“The $\mathcal{L}^p$ Moment Problem” for Operators in Banach Spaces

D. LEVIATAN & M. S. RAMANUJAN

Communicated by MAURICE HEINS

1. Introduction. Let E and F be Banach spaces and let $\mathfrak{L}(E, F)$ denote the space of continuous linear operators on E into F . Let $\mathbf{m}$ be a vector-valued measure which is defined on the Lebesgue measurable subsets of $[0, 1]$ and assumes values in $\mathfrak{L}(E, F)$ and let $1 \leq q \leq \infty$ and $1/p + 1/q = 1$. The q -variation $\tilde{\mathbf{m}}_q[0, 1]$ of $\mathbf{m}$ in $[0, 1]$ is defined in the following way (see [1; p. 241])

$$(1.1) \quad \tilde{\mathbf{m}}_q[0, 1] = \sup \sum \|\mathbf{m}(A_i)\| |\alpha_i|$$

where the α_i 's are scalars and the supremum is taken for all step functions $\varphi = \sum \chi_{A_i} \alpha_i$ (χ_A is the characteristic function of A) such that the A_i 's are pairwise disjoint measurable sets in $[0, 1]$ and $\int_0^1 |\varphi(t)|^p dt \leq 1$, (the integration is with respect to Lebesgue measure). The q -semi-variation $\tilde{\mathbf{m}}_q[0, 1]$ of $\mathbf{m}$ in $[0, 1]$ is defined in the following way (see [1; p. 246])

$$(1.2) \quad \tilde{\mathbf{m}}_q[0, 1] = \sup \|\sum \mathbf{m}(A_i)x_i\|$$

where the x_i 's are in E and the supremum is taken for all step functions $\Psi = \sum \chi_{A_i} x_i$ such that the A_i 's are pairwise disjoint measurable sets in $[0, 1]$ and $\int_0^1 \|\Psi(t)\|^p dt \leq 1$.

Let the sequence $\{\lambda_n\} (n \geq 0)$ satisfy

$$(1.3) \quad 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots \uparrow \infty, \quad \sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty,$$

and let $1 < q \leq \infty$. We are interested in the following problem. Given a sequence of continuous linear operators $\{\mu_n\} \subseteq \mathfrak{L}(E, F)$ what are the conditions, necessary and sufficient, on $\{\mu_n\} (n \geq 0)$ in order that it should possess the representation

$$(1.4) \quad \mu_n = \int_0^1 t^{\lambda_n} d\mathbf{m}(t), \quad n = 0, 1, 2, \dots,$$

where $\tilde{\mathbf{m}}_q[0, 1] < \infty$ or where $\tilde{\mathbf{m}}_q[0, 1] < \infty$.

In case E and F are the complex field, the q -variation and the q -semi-variation coincide. The problem in this case is the classical $\mathcal{L}^p$ moment problem which was first treated by Hausdorff for the sequence $\{\lambda_n = n\}$ ($n \geq 0$) (see [3], p. 81). Recently the problem for the general sequence $\{\lambda_n\}$ was discussed in [2].

An application concerning the Fourier coefficients of the measure $\mathbf{m}$ can be found in §4.

2. Preliminaries. Given a sequence $\{\mu_n\}$ ($n \geq 0$) of continuous linear operators on E into F define

(2.1)

$$\lambda_{nm} = (-1)^{n-m} \lambda_{m+1} \cdots \lambda_n \sum_{i=m}^n (1/\omega'_{nm}(\lambda_i)) \mu_i, \quad 0 \leq m \leq n = 0, 1, 2, \dots,$$

where $\omega_{nm}(x) = (x - \lambda_m) \cdots (x - \lambda_n)$, $0 \leq m \leq n = 0, 1, 2, \dots$. It is easily seen that

$$(2.2) \quad \mu_k = \sum_{m=0}^n (1 - \lambda_k/\lambda_{m+1}) \cdots (1 - \lambda_k/\lambda_n) \lambda_{nm}, \quad n \geq k = 0, 1, 2, \dots$$

