Nonlinear Parabolic Initial-Value Problems

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Communicated by R. PHILLIPS

Nonlinear parabolic initial-value problems have been studied by Browder [1] using the theory of monotone operators involving a densely defined operator in a Banach space X and by Lions [3] using monotone operators defined on all of X together with the method of elliptic regularization. The above methods, in spite of their generalities, cannot be applied to the study of the non-stationary Navier-Stokes Equations.

The purpose of this paper is to extend the results of [1], [3] (in the case when X is a Hilbert space) so as to include the Navier Stokes Equations. The method used is based on that of Lions [2].

The hypotheses are given in Section 1. An existence theorem is proved in Section 2. Applications are given in Section 3. First, a parabolic equation whose elliptic part is of divergence form $A(u) = \sum_{|\alpha| \leq m} D^\alpha(x, t, u, \dots, D^m u)$ with $A_\alpha(x, t, \xi_1, \dots, \xi_m)$ having at most a linear growth in $(\xi_1, \dots, \xi_m)$ is considered. Then the generalized Navier-Stokes Equations are studied. The usual Laplacian in the equations is replaced by a nonlinear elliptic operator of divergence form and the space-domain is a bounded open subset of R^n with $n \leq 4$.

Section 1 Let V and H be two real Hilbert spaces with a continuous natural injection mapping of V into H with V dense in H . Let $(\cdot, \cdot)$ denote the inner product in H and $((\cdot, \cdot))$ denote the pairing between V and its dual V^* .

Consider the Hilbert space $L^2(0, T; V)$ of equivalence classes of functions v from $[0, T)$ to V with $\int_0^T \|u(t)\|_V^2 dt < \infty$. The norm in this Hilbert space is

$$\|u\|_{L^2(0, T; V)} = \left\{ \int_0^T \|u(t)\|_V^2 dt \right\}^{1/2}.$$

Similarly, we define $L^2(0, T; H)$. Let $g \in L^2(0, T; V^*)$, $f \in L^2(0, T; V)$; then $\langle g, f \rangle$ denotes the pairing between $L^2(0, T; V^*)$ and $L^2(0, T; V)$. If $g \in L^2(0, T; H)$ and $f \in L^2(0, T; V)$, then

* Research sponsored by the U. S. Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant Nr. AFOSR-68-1531.

$$\langle g, f \rangle = \int_0^T \langle g(t), f(t) \rangle dt.$$

$\mathcal{H}_\gamma(-\infty, +\infty; V, H)$ is the space of elements $u \in L^2(-\infty, +\infty; V)$ such that $|\tau|^\gamma \hat{u}(\tau) \in L^2(-\infty, +\infty; H)$ with the norm

$$\|u\|_{\mathcal{H}_\gamma(-\infty, +\infty; V, H)} = \left\{ \int_{-\infty}^{\infty} \|u(t)\|_V^2 dt + \int_{-\infty}^{\infty} |\tau|^{2\gamma} \|\hat{u}(\tau)\|_H^2 d\tau \right\}^{1/2},$$

where $\hat{u}(\tau)$ is the Fourier transform of $u(t)$ with respect to t and $0 < \gamma$.

$\mathcal{H}_\gamma(0, T; V, H)$ is the space of restrictions to $(0, T)$ of elements in $\mathcal{H}_\gamma(-\infty, +\infty; V, H)$ with the norm

$$\|u\|_{\mathcal{H}_\gamma(0, T; V, H)} = \inf \|v\|_{\mathcal{H}_\gamma(-\infty, +\infty; V, H)}; v = u \text{ a.e. on } (0, T).$$

$D_t u \in L^2(0, T; H)$ for $0 < \gamma < 1$ means that there exists $U \in L^2(-\infty, +\infty; V)$ with $U = u$ a.e. on $(0, T)$ and $U \in \mathcal{H}_\gamma(-\infty, +\infty; V, H)$.

