \alpha_1 < \alpha$ and observe that the lemma will be established if we show the following:

$$(2.76) \quad u \text{ is in class } C^2(C[x^*, \alpha_1]), \text{ and } u \text{ is a classical solution of the heat equation } C[x^*, \alpha_1].$$

To show that (2.76) is indeed true, we observe that u is continuous in the closure of $C[x^*, \alpha_1]$ which we designate by $\bar{C}[x^*, \alpha_1]$. Consequently we can solve the first boundary value problem with respect to $u(x)$ on the boundary of $\bar{C}[x^*, \alpha_1]$ and obtain a function $v(x)$ which enjoys the following properties (see [4, p. 130] and [3, p. 226]):

$$(2.77) \quad \begin{aligned} & \text{(i)} \quad v(x) \text{ is continuous in } \bar{C}[x^*, \alpha_1]; \\ & \text{(ii)} \quad v(x) \text{ is in class } C^2(C[x^*, \alpha_1]); \\ & \text{(iii)} \quad v(x) \text{ is a classical solution of the heat} \\ & \quad \text{equation in } C[x^*, \alpha_1]; \\ & \text{(iv)} \quad v(x) = u(x) \text{ if } x = (x_1, \dots, x_N) \text{ is on the boundary of } \bar{C}[x^*, \alpha_1] \\ & \quad \text{and } x_N = x_N^* - \alpha_1 \text{ or there exists } j \text{ with } 1 \leq j \leq N - 1 \text{ such that} \\ & \quad x_j = x_j^* + \alpha_1 \text{ or } x_j = x_j^* - \alpha_1. \end{aligned}$$

But then it follows immediately from the Maximum Principle for the heat equation that $v(x) = u(x)$ in the region $\{x : |x_j - x_j^*| \leq \alpha_1 \text{ for } j = 1, \dots, N - 1 \text{ and } x_N^* - \alpha_1 \leq x_N \leq x_N^*\}$. Applying the maximum once again and using (2.77) (iv), we obtain that $v(x) = u(x)$ in the region $\{x : |x - x_j^*| < \alpha_1 \text{ for } j = 1, \dots, N - 1 \text{ and } x_N^* \leq x_N \leq x_N^* + \alpha_1\}$. Consequently, $u(x) = v(x)$ in $C[x^*, \alpha_1]$. But then (2.77) (ii) and (iii) imply (2.76), and the proof of the lemma is complete.

From Lemma 7 and Lemma 8, we obtain immediately the following lemma:

Lemma 9. *Let $f(x)$ be a continuous function in E_N of period 2π in each variable, and let $C[x^*, \alpha]$, $\alpha > 0$, be a given open N -cube. Also, let Q be a finite set contained in the open interval $(-\alpha + x_N^*, x_N^* + \alpha)$. For $t > 0$, set $f(x, t) = \sum_m \hat{f}(m) e^{i(m, x) - |m|^2 t}$. Suppose that the following facts hold:*

$$(2.78) \quad \limsup_{t \rightarrow 0} |\partial f(x, t) / \partial x_N| < \infty \text{ for } x \text{ in } C[x^*, \alpha] - Q^s;$$

$$(2.79) \quad \limsup_{t \rightarrow 0} |\partial^2 f(x, t)/\partial x_i^2| < \infty \text{ for } x \text{ in } C[x^*, \alpha] - Q^s \text{ and } j = 1, \dots, N;$$

(2.80)

$$\lim_{t \rightarrow 0} \left[\sum_{i=1}^{N-1} \partial^2 f(x, t)/\partial x_i^2 - \partial f(x, t)/\partial x_N \right] = 0 \text{ for almost every } x \text{ in } C[x^*, \alpha].$$

Then $f(x)$ is in class $C^2(C[x^*, \alpha])$ and is a classical solution of the heat equation in $C[x^*, \alpha]$.

We next come to the main lemma of the paper.

