On the Growth of Entire Functions of Bounded Index

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1. Introduction. Let $f(z)$ be a transcendental entire function. If there exists an integer N , independent of z , such that

$$\frac{|f^{(n)}(z)|}{n!} \leq \max_{0 \leq k \leq N} \left\{ \frac{|f^{(k)}(z)|}{k!} \right\}$$

for all n and all z then $f(z)$ is said to be of bounded index. The least such integer N is called the index of $f(z)$. It is known [3] that in this case the growth of $f(z)$ is at most exponential type $(1, N + 1)$. If $f(z)$ satisfies a linear differential equation with polynomial coefficients

$$(1.1) \quad P_0(z)f^{(k)}(z) + P_1(z)f^{(k-1)}(z) + \cdots + P_k(z)f(z) = Q(z)$$

where $P_j(z)$, $j = 0, 1, \dots, k$, $Q(z)$ are polynomials and $P_0(z) (\neq 0)$ is of degree not less than that of any $P_j(z)$, then $f(z)$ is of bounded index [4]. Pugh and Shah [2] have shown that if

$$f(z) = \Pi_1^\infty \left(1 - \frac{z}{z_j} \right), \quad |z_1| \geq 5, \quad |z_{j+1}| \geq 5^j |z_j| \quad (j = 1, 2, \dots),$$

then $f(z)$ is of bounded index. If the zeros a_n of the canonical product $f(z)$ are real and positive and $a_{n+1}/a_n \geq \gamma > 1$, then $f(z)$ is also of bounded index [1]. We extend this result and prove

Theorem 1. *Let*

$$f(z) = e^{\alpha z + \beta} \Pi_1^\infty \left(1 - \frac{z}{a_n} \right)$$

where α and β are any complex numbers, and

$$(1.2) \quad a_{n+1}/a_n \geq \gamma > 1, \quad a_1 > 0.$$

Then each $f^{(k)}(z)$, $k = 0, 1, \dots$, is of bounded index.

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The functions $f(z)$ satisfying (1.1) are necessarily of positive rational order and of perfectly regular growth. We now show that there exist entire functions f of bounded index and arbitrarily slow growth.

Theorem 2. *Given an indefinitely increasing sequence $\{\psi_n\}_1^\infty$ of positive numbers and a transcendental entire function $g(z)$, $g(0) \neq 0$, there exists an entire function $f(z)$ of bounded index such that, for all z and $k = 0, 1, \dots$, we have*

$$(1.3) \quad M(|z|, g) > M(|z|, f^{(k)}) > \psi_{k+1} M(|z|, f^{(k+1)}).$$

2. Proof of Theorem 1. We require the following lemmas.

Lemma 1. *Let $f(z)$ be an entire function and R and M be given positive numbers. Then there exists an integer p such that for $|z| \leq R$,*

$$\max_{0 \leq k \leq p} \left\{ \frac{|f^{(k)}(z)|}{k!} \right\} \geq M \frac{|f^{(j)}(z)|}{j!}, \quad j = p+1, p+2, \dots$$

The proof is similar to that of (2.7) of [4] and is omitted.

Lemma 2. ([1]) *Let*

$$(2.1) \quad P(z) = \Pi_1^\infty \left(1 - \frac{z}{a_n} \right)$$

where the zeros a_n satisfy the condition (1.2).

Then given any $M > 0$, there exists a number $R = R(M, P)$ such that for every z , $|z| > R$,

$$\max \{ |P(z)|, |P'(z)| \} > M \frac{|P^{(j)}(z)|}{j!}, \quad j = 2, 3, \dots$$

Let $\mathcal{E}$ denote the class of entire functions f such that given any $M > 0$, there exists $R = R(M, f)$ with the property that for every z , $|z| > R$,

$$\max \{ |f(z)|, |f'(z)| \} > M \frac{|f^{(j)}(z)|}{j!}, \quad j \geq 2.$$

Lemma 2 shows that the class $\mathcal{E}$ is not empty.