Let $A(t, u, v)$ be a nonlinear mapping of $L^2(0, T; V)$ into $L^2(0, T; V^*)$. We assume:

Assumption (I) (i) $((A(t, u, v) - A(t, u, w), v - w)) \geq 0$ for all u, v, w in V and for almost all $t \in (0, T)$.

(ii) $\|A(t, u, v)\|_{V^*} \leq M \|v\|_V$; u, v in V . M is a constant independent of t and u .

(iii) There exists a positive continuous function $c(r)$ with $c(r) \rightarrow +\infty$ as $r \rightarrow +\infty$ such that

$$\int_0^T ((A(t, u, u), u)) dt \geq c(\|u\|_{L^2(0, T; V)}) \|u\|_{L^2(0, T; V)}$$

for all u in $L^2(0, T; V)$.

(iv) For fixed u in V , (respectively in $L^2(0, T; V)$) $A(t, u, \cdot)$ is continuous from line segments in V (respectively in $L^2(0, T; V)$) to the weak topology of V^* (respectively to the weak topology of $L^2(0, T; V^*)$).

(v) For fixed v in V , if $u_n \rightarrow u$ weakly in $L^2(0, T; V)$ and if $D_t^\gamma u_n \rightarrow D_t^\gamma u$ weakly in $L^2(0, T; H)$ for some γ with $0 < \gamma \leq 1$, then $A(t, u_n, v) \rightarrow A(t, u, v)$ in $L^2(0, T; V^*)$.

We consider nonlinear mappings $B(u, v)$ of $V \times V$ into V^* satisfying the following conditions:

Assumption (II). (i) $((B(u, v), w)) + ((B(u, w), v)) = 0$ for all u, v, w in V .

(ii) $\|B(u, v)\|_{V^*} \leq M \|u\|_V \|v\|_V$ for all u, v in V .

(iii) $B(u) = B(u, u)$ is weakly continuous from V to V^* .

(iv) If $u_n \rightarrow u$ weakly in $L^2(0, T; V)$ and if $D_t^\gamma u_n \rightarrow D_t^\gamma u$ weakly in $L^2(0, T; H)$ for some γ with $0 < \gamma \leq 1$, then $B(u_n) \rightarrow B(u)$ weakly in $L^1(0, T; V^*)$.

Finally, we shall make the following assumption throughout the paper:

Assumption (III). Suppose that Assumptions (I), (II) are satisfied. Suppose further that: (i) There exists $\{u_n\}$ with $D_t u_n \in L^2(0, T; V^*)$, $u_n \rightarrow u$ weakly in $L^2(0, T; V)$, $D_t^\gamma u_n \rightarrow D_t^\gamma u$ weakly in $L^2(0, T; H)$ for some γ with $0 < \gamma \leq 1$ and $u_n(0) \rightarrow u(0)$ weakly in H . (ii) There exists $f \in L^2(0, T; V^*)$ such that

$$(a) \liminf \{ \|u_n(0)\|_H^2 + \langle D_t u_n + A(u_n, u_n), u_n \rangle \} \leq \langle f, u \rangle + \|u(0)\|_H^2,$$

$$(b) \langle D_t u_n + A(u_n, u_n), \varphi \rangle + \int_0^T \langle B(u_n), \varphi \rangle dt \rightarrow \langle f, \varphi \rangle$$

for all $\varphi \in C_c^\infty(0, T; V)$.

Then, $A(t, u_n, u_n) \rightarrow A(t, u, u)$ weakly in $L^2(0, T; V^*)$.

Before proving an existence theorem let us make the following remark about Assumption (III).

Lemma 1. Suppose that V is dense in H and $B(u, v) = 0$ for all u, v in V . Then Assumption (III) is verified.