Let $c_{kmn} = (1 - \lambda_k/\lambda_{m+1}) \cdots (1 - \lambda_k/\lambda_n)$ and $t_{nm} = [(1 - \lambda_1/\lambda_{m+1}) \cdots (1 - \lambda_1/\lambda_n)]^{1/\lambda_1}$ and let the points $\{(t_{nm}, c_{kmn})\}$ ($0 \leq m \leq n$, $n \geq k$) be the vertices of a polygon $P_k^{(n)}$. If $P_k^{(n)}(t)$ is the function describing this polygon, then (2.2) can be rewritten as

$$(2.3) \quad \mu_k = \sum_{m=0}^n P_k^{(n)}(t_{nm}) \lambda_{nm}, \quad n \geq k = 0, 1, 2, \dots$$

Also, it was proved by Schoenberg [5; (8.17) and (9.6)] that for each fixed k ,

$$(2.4) \quad P_k^{(n)}(t) \rightarrow t^{\lambda_k}, \text{ as } n \rightarrow \infty, \text{ uniformly in } [0, 1].$$

Define

$$(2.5) \quad \lambda_{nm}(t) = (-1)^{n-m} \lambda_{m+1} \cdots \lambda_n \sum_{i=m}^n (1/\omega'_{nm}(\lambda_i)) t^{\lambda_i}.$$

The following properties hold (see Lorentz [3; p. 46 (10) and (11) and Theorem 2.8.2])

$$(2.6) \quad \lambda_{nm}(t) \geq 0 \quad \text{for } 0 \leq m \leq n = 0, 1, 2, \dots \quad \text{and } 0 \leq t \leq 1,$$

$$\sum_{m=0}^n \lambda_{nm}(t) \equiv 1, \quad n \geq 0.$$

For every $f \in C[0, 1]$

$$(2.7) \quad \lim_{n \rightarrow \infty} \sum_{m=0}^n f(t_{nm}) \lambda_{nm}(t) = f(t) \quad \text{uniformly in } [0, 1].$$

Remark. In particular if $\lambda_n = n$ for all $n \geq 0$ then

$$(2.8) \quad \lambda_{nm} = \binom{n}{m} \Delta^{n-m} \mu_m, \quad 0 \leq m \leq n = 0, 1, 2, \dots,$$

where Δ is defined by $\Delta^0 \mu_m = \mu_m$, $\Delta \mu_m = \mu_m - \mu_{m+1}$ and $\Delta^k = \Delta(\Delta^{k-1})$.

In this case

$$\lambda_{nm}(t) = \binom{n}{m} t^m (1-t)^{n-m} \quad 0 \leq t \leq 1,$$

and the polynomials in the left-hand side of (2.7) are the well-known Bernstein polynomials.

3. Moment problems. For any functional $z \in F'$ define $\lambda_{nm}^* x = \langle \lambda_{nm} x, z \rangle$ $x \in E$. Our first result is

Theorem 1. Let $1 < p \leq \infty$. The sequence $\{\mu_n\} \subseteq \mathcal{L}(E, F)$ possesses the representation (1.4) where $\tilde{\mathbf{m}}_p[0, 1] < \infty$, if and only if

$$(3.1) \quad \sup_{||z|| \leq 1} \sup_{n \geq 0} \sum_{m=0}^n \frac{||\lambda_{nm}^z||^p}{\left(\int_0^1 \lambda_{nm}(t) dt \right)^{p-1}} = (H_p)^p < \infty \quad \text{if } 1 < p < \infty,$$

or

$$(3.2) \quad \sup_{\substack{0 \leq m \leq n \\ n \geq 0}} \frac{||\lambda_{nm}||}{\int_0^1 \lambda_{nm}(t) dt} = H_\infty < \infty \quad \text{if } p = \infty.$$

Moreover $\tilde{\mathbf{m}}_p[0, 1] = H_p$, $1 < p \leq \infty$.

Proof. If $\mathcal{L}_E^q[0, 1]$ denotes the space of functions on $[0, 1]$ into E with $\int_0^1 ||f||^q dt < \infty$, then the linear combinations, with coefficients in E , of the t^{λ_i} 's are dense in the set of step functions assuming values in E in the topology of pointwise convergence and consequently dense in $\mathcal{L}_E^q[0, 1]$, $1 \leq q \leq \infty$.