Lemma 10. *Lemma 9 is also true under the weaker assumption that Q is a countable set.*

In order to prove Lemma 10, we define for each positive integer k the following $(N + 1)$ sets:

$$(2.81) \quad \alpha_N^1(k) = \{x : x \text{ in } C[x^*, \alpha] \text{ and } \sup_{0 < t < 1} |f_{x_N}(x, t)| > k\},$$

$$(2.82) \quad \alpha_j^2(k) = \{x : x \text{ in } C[x^*, \alpha] \text{ and } \sup_{0 < t < 1} |f_{x_j x_j}(x, t)| > k\}$$

where $j = 1, \dots, N$ in (2.82).

Since $f_{x_N}(x, t)$ and $f_{x_j x_j}(x, t)$ are continuous functions for x in E_N and $0 < t < 1$, it follows that $\alpha_N^1(k)$ and $\alpha_j^2(k)$ are open sets for each positive integer k and $j = 1, \dots, N$.

Next we define

$$(2.83) \quad \begin{aligned} \alpha_N^1 &= \bigcap_{k=1}^{\infty} \alpha_N^1(k); \\ \alpha_j^2 &= \bigcap_{k=1}^{\infty} \alpha_j^2(k) \quad \text{for } j = 1, \dots, N; \\ \alpha &= \alpha_N^1 \cup \bigcup_{j=1}^N \alpha_j^2. \end{aligned}$$

Now α_N^1 is a G_δ -set and α_j^2 is a G_δ -set for $j = 1, \dots, N$. Also since a finite union of G_δ -sets is a G_δ -set, we conclude that α is a G_δ -set contained in $C[x^*, \alpha]$. Consequently we can write

$$(2.84) \quad \alpha = \bigcap_{i=1}^{\infty} \mathcal{B}_i,$$

where $\mathcal{B}_i \supset \mathcal{B}_{i+1}$ and $\mathcal{B}_i$ is an open set contained in $C[x^*, \alpha]$ for $j = 1, 2, \dots$.

Next, we let Z be the projection of α on the x_N -axis, i.e.,

$$(2.85) \quad Z = \{x_N : x \text{ in } \alpha\}.$$

From the hypothesis of the lemma, i.e., from (2.78) and (2.79), it follows from (2.81), (2.82), and (2.83) that $\alpha \subset Q^s$. Consequently it follows from (2.85) that

$Z \subset Q$. Therefore Z is a countable set. If Z contains only a finite number of elements, the conclusion to the lemma follows immediately from Lemma 9. Consequently, we shall assume that Z is of the form

$$(2.86) \quad Z = \bigcup_{i=1}^{\infty} \{z_i\} \text{ where } z_i \neq z_k \text{ for } i \neq k.$$

If we can show that (2.86) leads to a contradiction, it will then follow that Z must be the empty set and the proof to the lemma will be complete.

We now show that (2.86) leads to a contradiction. In the first place, it follows from (2.85) and (2.86) that we can find y^1 and r_1 with following properties

$$(2.87) \quad \begin{aligned} & \text{(i)} \quad y^1 \text{ is in } \mathfrak{A}, \\ & \text{(ii)} \quad z_1 \text{ is not in the interval } (y_N^1 - r_1, y_N^1 + r_1), \\ & \text{(iii)} \quad C[y^1, r_1] \subset \mathfrak{B}_1, \\ & \text{(iv)} \quad 0 < r_1 < 2^{-1}. \end{aligned}$$

We set

$$(2.88) \quad Z^1 = \{x_N : x \text{ in } \mathfrak{A} \cap C[y^1, r_1/2]\}.$$

It follows from (2.85) and (2.88) that $Z^1 \subset Z$ and from (2.87) (iii) that $C[y^1, r_1/2] \subset C[x^*, \alpha]$. If Z^1 contained only a finite number of elements, then it would follow from (2.88) and Lemma 9 that f would be in $C^2(C[y^1, r_1/2])$ and consequently from Lemma 2 that y^1 would not be in $\mathfrak{A}$ which would contradict (2.88) (i). Therefore Z^1 has infinite number of elements. But then we can find y^2 and r_2 with the following properties:

$$(2.89) \quad \begin{aligned} & \text{(i)} \quad y^2 \text{ is in } \mathfrak{A} \cap C[y^1, r_1/2], \\ & \text{(ii)} \quad z_k \text{ is not in the interval } (y_N^2 - r_2, y_N^2 + r_2) \text{ for } k = 1, 2, \\ & \text{(iii)} \quad C[y^2, r_2] \subset \mathfrak{B}_2 \cap C[y^1, r_1/2], \\ & \text{(iv)} \quad 0 < r_2 < 2^{-2}. \end{aligned}$$