Proof. 1.) Let $\varphi \in C_c^\infty(0, T; V)$, then

$$\langle D_t u_n + A(u_n, u_n), \varphi \rangle = -\langle u_n, D_t \varphi \rangle + \langle A(u_n, u_n), \varphi \rangle.$$

Since $\|u_n\|_{L^2(0, T; V)} \leq M$, it follows from Assumption (I) and the weak compactness of the unit ball in a Hilbert space that $A(u_n, u_n) \rightarrow g$ weakly in $L^2(0, T; V^*)$ as $n \rightarrow +\infty$.

Hence

$$-\langle u, D_t \varphi \rangle + \langle g, \varphi \rangle = \langle f, \varphi \rangle \quad \text{for all } \varphi \in C_c^\infty(0, T; V).$$

Therefore

$$D_t u = f - g \in L^2(0, T; V^*).$$

2.) Let $\varphi \in L^2(0, T; V)$ with $D_t \varphi \in L^2(0, T; V^*)$. It is well-known that

$$\langle D_t(u_n - \varphi), u_n - \varphi \rangle = \frac{1}{2} \|u_n(T) - \varphi(T)\|_H^2 - \frac{1}{2} \|u_n(0) - \varphi(0)\|_H^2.$$

Hence $\langle D_t(u_n - \varphi) + A(u_n, u_n) - A(u_n, \varphi), u_n - \varphi \rangle + \|u_n(0) - \varphi(0)\|_H^2 \geq 0$. The inequality may be rewritten as:

$$\begin{aligned} & \langle D_t u_n + A(u_n, u_n), u_n \rangle + \|u_n(0)\|_H^2 - \langle D_t u_n, \varphi \rangle - \langle A(u_n, u_n), \varphi \rangle \\ & - \langle D_t \varphi + A(u_n, \varphi), u_n - \varphi \rangle + \|\varphi(0)\|_H^2 - 2\langle u_n(0), \varphi(0) \rangle \geq 0. \end{aligned}$$

Taking into account Assumption (I), we get:

$$\|u(0)\|_H^2 + \langle f, u - \varphi \rangle - \langle D_t \varphi + A(u, \varphi), u - \varphi \rangle + \|\varphi(0)\|_H^2 - 2\langle u(0), \varphi(0) \rangle \geq 0.$$

Thus $\langle f - D_t \varphi - A(u, \varphi), u - \varphi \rangle + \|u(0) - \varphi(0)\|_H^2 \geq 0$. But $f = D_t u + g$, hence:

$$\langle D_t(u - \varphi) + g - A(u, \varphi), u - \varphi \rangle + \|u(0) - \varphi(0)\|_H^2 \geq 0.$$

By a standard argument of the theory of monotone operators (e.g. cf. Lemma 2.1 of [1], p. 345) we obtain: $g = A(u, u)$. Q.E.D.

Section 2. The main result of the paper is the following theorem,

Theorem 1. Suppose that Assumptions (I), (II), (III) are satisfied. Then for each f in $L^2(0, +\infty; V^*)$ and for each u_0 in V_0 , the closure of V in the H -norm, there exists $u \in L^2(0, +\infty; V)$ with $D_t^\gamma u \in L^2(0, +\infty; H)$ for any γ , $0 < \gamma < \frac{1}{4}$, satisfying the equation:

$$\begin{aligned} \int_0^\infty (u, D_t \varphi) dt + \int_0^\infty ((A(t, u, u), \varphi)) dt + \int_0^\infty ((B(u, u), \varphi)) dt \\ = \int_0^\infty ((f, \varphi)) dt + (u_0, \varphi(0)), \end{aligned}$$

for all continuous functions $\varphi(t)$ from R^+ to V with compact support and such that $D_t \varphi \in L^2(0, +\infty; H)$.

Moreover if $B(u, v) = 0$ for all u, v in V then $D_t u \in L^2(0, +\infty; V^*)$.