Lemma 3. *If $Q(z) \in \mathcal{E}$, $Q'(z) \in \mathcal{E}$ then $Q'(z) + \lambda Q(z) \in \mathcal{E}$ where λ is any complex number.*

Proof. We may assume $\lambda \neq 0$. Let $M > 2$ be given. Denote

$$H(z) = Q'(z) + \lambda Q(z),$$

and

$$M_0 = \left(1 + \frac{1}{|\lambda|} \right)^2 M + \frac{2}{|\lambda|^2}.$$

Since $Q \in \mathcal{E}$ there exists $R_0 = R_0(M_0)$ such that for every z , $|z| > R_0$,

$$(2.2) \quad \max \{|Q(z)|, |Q'(z)|\} > M_0 \frac{|Q^{(j)}(z)|}{j!}, \quad j \geq 2.$$

We consider two cases. Let z_0 be a given point such that $|z_0| > R_0$ and suppose first that $|Q'(z_0)| \geq |Q(z_0)|$. Then

$$\begin{aligned} |H'(z_0)| &\geq |\lambda| |Q'(z_0)| - |Q''(z_0)| \\ &\geq |\lambda| |Q'(z_0)| - \frac{2}{M_0} |Q'(z_0)| \\ &= \left(|\lambda| - \frac{2}{M_0}\right) \max \{|Q(z_0)|, |Q'(z_0)|\}. \end{aligned}$$

If $|Q'(z_0)| < |Q(z_0)|$, then

$$\lambda H(z_0) - H'(z_0) = \lambda^2 Q(z_0) - Q''(z_0).$$

Hence

$$\begin{aligned} (|\lambda| + 1) \max \{|H(z_0)|, |H'(z_0)|\} &> |\lambda H(z_0) - H'(z_0)| \\ &= |\lambda^2 Q(z_0) - Q''(z_0)| \\ &> |\lambda^2 Q(z_0)| - \frac{2}{M_0} \max \{|Q(z_0)|, |Q'(z_0)|\} \\ &= \left(|\lambda|^2 - \frac{2}{M_0}\right) \max \{|Q(z_0)|, |Q'(z_0)|\}. \end{aligned}$$

Consequently

$$\max \{|H(z_0)|, |H'(z_0)|\} > \left[\frac{|\lambda|^2 - \frac{2}{M_0}}{|\lambda| + 1} \right] \max \{|Q(z_0)|, |Q'(z_0)|\}.$$

Since

$$|\lambda| - \frac{2}{M_0} > \frac{|\lambda|^2 - \frac{2}{M_0}}{|\lambda| + 1},$$

we have in both cases

$$(2.3) \quad \max \{|H(z)|, |H'(z)|\} > \left[\frac{|\lambda|^2 - \frac{2}{M_0}}{|\lambda| + 1} \right] \max \{|Q(z)|, |Q'(z)|\}.$$

Now $H^{(j)}(z) = Q^{(j+1)}(z) + \lambda Q^{(j)}(z)$ and hence for every z ,

$$(2.4) \quad M_0 \frac{|H^{(j)}(z)|}{j!} \leq (|\lambda| + 1) \max \left\{ M_0 \frac{|Q^{(j+1)}(z)|}{j!}, M_0 \frac{|Q^{(j)}(z)|}{j!} \right\}, \quad j \geq 1.$$

Since $Q'(z) \neq 0$ we have for $|z| > R_1 = R_1(M_0)$

$$(2.5) \quad \max \{|Q'(z)|, |Q''(z)|\} > M_0 \frac{|Q^{(j+1)}(z)|}{j!}, \quad j \geq 2.$$

Let $R = \max(R_0, R_1)$. Then (2.2) and (2.5) imply, for $|z| > R$,

$$\max \{|Q(z)|, |Q'(z)|\} > \frac{M_0}{2} |Q''(z)| > |Q''(z)|,$$

and hence

$$(2.6) \quad \max \{|Q(z)|, |Q'(z)|\} \geq M_0 \max \left\{ \frac{|Q^{(j)}(z)|}{j!}, \frac{|Q^{(j+1)}(z)|}{j!} \right\}, \quad j \geq 2.$$

For $|z| > R$ and $j \geq 2$, we have from (2.3) (2.6) and (2.4),

$$\max \{|H(z)|, |H'(z)|\} > M \frac{|H^{(j)}(z)|}{j!}.$$

This proves the lemma.