Proceeding inductively in the same manner in which we have established (2.89), we see that we can define sequences $\{y^i\}_{i=1}^{\infty}$ and $\{r_i\}_{i=1}^{\infty}$ with the following properties:

$$(2.90) \quad \begin{aligned} & \text{(i)} \quad y^{i+1} \text{ is in } \mathfrak{A} \cap C[y^i, r_i/2], \\ & \text{(ii)} \quad z_k \text{ is not in the interval } (y_N^{i+1} - r_{i+1}, y_N^{i+1} + r_{i+1}) \text{ for } k = 1, \\ & \quad \quad \quad \dots, j+1, \\ & \text{(iii)} \quad C[y^{i+1}, r_{i+1}] \subset \mathfrak{B}_{i+1} \cap C[y^i, r_i/2] \\ & \text{(iv)} \quad 0 < r_{i+1} < 2^{-(i+1)}. \end{aligned}$$

Using the Cantor Intersection Theorem, [7, p. 54], we see from (2.90) (iii) and (iv) that y^0 defined by $y^0 = \bigcap_{i=1}^{\infty} \bar{C}[y^i, r_i/2]$ is such that y^0 is in $\bigcap_{i=1}^{\infty} \mathfrak{B}_i = \mathfrak{A}$. But then y_N^0 is in Z . Therefore there is a j_0 such that $y_N^0 = z_{j_0}$. But then by (2.90) (ii) and (2.87) (ii), y^0 is not in $\bar{C}[y^j, r_j/2]$ for $j = j_0, j_0 + 1, j_0 + 2, \dots$, and we

have a contradiction to the definition of y^0 . Therefore Z cannot have an infinite number of elements and the proof of the lemma is complete.

3. Proof of the sufficiency condition of the theorem. To prove the theorem, with no loss in generality we can assume that $x^* = 0$ and $0 < \alpha < 1$.

Next, we let α_1 be such that $0 < \alpha_1 < \alpha$ and observe that theorem will be established if we can show that u is in $C^2(C[0, \alpha_1])$ and is a classical solution of the heat equation in $C[0, \alpha_1]$. In order to do this, we select α_2 such that $0 < \alpha_1 < \alpha_2 < \alpha$ and introduce a localizing function $\lambda(x)$ enjoying the following properties:

- (3.1) (i) $\lambda(x)$ is in $C^\infty(E_N)$;
(ii) $\lambda(x) = 1$ for x in $C[0, \alpha_1]$;
(iii) $\lambda(x) = 0$ for x not in $C[0, \alpha_2]$.

We then define $f(x)$ as follows:

$$(3.2) \quad \begin{aligned} f(x) &= \lambda(x)u(x) \quad \text{for } x \text{ in } C[0, \alpha], \\ &= 0 \quad \text{for } x \text{ in } T_N, \end{aligned}$$

$f(x)$ is defined in E_N by periodicity of period 2π in each variable.

It follows from (3.1) and (3.2) that

- (3.3) $f(x)$ is continuous in E_N and periodic of period 2π in each variable, and furthermore that

$$(3.4) \quad f(x) = u(x) \quad \text{for } x \text{ in } C[0, \alpha_1].$$

From (3.4) and the hypothesis of the theorem; namely, (ii), (iii), and (iv) in §1, we then obtain that

$$(3.5) \quad f \text{ is in } T_2^1(x) \text{ for } x \text{ in } C[0, \alpha_1] - Q^s,$$

$$(3.6) \quad f \text{ is in } t_2^1(x) \text{ almost everywhere in } C[0, \alpha_1],$$

$$(3.7) \quad f \text{ is a pointwise solution of the heat equation almost everywhere in } C[0, \alpha_1].$$