Proof. 1.) Let $v_1, v_2, \dots, v_n, \dots$ be a basis for V and set

$$u_i(t) = \sum_{i=1}^j g_{ii}(t) v_i.$$

Since $u_0 \in V_0$, there exist real numbers α_{ii} such that

$$\sum_{i=1}^j \alpha_{ii} v_i \rightarrow u_0 \quad \text{in } H.$$

Consider the system of nonlinear ordinary differential equations in $g_{ii}(t)$:

$$(2.1) \quad \begin{cases} (D_t u_i, v_k) + ((A(t, u_i, u_i), v_k)) + ((B(u_i, u_i), v_k)) = ((f, v_k)) \\ g_{ii}(0) = \alpha_{ii}; \quad k = 1, \dots, j. \end{cases}$$

2.) We show the existence of a local solution of (2.1) by applying the Caratheodory's Theorem.

Let $F(t, w) = ((A(t, w, w) + B(w, w) - f(t), v_k))$ for w in V . Taking into account Assumptions (I), (II), we have

$$|F(t, w)| \leq \{ \|f(t)\|_{V^*} + M \|w\|_V^2 + M' \|w\|_V \} \|v_k\|_V.$$

Hence $F(t, w)$ is bounded by a Lebesgue integrable function on $[0, T]$ for $T < \infty$.

$F(t, w)$ is continuous in w . Indeed suppose $w_n \rightarrow w$ in V . Then it follows from Assumption (II) that $((B(w_n, w_n), v_k)) \rightarrow ((B(w, w), v_k))$. On the other hand $\|A(t, w_n, w_n)\|_{V^*} \leq M \|w_n\|_V \leq C$. So from the weak compactness of the unit ball in a Hilbert space we get: $A(t, w_n, w_n) \rightarrow g$ weakly in V^* .

Consider $((A(t, w_n, w_n) - A(t, w_n, v), w_n - v)) \geq 0$. Since w_n and w are independent of t , we have by using Assumption (I) $((g - A(t, w, v), w - v)) \geq 0$ for all v in V and for almost all $t \in [0, T]$ with $T < \infty$. A standard argument of the theory of monotone operators gives $g = A(t, u, u)$. Hence $|F(t, w_n) - F(t, w)| \leq \epsilon$ when $\|w_n - w\|_V \leq \delta(\epsilon)$.

Taking

$$w_n = \sum_{i=1}^j \xi_{ii}^{(n)} v_i, \quad w = \sum_{i=1}^j \xi_{ii} v_i,$$

where $\zeta_{ij}^{(n)}$, ζ_{ij} are numbers and with $\zeta_{ij}^{(n)} \rightarrow \zeta_{ij}$, we see that the hypotheses of the Caratheodory's Theorem are satisfied. The initial-value problem (2.1) has a solution $g_{ij}(t)$ defined on $[0, T_i]$ for $T_i < T$.

3.) We show that the local solution of (2.1) is in fact a global solution. Multiplying the two sides of (2.1) by g_{ik} and summing with respect to k from 1 to j , we get by taking into account Assumption (II)

$$\frac{1}{2} \frac{d}{dt} \|u_i(t)\|_H^2 + ((A(t, u_i, u_i), u_i)) = ((f(t), u_i)).$$

Thus

$$\begin{aligned} \frac{1}{2} \|u_i(t)\|_H^2 + \int_0^t ((A(\sigma, u_i, u_i), u_i(\sigma))) d\sigma &\leq \frac{1}{2} \|u_i(0)\|_H^2 \\ &+ \int_0^t ((f(\sigma), u_i(\sigma))) d\sigma \quad \text{for } t < T_i. \end{aligned}$$

Hence $c(\|u_i\|_{L^2(0,t;V)}) \leq \|f\|_{L^2(0,+\infty;V^*)} + C \|u_0\|_H^2 / \|u_i\|_{L^2(0,t;V)}$. If $\|u_i\|_{L^2(0,t;V)} \rightarrow +\infty$ as $t \rightarrow +\infty$; then $\lim c(\|u_i\|_{L^2(0,t;V)}) \leq M$. This is impossible since by hypothesis $c(r) \rightarrow +\infty$ as $r \rightarrow +\infty$. Therefore $\|u_i\|_{L^2(0,+\infty;V)} < \infty$.