Let $1 < p \leq \infty$ and $1/p + 1/q = 1$ and assume that (3.1) or (3.2) holds for this p . Define a linear mapping on the set of linear combinations, with coefficients in E , of the t^{λ_i} 's, into F in the following way. Let $\mathbf{P}(t) = \sum_{i=0}^k t^{\lambda_i} x_i$ where $x_i \in E$; define $U(\mathbf{P}) = \sum_{i=0}^k \mu_i x_i$. We shall prove that $||U(\mathbf{P})|| \leq H_p (\int_0^1 ||\mathbf{P}(t)||^q dt)^{1/q}$ which implies that U can be extended by continuity to $\mathcal{L}_E^q[0, 1]$ and that the extended U is a continuous linear mapping on $\mathcal{L}_E^q[0, 1]$ into F . By (2.3) it follows that

$$U(\mathbf{P}) = \sum_{m=0}^n \lambda_{nm} \left[\sum_{i=0}^k P_i^{(n)}(t_{nm}) x_i \right].$$

Now let μ be the positive measure defined on $\{0, \dots, n\}$ by $\mu(m) = \int_0^1 \lambda_{nm}(t) dt$, then by Dinculeanu [1; p. 246 (9) and p. 249 Proposition 1] it follows by (3.1) or (3.2) that

$$\begin{aligned}
 (3.3) \quad ||U(\mathbf{P})|| &\leq H_p \left(\sum_{m=0}^n \left\| \sum_{i=0}^k P_i^{(n)}(t_{nm}) x_i \right\|^q \int_0^1 \lambda_{nm}(t) dt \right)^{1/q} \\
 &= H_p \left(\int_0^1 \sum_{m=0}^n \left\| \sum_{i=0}^k P_i^{(n)}(t_{nm}) x_i \right\|^q \lambda_{nm}(t) dt \right)^{1/q}.
 \end{aligned}$$

By (2.4), (2.6) and (2.7) we now obtain that

$$\sum_{m=0}^n \left\| \sum_{i=0}^k P_i^{(n)}(t_{nm}) x_i \right\|^q \lambda_{nm}(t) \rightarrow ||\mathbf{P}(t)||^q, \quad \text{as } n \rightarrow \infty$$

uniformly in $[0, 1]$. Thus by (3.3)

$$||U(\mathbf{P})|| \leq H_p \left(\int_0^1 ||\mathbf{P}(t)||^q dt \right)^{1/q}.$$

By [1; p. 259 Theorem 1] there exists a vector measure $\mathbf{m}$ with $\tilde{\mathbf{m}}_p[0, 1] < \infty$ such that

$$(3.4) \quad U(\mathbf{f}) = \int_0^1 \mathbf{f}(t) d\mathbf{m}(t), \quad \mathbf{f} \in \mathcal{L}_E^q[0, 1],$$

and by [1; p. 261 Theorem 2] there exists $U' : \mathcal{L}^q[0, 1] \rightarrow \mathcal{L}(E, F)$ such that $U'(\varphi) = \int_0^1 \varphi(t) d\mathbf{m}(t)$, $\varphi \in \mathcal{L}^q[0, 1]$ and $U'(\varphi)x = U(\varphi x)$, whence in particular $\mu_n = U'(t^n) = \int_0^1 t^n d\mathbf{m}(t)$, $n = 0, 1, 2, \dots$. This completes the proof of the sufficiency part.

Conversely, assume that $\{\mu_n\} \subseteq \mathcal{L}(E, F)$ possesses the representation (1.4) where $\tilde{\mathbf{m}}_p[0, 1] < \infty$ for some $1 < p \leq \infty$. Then it follows that

$$(3.5) \quad \lambda_{nm} = \int_0^1 \lambda_{nm}(t) d\mathbf{m}(t), \quad 0 \leq m \leq n = 0, 1, 2, \dots$$

If $p = \infty$, then it follows by the definition of the ∞ -semi-variation that

$$||\lambda_{nm}|| \leq \tilde{\mathbf{m}}_\infty[0, 1] \int_0^1 \lambda_{nm}(t) dt, \quad 0 \leq m \leq n = 0, 1, 2, \dots,$$

which proves (3.2).