Lemma 4. *If $Q(z) \in \mathcal{E}$ and λ is any complex number then the entire function $e^{\lambda z} Q(z)$ is of bounded index.*

Proof. Write $f(z) = e^{\lambda z} Q(z)$ and choose $M \geq 2(1 + |\lambda|)e^{|\lambda|}$. Then

$$\frac{|f^{(k)}(z)|}{k!} \leq |e^{\lambda z}| \sum_{j=0}^k \frac{k!}{j! (k-j)!} \frac{|\lambda|^j |Q^{(k-j)}(z)|}{k!}.$$

Hence, by Lemma 2, we have for $|z| > R$

$$\begin{aligned} \frac{|f^{(k)}(z)|}{k!} &\leq |e^{\lambda z}| \left(\frac{e^{|\lambda|}}{M} + \frac{|\lambda|^{(k-1)}}{(k-1)!} + \frac{|\lambda|^k}{k!} \right) \max \{|Q(z)|, |Q'(z)|\} \\ &< \left(\frac{e^{|\lambda|}}{M} + \frac{|\lambda|^{(k-1)}}{(k-1)!} + \frac{|\lambda|^k}{k!} \right) (1 + |\lambda|) \max \{|f(z)|, |f'(z)|\}. \end{aligned}$$

Choose L such that, for $k \geq L$,

$$\left\{ \frac{|\lambda|^{k-1}}{(k-1)!} + \frac{|\lambda|^k}{k!} \right\} (1 + |\lambda|) \leq 1/2.$$

Then for $k \geq N = \max(M, L)$ and $|z| > R$.

$$\max \{|f(z)|, |f'(z)|\} \geq \frac{|f^{(k)}(z)|}{k!}.$$

This inequality and Lemma 1 complete the proof.

Proof of Theorem 1.

Write

$$Q_0(z) = P(z),$$

$$Q_n(z) = Q'_{n-1}(z) + \alpha Q_{n-1}(z), \quad n = 1, 2, \dots$$

We may suppose that $\beta = 0$. Then Lemmas 2 and 4 prove that $f(z)$ is of bounded index. Furthermore by Laguerre's Theorem we see that the zeros of $P'(z)$ satisfy

a condition of the type (1.2) with γ replaced by another number $\gamma' > 1$ (cf: (2.3) of [1]). Hence $P'(z) \in \mathcal{E}$ and so Lemmas 3 and 4 imply that $Q_1(z) \in \mathcal{E}$ and $e^{\alpha z} Q_1(z) = f'(z)$ is of bounded index.

Similarly $P^{(k)}(z) \in \mathcal{E}$ for every $k \geq 1$ and so $Q_1^{(k)}(z) = \alpha P^{(k)}(z) + P^{(k+1)}(z) \in \mathcal{E}$. An induction argument on n shows that $Q_n^{(k)}(z) \in \mathcal{E}$. Hence $f^{(n)}(z) = e^{\alpha z} Q_n(z)$ ($n \geq 1$) is of bounded index.

3. Proof of Theorem 2. We require a lemma.

Lemma 5. ([1]) Suppose $P(z)$ is defined by (2.1) where $a_{n+1}/a_n \geq n + 1$, $n = 2, 3, \dots$, $a_2 > a_1 > 0$. Write $B = \max(A, 2A^2)$ where $A = (7.5)/(a_2 - a_1)$. Then for every z

$$B \max \{|P(z)|, |P'(z)|\} > |P''(z)|.$$

We now prove Theorem 2. We may assume that $1 < \psi_1 < \psi_2 < \dots$. Since $g(z)$ is transcendental, $\log M(r, g)/\log r$ is a strictly increasing function of r for $r \geq r_0(g)$.

