On the Commutativity of Homotopy Groups

GERALD J. PORTER*

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Let $[X, Y]$ denote the set of based homotopy classes of continuous maps from X to Y . We assume all spaces have base points and use $*$ to denote both the base point and the constant map. The reduced suspension of X is denoted by ΣX and the loop space of X by ΩX . All spaces are assumed to be connected.

The one point union, $X \vee Y$, is the subset of $X \times Y$ consisting of points with at least one coordinate equal to the base point. The folding map, $\nabla: X \vee X \rightarrow X$ is defined by $\nabla(x, *) = \nabla(*, x) = x$. X is called an H -space if ∇ can be extended to $X \times X$. Such an extension is called a multiplication on X . Dually Y is called an H' -space if there is $\varphi: Y \rightarrow Y \vee Y$ such that φ followed by inclusion into the product is homotopic to the diagonal, $\Delta: Y \rightarrow Y \times Y$. In this case φ is called a comultiplication. Composition of loops gives ΩX an H -space structure and the "pinching map" gives ΣY an H' -space structure. These structures are taken as canonical.

Since $[\Sigma X, Y] \approx [X, \Omega Y]$ it is easily seen that $[\Sigma X, Y]$ is an abelian group if ΩY is homotopy-abelian and dually $[X, \Omega Y]$ is an abelian group if ΣX is homotopy-abelian. We show that the first result holds if ΣX is replaced by an arbitrary H' -space and dually the second result holds if ΩY is replaced by an arbitrary H -space.

Theorem. $[X, Y]$ is an abelian group if either

- (a) X is an H' -space and ΩY is homotopy abelian, or
- (b) Y is an H -space and ΣX is homotopy abelian.

Proof. Let $e: \Sigma \Omega X \rightarrow X$ and $\lambda: Y \rightarrow \Omega \Sigma Y$ be the adjoints of $1_{\Omega X}$ and $1_{\Sigma Y}$ respectively. If X is an H' -space, $e^*: [X, Y] \rightarrow [\Sigma \Omega X, Y]$ is one-one. Dually, if Y is an H -space, $\lambda_*: [X, Y] \rightarrow [X, \Omega \Sigma Y]$ is one-one. In both cases the range is an abelian group. To prove the theorem it therefore suffices to show that in case (a), e^* is a homomorphism, and in case (b), λ_* is a homomorphism.

We first prove (a). Let $\bar{\varphi}: X \rightarrow X \vee X$ and $\varphi: \Sigma \Omega X \rightarrow \Sigma \Omega X \vee \Sigma \Omega X$ be the comultiplications in X and $\Sigma \Omega X$ respectively. e^* is a homomorphism if and only

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if $\nabla(f \vee g)(\bar{\varphi}e - (e \vee e)\varphi) \sim *$ for all $f, g \in [X, Y]$. Let $i: X \vee X \rightarrow X \times X$ be inclusion. The fibre of i has the homotopy type of $\Omega X * \Omega X$, where $*$ is the join operation. The map from $\Omega X * \Omega X$ to $X \vee X$ is the generalized Whitehead product, $[e_1, e_2]$. ($e_1, e_2: \Sigma \Omega X \rightarrow X \vee X$ are the adjoints of $\Omega i_1, \Omega i_2$ where $i_1, i_2: X \rightarrow X \vee X$ are the two inclusions.) Since $i(\bar{\varphi}e - (e \vee e)\varphi) \sim *$ we may write $\bar{\varphi}e - (e \vee e)\varphi \sim [e_1, e_2]H(e)$ for $H(e): \Sigma \Omega X \rightarrow \Omega X * \Omega X$. $H(e)$ is called the total Hopf invariant of e . Since ΩY is homotopy abelian all Samelson products in ΩY are zero. Whitehead products in Y are the adjoints of Samelson products in ΩY . Thus all Whitehead products in Y are zero. Therefore $\nabla(f \vee g)[e_1, e_2] = [fe, ge] \sim *$ and (a) follows.

The proof of (b) is not precisely dual. Let $\bar{\mu}: Y \times Y \rightarrow Y$ and $\mu: \Omega \Sigma Y \times \Omega \Sigma Y$ be the multiplications on Y and $\Omega \Sigma Y$ respectively. λ_* is a homomorphism if and only if $(\mu(\lambda \times \lambda) - \lambda\bar{\mu})(f \times g)\Delta \sim *$ for all $f, g \in [X, Y]$ where Δ is the diagonal map. $(\mu(\lambda \times \lambda) - \lambda\bar{\mu})(f \times g) \mid (X \vee X)$ is null-homotopic and hence may (up to homotopy) be factored through $X \wedge X = (X \times X)/(X \vee X)$. Let $q: X \times X \rightarrow X \wedge X$ be the quotient map and let $\psi: X \wedge X \rightarrow \Omega \Sigma Y$ be such that $(\mu(\lambda \times \lambda) - \lambda\bar{\mu})(f \times g)\Delta \sim \psi q\Delta$. Let $\lambda': X \wedge X \rightarrow \Omega \Sigma(X \wedge X)$ be the adjoint of $1_{\Sigma(X \wedge X)}$. A direct calculation shows that $\psi = (\Omega e')(\Omega \Sigma \psi)\lambda'$ where $e': \Sigma \Omega \Sigma Y \rightarrow \Sigma Y$ is the adjoint of $1_{\Omega \Sigma Y}$. Thus $\psi q\Delta \sim (\Omega e')(\Omega \Sigma \psi)\lambda' q\Delta$. Theorem 4.1 of [1] states that ΣX is homotopy abelian if and only if $\lambda' q\Delta \sim *$. Hence if ΣX is homotopy abelian, λ_* is a homomorphism and (b) follows.

Remark. Theorem 2 of [2] and Theorem 1 of [3] show that with an additional hypothesis our theorem is true. Our proof shows that this hypothesis is superfluous.

REFERENCES

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University of Pennsylvania
Philadelphia, Pennsylvania

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