If $1 < p < \infty$, let $A_i = [i/(k+1), (i+1)/(k+1))$ for $0 \leq i < k$ and $A_k = [k/(k+1), 1]$. For a fixed n , define the step function $\varphi_m = \sum_{i=0}^k \lambda_{nm}(i/(k+1)) \chi_{A_i}$, $0 \leq m \leq n$ (χ_A is the characteristic function of A). For each fixed m , $0 \leq m \leq n$, it is readily seen that $\varphi_m(t) \rightarrow \lambda_{nm}(t)$, as $k \rightarrow \infty$, in the $\mathcal{L}^q[0, 1]$ topology. For $z \in F'$ define the vector valued measure, $\mathbf{m}_z$, on the measurable sets in $[0, 1]$ by $\mathbf{m}_z(A)x = \langle \mathbf{m}(A)x, z \rangle x \in E$. Then it follows by (3.5) that

$$(3.6) \quad \lambda_{nm}^z = \int_0^1 \lambda_{nm}(t) d\mathbf{m}_z(t), \quad 0 \leq m \leq n = 0, 1, 2, \dots,$$

and

$$(3.7) \quad \int_0^1 \varphi_m(t) d\mathbf{m}_z(t) \rightarrow \lambda_{nm}^z \quad \text{as } k \rightarrow \infty.$$

Now let $||z|| \leq 1$; then

$$\begin{aligned} \sum_{m=0}^n \frac{\left\| \int_0^1 \varphi_m(t) dm_z(t) \right\|^p}{\left(\int_0^1 \varphi_m(t) dt \right)^{p-1}} &\leq (k+1)^{p-1} \sum_{m=0}^n \frac{\left(\sum_{i=0}^k \lambda_{nm} \left(\frac{i}{k+1} \right) ||\mathbf{m}_z(A_i)|| \right)^p}{\left(\sum_{i=0}^k \lambda_{nm} \left(\frac{i}{k+1} \right) \right)^{p-1}} \\ &= (k+1)^{p-1} \sum_{m=0}^n \left(\sum_{i=0}^k \lambda_{nm} \left(\frac{i}{k+1} \right) \right) \left[\frac{\sum_{i=0}^k \lambda_{nm} \left(\frac{i}{k+1} \right) ||\mathbf{m}_z(A_i)||}{\sum_{i=0}^k \lambda_{nm} \left(\frac{i}{k+1} \right)} \right]^p \end{aligned}$$

[and by Jensen's Inequality (see [6; p. 23-24])]

$$\begin{aligned} &\leq (k+1)^{p-1} \sum_{m=0}^n \left(\sum_{i=0}^k \lambda_{nm} \left(\frac{i}{k+1} \right) \right) \frac{\sum_{i=0}^k \lambda_{nm} \left(\frac{i}{k+1} \right) ||\mathbf{m}_z(A_i)||^p}{\sum_{i=0}^k \lambda_{nm} \left(\frac{i}{k+1} \right)} \\ &= (k+1)^{p-1} \sum_{m=0}^n \sum_{i=0}^k \lambda_{nm} \left(\frac{i}{k+1} \right) ||\mathbf{m}_z(A_i)||^p \\ &= \sum_{i=0}^k \frac{||\mathbf{m}_z(A_i)||^p}{\left(\frac{1}{k+1} \right)^{p-1}} \sum_{m=0}^n \lambda_{nm} \left(\frac{i}{k+1} \right). \end{aligned}$$

By (2.6) $\sum_{m=0}^n \lambda_{nm}(i/(k+1)) = 1$ for $0 \leq i \leq k$; thus by [1; p. 249 Proposition 1] the above expression

$$= \sum_{i=0}^k \frac{||\mathbf{m}_z(A_i)||^p}{\left(\frac{1}{k+1} \right)^{p-1}} \leq (\tilde{\mathbf{m}}_p[0, 1])^p.$$

Therefore for every $z \in F'$, $||z|| \leq 1$, and $n \geq 0$ we obtain by (3.7) that

$$\sum_{m=0}^n \frac{||\lambda_{nm}^z||}{\left(\int_0^1 \lambda_{nm}(t) dt \right)^{p-1}} \leq (\tilde{\mathbf{m}}_p[0, 1])^p.$$

This completes our proof.