4.) We prove that $\|u_i\|_{L^2(0,+\infty;V)}$ are uniformly bounded. It follows from the previous part that

$$c(\|u_i\|_{L^2(0,+\infty;V)}) \leq \|f\|_{L^2(0,+\infty;V^*)} + C \|u_0\|_H^2 / \|u_i\|_{L^2(0,+\infty;V)}.$$

As above we get $\|u_i\|_{L^2(0,+\infty;V)} \leq M$ where M is a constant independent of j .

5.) We establish estimates for fractional derivatives of u_i . Extend $u_i(t)$ to be zero for $t < 0$ and denote the extension by $\tilde{u}_i(t)$. Since by hypotheses $A(v, 0) = B(v, 0) = 0$, we have

$$\begin{aligned} (2.2) \quad \frac{d}{dt} (\tilde{u}_i, v_k) + ((A(t, \tilde{u}_i, \tilde{u}_i), v_k)) + ((B(\tilde{u}_i, \tilde{u}_i), v_k)) \\ = ((\tilde{f}, v_k)) + (u_i(0), v_k) \delta. \end{aligned}$$

$\tilde{f}(t)$ is the function f extended to be zero for $t < 0$. Set $a_i = A(t, \tilde{u}_i, \tilde{u}_i)$; $b_i = B(\tilde{u}_i, \tilde{u}_i)$.

Then

$$\int_{-\infty}^{\infty} \|a_i\|_{V^*}^2 dt \leq M \int_{-\infty}^{\infty} \|\tilde{u}_i\|_V^2 dt \leq M_2,$$

and

$$\int_{-\infty}^{\infty} \|b_i\|_{V^*}^2 dt \leq M \int_{-\infty}^{\infty} \|\tilde{u}_i\|_V^2 dt \leq M_3.$$

Taking the Fourier transform with respect to t of both sides of (2.2) we get

$$i\tau(\hat{u}_i(\tau), v_k) + ((\hat{a}_i(\tau) + \hat{b}_i(\tau), v_k)) = ((\hat{f}(\tau), v_k)) + (u_i(0), v_k).$$

Multiplying the two sides of the above equation by $g_{k_i}(\tau)$ and summing with respect to k , we obtain

$$i\tau \|\hat{u}_i(\tau)\|_H^2 + ((\hat{a}_i(\tau) + \hat{b}_i(\tau), \hat{u}_i(\tau))) = ((\hat{f}(\tau), \hat{u}_i(\tau))) + (u_i(0), \hat{u}_i(\tau)).$$

Hence

$$|\tau| \|\hat{u}_i(\tau)\|_H^2 \leq C \|u_0\|_H \|\hat{u}_i(\tau)\|_H + \{\|\hat{f}(\tau)\|_{V^*} + M_3 + \|\hat{a}_i(\tau)\|_{V^*}\} \|\hat{u}_i(\tau)\|_V.$$

Let γ be a number with $0 < \gamma < \frac{1}{4}$. Then we may choose β with $\frac{1}{2} < \beta < 1$ and such that

$$|\tau|^{2\gamma} \leq c(1 + |\tau|)(1 + |\tau|^\beta)^{-1}.$$

Thus

$$\begin{aligned} \int_{-\infty}^{\infty} |\tau|^{2\gamma} \|\hat{u}_i(\tau)\|_H^2 d\tau &\leq C \int_{-\infty}^{\infty} (1 + |\tau|^\beta)^{-1} \|\hat{u}_i(\tau)\|_H^2 d\tau \\ &+ C \int_{-\infty}^{\infty} (1 + |\tau|^\beta)^{-1} \{\|\hat{u}_i(\tau)\|_H + [M_3 + \|\hat{f}(\tau)\|_{V^*} + \|\hat{a}_i\|_{V^*}]\} \|\hat{u}_i(\tau)\|_V d\tau. \end{aligned}$$

Applying the Cauchy-Schwartz Inequality and noting that $2\beta > 1$, we get by taking into account Part 4

$$\int_{-\infty}^{\infty} |\tau|^{2\gamma} \|\hat{u}_i(\tau)\|_H^2 d\tau \leq M.$$

M is a constant independent of j .