Next, with $\hat{f}(m)$ designating the m -th Fourier coefficient of f , we introduce for $t > 0$ the function $f(x, t)$ defined as follows:

$$(3.8) \quad f(x, t) = \sum_m \hat{f}(m) e^{i(m, x) - |m|^2 t}.$$

From (3.5) it follows that f is in $t_1^1(x)$ for x in $C[0, \alpha_1] - Q^s$. We therefore obtain from (2.6) in Lemma 2 that $\lim_{t \rightarrow 0} f_{x_N}(x, t) = D_N f(x)$ for x in $C[0, \alpha_1] - Q^s$. In particular we have that

$$(3.9) \quad \limsup_{t \rightarrow 0} |\partial f(x, t) / \partial x_N| < \infty \quad \text{for } x \text{ in } C[0, \alpha_1] - Q^s,$$

From (3.5) and Lemma 1, we obtain

$$(3.10) \quad \limsup_{t \rightarrow 0} |\partial^2 f(x, t)/\partial x_i^2| < \infty \text{ for } x \text{ in } C[0, \alpha_1] - Q^s \text{ and } j = 1, \dots, N.$$

From (3.6) and (3.7) and from (2.6) and (2.7) of Lemma 2, we obtain

(3.11)

$$\lim_{t \rightarrow 0} \left[\sum_{j=1}^{N-1} \partial^2 f(x, t)/\partial x_j^2 - \partial f(x, t)/\partial x_N \right] = 0 \text{ for almost every } x \text{ in } C[0, \alpha_1].$$

But then it follows from Lemma 9 and Lemma 10 and from (3.3), (3.8), (3.9), (3.10) and (3.11) that

$$(3.12) \quad f(x) \text{ is in } C^2(C[0, \alpha_1]) \text{ and is a classical solution of the heat equation in } C[0, \alpha_1].$$

However, from (3.4) we have that $f(x) = u(x)$ in $C[0, \alpha_1]$. Consequently, (3.12) holds with $f(x)$ replaced by $u(x)$, and the proof of the theorem is complete.

4. Appendix. We close this paper by showing that in the sufficiency condition of the theorem the fact that Q^s consists of characteristic planes is crucial. In particular, we shall show that if a plane is non-characteristic then it is not a removable set for the heat equation. To be specific we shall establish the following remark.

Remark. Let Q^P be a non-characteristic plane for the heat equation in E_N . Then there exists a function $u(x)$ with the following properties:

- (i) $u(x)$ is continuous in E_N ;
- (ii) $u(x)$ is in class $C^\infty(E_N - Q^P)$;
- (iii) $u(x)$ is a classical solution of the heat equation in $E_N - Q^P$;
- (iv) $u(x)$ is not in class $C^1(E_N)$.

If Q^P is a non-characteristic plane for the heat equation in E_N , then Q^P has following form:

$$(4.1) \quad Q^P = \left\{ x : \sum_{i=1}^N \beta_i x_i = \gamma, \sum_{i=1}^N \beta_i^2 = 1, \beta_N^2 \neq 1 \right\}.$$

Two cases then occur; namely, Case I when $\beta_N = 0$ and Case II when $\beta_N \neq 0$.

For Case I we define $u(x)$ as follows:

$$(4.2) \quad \begin{aligned} u(x) &= \sum_{i=1}^N \beta_i x_i - \gamma \quad \text{for} \quad \sum_{i=1}^{N-1} \beta_i x_i \geq \gamma, \\ u(x) &= 0 \quad \text{for} \quad \sum_{i=1}^{N-1} \beta_i x_i \leq \gamma. \end{aligned}$$

For Case II we define $u(x)$ as follows:

$$(4.3) \quad u(x) = \exp \left[\beta_N \sum_{i=1}^{N-1} \beta_i x_i / (1 - \beta_N^2) \right] - \exp [\beta_N \gamma / (1 - \beta_N^2)]$$

$$\text{for } \sum_{i=1}^N \beta_i x_i \geq \gamma,$$

$$u(x) = 0 \quad \text{for } \sum_{i=1}^N \beta_i x_i \leq \gamma.$$

Easy computations using (4.1), (4.2), and (4.3) show that in both cases $u(x)$ meets conditions (i), (ii), (iii), and (iv) above, and proof of the Remark is complete.

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