Theorem 2. Let $1 < p \leq \infty$. The sequence $\{\mu_n\} \subseteq \mathcal{L}(E, F)$ possesses the representation (1.4) where $\tilde{\mathbf{m}}_p[0, 1] < \infty$, if and only if

$$(3.8) \quad \sup_{n \geq 0} \sum_{m=0}^n \frac{||\lambda_{nm}||^p}{\left(\int_0^1 \lambda_{nm}(t) dt \right)^{p-1}} = (H_p)^p < \infty \quad \text{if } 1 < p < \infty$$

or (3.2) holds if $p = \infty$. Moreover $\tilde{\mathbf{m}}_p[0, 1] = H_p$, $1 < p \leq \infty$.

Proof. For $p = \infty$ the theorem follows by Theorem 1 and the fact that $\tilde{\mathbf{m}}_\infty[0, 1] = \bar{\mathbf{m}}_\infty[0, 1]$. Let $1 < p < \infty$. Since condition (3.8) is stronger than condition (3.1) it follows by the proof of Theorem 1 that there exists a continuous linear mapping U on $\mathcal{L}_E^p[0, 1]$ into F such that

$$U(t^{\wedge n}x) = \mu_n x, n = 0, 1, 2, \dots, x \in E.$$

We prove now that if

$$|||U|||_q = \sup \sum ||U(\chi_{A_i}x_i)||$$

where the supremum is taken for all step functions $\Psi = \sum \chi_{A_i}x_i$ with A_i 's pairwise disjoint and $\int_0^1 ||\Psi(t)||^q dt \leq 1$, then $|||U|||_q \leq H_p$. Let $\Psi = \sum_{i=0}^r \chi_{A_i}x_i$ and let $\epsilon > 0$. It is possible to find $\mathbf{P}_i(t) = \sum_{j=0}^k t^{\wedge j} x_{ij}$ such that

$$(3.9) \quad \int_0^1 ||\chi_{A_i}(t)x_i - \mathbf{P}_i(t)||^q dt \leq \left(\frac{\epsilon}{r+1}\right)^q, \quad i = 0, 1, \dots, r,$$

and

$$(3.10) \quad ||U(\chi_{A_i}x_i) - U(\mathbf{P}_i)|| \leq \frac{\epsilon}{r+1}, \quad i = 0, 1, \dots, r.$$

Now by (3.10)

$$(3.11) \quad \begin{aligned} \sum_{i=0}^r ||U(\chi_{A_i}x_i)|| &\leq \epsilon + \sum_{i=0}^r ||U(\mathbf{P}_i)|| \\ &= \epsilon + \sum_{i=0}^r \left\| \sum_{j=0}^k \mu_j x_{ij} \right\|. \end{aligned}$$

By (2.3) it follows that

$$\begin{aligned} \sum_{i=0}^r \left\| \sum_{j=0}^k \mu_j x_{ij} \right\| &= \sum_{i=0}^r \left\| \sum_{m=0}^n \lambda_{nm} \sum_{j=0}^k P_i^{(n)}(t_{nm})x_{ij} \right\| \\ &\leq \sum_{m=0}^n ||\lambda_{nm}|| \sum_{i=0}^r \left\| \sum_{j=0}^k P_i^{(n)}(t_{nm})x_{ij} \right\|, \end{aligned}$$

and using the measure μ defined on $\{0, \dots, n\}$ by $\mu(m) = \int_0^1 \lambda_{nm}(t)dt$, together with [1; p. 242 (9) and p. 249 Proposition 1] we obtain by (3.8) that

$$\begin{aligned} \sum_{i=0}^r \left\| \sum_{j=0}^k \mu_j x_{ij} \right\| &\leq H_p \left(\sum_{m=0}^n \left[\sum_{i=0}^r \left\| \sum_{j=0}^k P_i^{(n)}(t_{nm})x_{ij} \right\|^q \int_0^1 \lambda_{nm}(t) dt \right]^{1/q} \right. \\ &\quad \left. = H_p \left(\int_0^1 \sum_{m=0}^n \left[\sum_{i=0}^r \left\| \sum_{j=0}^k P_i^{(n)}(t_{nm})x_{ij} \right\|^q \lambda_{nm}(t) dt \right]^{1/q} \right). \end{aligned}$$