6.) We show that $A(u_i, u_i) \rightarrow A(u, u)$ weakly in $L^2(0, T; V^*)$ by verifying Assumption (III).

Let $\varphi_k(t) = \psi_k(t)v_k$ with $\psi_k \in C^\infty(0, T)$, $\psi_k(T) = 0$ and $k < j$. We have

$$-\int_0^T (u_i, D_t \varphi_k) dt + \int_0^T ((A(u_i, u_i) + B(u_i, u_i) - f, \varphi_k)) dt = (u_i(0), \varphi_k(0)).$$

Let $\varphi \in C^\infty(0, T; V)$ with $\varphi(T) = 0$. Set $\theta_N = \sum_{k=1}^N \varphi_k$. It is standard to show that $\|\theta_N - \varphi\|_V \rightarrow 0$ uniformly on $[0, T]$ and $\|D_t(\theta_N - \varphi)\|_V \rightarrow 0$ uniformly on $[0, T]$ (e.g. cf. proof of Lemma 2.1, of [2], p. 45). We have

$$\begin{aligned} \left| -\int_0^T (u_i, D_t \varphi) dt + \int_0^T ((A(u_i, u_i) + B(u_i, u_i) - f, \varphi)) dt - (u_i(0), \varphi(0)) \right| \\ \leq \{ \|u_i\|_{L^2(0, T; H)} \|D_t(\varphi - \theta_N)\|_{L^2(0, T; H)} \\ + \|A(u_i, u_i)\|_{L^2(0, T; V^*)} \|\varphi - \theta_N\|_{L^2(0, T; V)} \\ + \|B(u_i, u_i)\|_{L^2(0, T; V^*)} \|\varphi - \theta_N\|_{L^\infty(0, T; V)} \\ + \|f\|_{L^2(0, T; V^*)} \|\varphi - \theta_N\|_{L^2(0, T; V)} + \|u_i(0)\|_H \|\varphi - \theta_N\|_H \} \end{aligned}$$

for $N < j$.

By taking into account our hypotheses on A, B and the results of Part 4 we obtain

$$\begin{aligned}
& \left| -\int_0^T (u_i, D_t \varphi) dt + \int_0^T ((A(u_i, u_i) + B(u_i, u_i) - f, \varphi)) dt - (u_i(0), \varphi(0)) \right| \\
& \leq M \{ \|\varphi - \theta_N\|_{L^2(0, T; V)} + \|D_t(\varphi - \theta_N)\|_{L^2(0, T; H)} \\
& \quad + \|\varphi - \theta_N\|_{L^\infty(0, T; V)} + \|\varphi(0) - \theta_N(0)\|_H \}.
\end{aligned}$$

M is a constant independent of j and N with $N < j$. Let j then N go to infinity, we get

$$\begin{aligned}
& \lim \left\{ -\int_0^T (u_i, D_t \varphi) dt + \int_0^T ((A(u_i, u_i) + B(u_i, u_i), \varphi)) dt \right\} \\
& = \int_0^T ((f, \varphi)) dt + (u_0, \varphi(0)).
\end{aligned}$$

On the other hand we know that

$$\left(\frac{du_i}{dt}, u_i \right) + (A(u_i, u_i), u_i) = (f, u_i).$$

So

$$\langle D_t u_i, u_i \rangle + \langle A(u_i, u_i), u_i \rangle = \langle f, u_i \rangle,$$

and

$$\lim \langle D_t u_i + A(u_i, u_i), u_i \rangle = \langle f, u \rangle.$$

Moreover since $u_i(0) \rightarrow u_0$ in H , we see by applying Assumption (III) that $A(u_i, u_i) \rightarrow A(u, u)$ weakly in $L^2(0, T; V^*)$.