We take $r_0 > 1$ and let

$$(3.1) \quad h(r) = \begin{cases} \frac{\log M(r, g)}{2 \log r}, & r \geq r_0 \\ h(r_0), & r < r_0. \end{cases}$$

Then $h(r) \rightarrow \infty$ as $r \rightarrow \infty$. Choose $a_1 \geq \max(8, r_0, 2\psi_1)$ such that $h(a_1) \geq 1$. Having chosen $a_1, a_2, \dots, a_n$ choose

$$(3.2) \quad a_{n+1} \geq \max \{(n+2)a_n, 2(1+n)(2+n)\psi_{n+1}\}, n \geq 1$$

such that $h(a_{n+1}) \geq n + 1$. Let

$$(3.3) \quad F(z) = \Pi_1^\infty \left(1 - \frac{z}{a_n}\right).$$

Then $\log M(r, F) \sim N(r, 1/F) \leq n(r, 1/F) \log r$

$$\begin{aligned} &\leq h(r) \log r \\ &\leq \frac{\log M(r, g)}{2}. \end{aligned}$$

Hence $M(r, F) \leq M(r, g)$, $r \geq r_1 > 0$. Write

$$c = |g(0)|, k = \max(1, c)$$

and

$$(3.4) \quad \begin{aligned} f(z) &= \frac{c}{k} \frac{F(z)}{F(-r_1)} \\ &= f(0) \Pi_1^\infty \left(1 - \frac{z}{a_n}\right). \end{aligned}$$

Then $f(z)$ is an entire function and

$$(3.5) \quad M(r, f) < \frac{c}{k} M(r, g) \quad \text{for } r \leq r_1 .$$

When $r > r_1$,

$$(3.5)' \quad M(r, f) = \frac{c}{k} \frac{1}{F(-r_1)} M(r, F) < M(r, g).$$

Denote the zeros of $f^{(k)}(z)$ by $\{a_n^{(k)}\}_{n=1}^\infty$. Then we have the following

Lemma 6. (i) If $a_{n+1}/a_n \geq n + 2$ then for every $k \geq 1$, $a_{n+1}^{(k)}/a_n^{(k)} \geq n + 2$.
(ii) For every $k \geq 0$, $n \geq 1$

$$a_n^{(k)} > \frac{n(n+1)}{(n+k)(n+k+1)} a_{n+k} .$$

We omit the proof since these results are similar to those in Lemma 4 (ii) and (5.7) of [1].

Now for $r \geq 0$,

$$\begin{aligned} M(r, f') &= |f(0)| \left(\sum_1^\infty \frac{1}{a_n} \right) \Pi_1^\infty \left(1 + \frac{r}{a_n^{(1)}} \right) \\ &\leq \left(\sum_1^\infty \frac{1}{a_n} \right) M(r, f) < \frac{2}{a_1} M(r, f) \leq \frac{1}{\psi} M(r, f). \end{aligned}$$

By Lemma 5 and Lemma 6 we have for every z

$$\begin{aligned} \max \{ |F(z)|, |F'(z)| \} &> \frac{a_2}{12} |F''(z)| \geq \psi_2 |F''(z)|; \\ \max \{ |F^{(k)}(z)|, |F^{(k+1)}(z)| \} &> \frac{a_2^{(k)}}{12} |F^{(k+2)}(z)| \\ &> |F^{(k+2)}(z)| \frac{a_{2+k}}{2(2+k)(3+k)} \\ &> |F^{(k+2)}(z)| \psi_{k+2} . \end{aligned}$$

This proves that, for every z and $k \geq 0$,

$$(3.6) \quad \max \{ |f^{(k)}(z)|, |f^{(k+1)}(z)| \} > |f^{(k+2)}(z)| \psi_{k+2} .$$

Now

$$f^{(k+1)}(z) = f^{(k+1)}(0) \Pi_1^\infty \left(1 - \frac{z}{a_n^{(k+1)}} \right) ,$$

and so by Lemma 6,

$$\begin{aligned}
M(r, f^{(k+1)}) &= |f^{(k)}(0)| \left(\sum_{n=1}^{\infty} \frac{1}{a_n^{(k)}} \right) \Pi_1^{\infty} \left(1 + \frac{r}{a_n^{(k+1)}} \right) \\
&< \frac{2}{a_1^{(k)}} M(r, f^{(k)}) \\
&< \frac{(k+1)(k+2)}{a_{k+1}} M(r, f^{(k)}) < \frac{M(r, f^{(k)})}{\psi_{k+1}};
\end{aligned}$$

that is,

$$(3.7) \quad M(r, f^{(k)}) > \psi_{k+1} M(r, f^{(k+1)}).$$

The inequality (3.6) shows that each $f^{(k)}(z)$ is of bounded index $N_k = 1$. The inequality (1.3) follows from (3.5), (3.5)' and (3.7).

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