Thus by (2.4), (2.6) and (2.7)

$$(3.12) \quad \sum_{i=0}^r \left\| \sum_{j=0}^k \mu_j x_{ij} \right\| \leq H_p \left(\int_0^1 \left[\sum_{i=0}^r ||\mathbf{P}_i(t)||^q \right]^{1/q} dt \right).$$

Combining (3.9), (3.11) and (3.12) it follows that

$$\begin{aligned} \sum_{i=0}^r ||U(\chi_{\lambda, x_i})|| &\leq \epsilon(1 + H_p) + H_p \left(\int_0^1 \left[\sum_{i=0}^r ||\chi_{\lambda, i}(t)x_i|| \right]^q dt \right)^{1/q} \\ &= \epsilon(1 + H_p) + H_p \left(\int_0^1 ||\Psi(t)||^q dt \right)^{1/q}. \end{aligned}$$

Since ϵ is arbitrary we have

$$(3.13) \quad |||U|||_q \leq H_p.$$

We complete the proof of the sufficiency part as in the proof of Theorem 1 using [1; p. 261 Corollary 1 and p. 262 Theorem 3] instead of [1; p. 259 Theorem 1 and p. 261 Theorem 2].

The proof of the necessity is similar to the proof of the same part in Theorem 1. This completes our proof.

Theorem 3. *Let Z be a norming subspace of F' and let $1 < p \leq \infty$. Given the sequence $\{\mu_n\} \subseteq \mathcal{L}(E, F)$, there exist a function G on $[0, 1]$ into $\mathcal{L}(E, Z')$ with $||G|| \in \mathcal{L}^p[0, 1]$, such that*

$$(3.14) \quad \langle \mu_n x, z \rangle = \int_0^1 \langle G(t) t^n x, z \rangle dt, \quad x \in E, \quad z \in Z, \quad n = 0, 1, 2, \dots,$$

if and only if (3.8) holds if $1 < p < \infty$ or (3.2) holds if $p = \infty$. Moreover $(\int_0^1 ||G(t)||^p dt)^{1/p} = H_p$ if $1 < p < \infty$ or $\text{esssup}_{0 \leq t \leq 1} ||G(t)|| = H_\infty$ if $p = \infty$.

Proof. The proof is similar to that of Theorem 2 using [1; p. 279 Theorem 8] instead of [1; p. 261 Corollary 1].

4. Application. Let $\mathbf{m}$ be a vector measure defined on the Lebesgue measurable subsets of $[0, 1]$, into $\mathcal{L}(E, F)$ such that $\hat{\mathbf{m}}_p[0, 1] < \infty$ for some $1 < p \leq \infty$. Define the Fourier coefficients of $\mathbf{m}$ by $C_n = \int_0^1 e^{-2\pi i n t} d\mathbf{m}(t)$, $n = 0, \pm 1, \pm 2, \dots$, and denote $\gamma_{nzz} = \langle C_n x, z \rangle$, $n = 0, \pm 1, \pm 2, \dots$, $x \in E$, $z \in F'$. By [1; p. 279, Theorem 8] there exists a function G on $[0, 1]$ into $\mathcal{L}(E, F'')$ with $||G|| \in \mathcal{L}^p[0, 1]$ such that $\gamma_{nzz} = \int_0^1 e^{-2\pi i n t} \langle G(t)x, z \rangle dt$, $n = 0, \pm 1, \pm 2, \dots$, $x \in E$, $z \in F'$. Since for every fixed $x \in E$ and $z \in F'$ the function $\langle G(t)x, z \rangle$ belongs to $\mathcal{L}^p[0, 1]$ it follows that for $1 < p \leq 2$ the sequence $\{\gamma_{nzz}\}$, $n = 0, \pm 1, \pm 2, \dots$, (x and z being fixed) is in ℓ_q where $1/p + 1/q = 1$. It is known, however, that if $2 < q < \infty$ the converse is not true even if $\mathbf{m}$ is a scalar measure; i.e., there are sequences in ℓ_q , $2 < q < \infty$, which are not sequences of Fourier coefficients of a measure $\mathbf{m}$ with $\hat{\mathbf{m}}_p[0, 1] < \infty$.

Our purpose is to determine which sequences $\{C_n\}$, with $\{\gamma_{nzz}\}$ in ℓ_q for every $x \in E$ and $z \in F'$, are sequences of Fourier coefficients of a measure $\mathbf{m}$ with $\hat{\mathbf{m}}_p[0, 1] < \infty$. We are able to give the answer only in the case where $\mathbf{m}$ assumes values in $\mathcal{L}(E, F'')$. The result in the scalar case is due to Rooney [4] and we follow his technique. Let

$$a_{rnm} = \binom{n}{m} \int_0^1 t^m (1-t)^{n-m} e^{2\pi i r t} dt \quad r = 0, \pm 1, \pm 2, \dots,$$

$0 \leq m \leq n = 0, 1, 2, \dots$, and suppose that $\{\gamma_n\}$, $n = 0, \pm 1, \pm 2, \dots$, is a sequence in ℓ_q for some $1 \leq q < \infty$, then it was proved by Rooney [4; Lemma 3] that $\sum_{r=-\infty}^{\infty} \gamma_r a_{rnm}$ is convergent. Let $\{C_n\}$, $n = 0, \pm 1, \pm 2, \dots$, be a sequence of continuous linear operators in $\mathcal{L}(E, F'')$ such that for every $x \in E$ and $z \in F'$ $\{\gamma_{nxx}\}$, $n = 0, \pm 1, \pm 2, \dots$, is in ℓ_q for some $1 \leq q < \infty$. Define the operator $\mu_{nm}(C)$ by $\langle \mu_{nm}(C)x, z \rangle = \sum_{r=-\infty}^{\infty} \gamma_{rxz} a_{rnm}$; then $\mu_{nm}(C)$ is a linear operator on E into F'' , not necessarily continuous.

Theorem 4. *Necessary and sufficient conditions in order that a sequence $\{C_n\}$, $n = 0, \pm 1, \pm 2, \dots$, of operators in $\mathcal{L}(E, F'')$ with $\{\gamma_{nxx}\} \in \ell_q$ for some $1 \leq q < \infty$ for every $x \in E$ and $z \in F'$, be a Fourier sequence of a vector measure $\mathbf{m}$ with values in $\mathcal{L}(E, F'')$ and such that $\bar{\mathbf{m}}_p[0, 1] < \infty$ where $1/p + 1/q = 1$, are that $\mu_{nm}(C)$ is continuous for $0 \leq m \leq n = 0, 1, 2, \dots$ and that*

$$(4.1) \quad \sup_{n \geq 0} (n+1)^{p-1} \sum_{m=0}^n \|\mu_{nm}(C)\|^p < \infty.$$

Proof. Assume first that

$$C_n = \int_0^1 e^{-2\pi i n t} d\mathbf{m}(t), \quad n = 0, \pm 1, \pm 2, \dots,$$

where $\mathbf{m}$ assume values in $\mathcal{L}(E, F'')$ and $\bar{\mathbf{m}}_p[0, 1] < \infty$. Since F' is a norming subspace of F'''' it follows by [1; p. 279, Theorem 8] that there exists a function $\mathbf{G}$ on $[0, 1]$ into $\mathcal{L}(E, F'')$ with $\|\mathbf{G}\| \in \mathcal{L}^p[0, 1]$ such that $\gamma_{nxx} = \int_0^1 e^{-2\pi i n t} \langle \mathbf{G}(t)x, z \rangle dt$, $n = 0, \pm 1, \pm 2, \dots$, $x \in E$, $z \in F'$. Since by our assumption $\{\gamma_{nxx}\}$, $n = 0, \pm 1, \pm 2, \dots$, is in ℓ_q for every $x \in E$ and $z \in F'$ we obtain, as in Rooney [4; p. 766],

$$|\langle \mu_{nm}(C)x, z \rangle| \leq (n+1)^{-1/q} \binom{n}{m} \left(\int_0^1 t^m (1-t)^{n-m} |\langle \mathbf{G}(t)x, z \rangle|^p dt \right)^{1/p}$$

and thus

$$(4.2) \quad \begin{aligned} \|\mu_{nm}(C)\| &= \sup_{\substack{\|x\| \leq 1 \\ \|z\| \leq 1}} |\langle \mu_{nm}(C)x, z \rangle| \\ &\leq (n+1)^{-1/q} \binom{n}{m} \int_0^1 t^m (1-t)^{n-m} \|\mathbf{G}(t)\|^p dt \end{aligned} < \infty.$$

Moreover by (4.2)

$$\begin{aligned} (n+1)^{p-1} \sum_{m=0}^n \|\mu_{nm}(C)\|^p &\leq \int_0^1 \sum_{m=0}^n \binom{n}{m} t^m (1-t)^{n-m} \|\mathbf{G}(t)\|^p dt \\ &= \int_0^1 \|\mathbf{G}(t)\|^p dt < \infty. \end{aligned}$$

This completes the proof of the necessity.

Conversely, assume that the $\mu_{nm}(C)$ are continuous and that (4.1) holds.

If $\{\mu_n\} \subseteq \mathcal{L}(E, F'')$ is defined by

$$\langle \mu_n x, z \rangle = \sum_{r=-\infty}^{\infty} \gamma_{rz} a_{rnn}, \quad n = 0, 1, 2, \dots,$$

and if

$$\lambda_{nm} = \binom{n}{m} \Delta^{n-m} \mu_m, \quad 0 \leq m \leq n = 0, 1, 2, \dots$$

(see (2.8)), then by [4; p. 767]

$$\langle \lambda_{nm} x, z \rangle = \langle \mu_{nm}(C)x, z \rangle \text{ for all } x \in E, z \in F''$$

and thus

$$\lambda_{nm} = \mu_{nm}(C), \quad 0 \leq m \leq n = 0, 1, 2, \dots$$

By (4.1) it follows that

$$(4.3) \quad \sup_{n \geq 0} (n+1)^{p-1} \sum_{m=0}^n \|\lambda_{nm}\|^p < \infty$$

and since

$$\int_0^1 \binom{n}{m} t^m (1-t)^{n-m} dt = \frac{1}{n+1},$$

it follows by (4.3) and Theorem 2 that there exists a vector measure $\mathbf{m}$ with values in $\mathcal{L}(E, F'')$ and such that $\|\mathbf{m}\|_p[0, 1] < \infty$, such that

$$(4.4) \quad \mu_n = \int_0^1 t^n d\mathbf{m}(t), \quad n = 0, 1, 2, \dots$$

We complete the proof, now, exactly as in Rooney [4].

REFERENCES

- [1] N. DINULEANU, *Vector Measures*, Inter. Ser. of Mono. in Pure and Appl. Math., Vol. 95, Pergamon Press, 1967.
- [2] D. LEVIATAN, A generalized moment problem, *Israel J. Math.*, 5 (1967) 97-103.
- [3] G. G. LORENTZ, *Bernstein Polynomials*, Univ. of Toronto Press, Toronto, 1953.
- [4] P. G. ROONEY, On the representation of sequences as Fourier coefficients, *Proc. Amer. Math. Soc.*, 11 (1960) 762-768.
- [5] I. J. SCHOENBERG, On finite-rowed systems of linear inequalities in infinitely many variables, *Trans. Amer. Math. Soc.*, 34 (1932) 594-619.
- [6] A. ZYGMUND, *Trigonometric Series*, Cambridge Univ. Press, Cambridge, 1959.

University of Illinois

University of Michigan

Date communicated: SEPTEMBER 19, 1969