7.) Let $\varphi_k(t) = \psi_k(t)v_k$ with $\psi_k(t) \in C^1(0, T)$ and such that $\psi_k(T) = 0$, $k < j$. We have

$$\begin{aligned}
& -\int_0^T (u_i, D_t \varphi_k) dt + \int_0^T ((A(u_i, u_i) + B(u_i, u_i), \varphi_k)) dt \\
& = \int_0^T ((f, \varphi_k)) dt + (u_i(0), \varphi_k(0)).
\end{aligned}$$

Let $j \rightarrow +\infty$, we get by taking into account Assumption (II) and the previous part

$$\begin{aligned}
& -\int_0^T (u, D_t \varphi_k) dt + \int_0^T ((A(u, u) + B(u, u), \varphi_k)) dt \\
& = \int_0^T ((f, \varphi_k)) dt + (u_0, \varphi_k(0)).
\end{aligned}$$

The above equation is true for $\varphi_k(t) = \psi_k(t)v_k$, finite sum, where $\psi_k(t) \in C_c^1(0, \infty)$. Then by continuity, the equation remains valid for all $\varphi(t) \in C_c(0, \infty; V)$ with $D_t \varphi \in L^2(0, \infty; H)$.

Finally since $\|u_i\|_{\mathcal{H}_\gamma(0, \infty; \mathbf{v}, H)} \leq M$ and $\mathcal{H}_\gamma(0, \infty; V, H)$ is a Hilbert space we get $u \in \mathcal{H}_\gamma(0, \infty; V, H)$. The theorem is proved.

Section 3. Applications. Let Ω be a bounded open subset of R^n with a smooth boundary. The points of Ω will be denoted by $x = (x_1, \dots, x_n)$. Set $D_i = \partial x_i$, $j = 1, \dots, n$. For each n -tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ of non-negative integers we write

$$D^\alpha = \prod_{i=1}^n D_{i^{\alpha_i}} \quad \text{with} \quad |\alpha| = \sum_{i=1}^n \alpha_i.$$

The points of E^1 will be denoted by t and differentiation in t by D_t . Let r be an integer, $r \geq 1$. By functions, we mean r -vector functions $u = (u_1, \dots, u_r)$ where each u_k is a real valued function on Ω or on $\Omega \times E^1$. Thus

$$D^\alpha u = (D^\alpha u_1, \dots, D^\alpha u_r) \quad \text{and} \quad D_t u = (D_t u_1, \dots, D_t u_r).$$

Let k be a positive integer, we denote by $W^{k,2}(\Omega)$ the space

$$W^{k,2}(\Omega) = \{u : u \text{ in } L^2(\Omega), D^\alpha u \text{ in } L^2(\Omega); |\alpha| \leq k\},$$

(the derivatives are taken in the sense of the theory of distributions). $W^{k,2}(\Omega)$ is a Hilbert space with the norm

$$\|u\|_{k,2} = \left\{ \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^2(\Omega)}^2 \right\}^{1/2},$$

and the obvious inner product.

$W_0^{k,2}(\Omega)$ is the closure of $C_c^\infty(\Omega)$, the family of infinitely differentiable functions with compact support in Ω , with respect to the $\|\cdot\|_{k,2}$ norm.

We consider systems of partial differential operators on Ω with coefficients depending on t , of the form:

$$A(t)u = \sum_{|\alpha| \leq m} D^\alpha A_\alpha(x, t, u, \dots, D^m u),$$

where for each α , A_α is an r -vector function of (x, t) on $\Omega \times E^1$.

Assumption (IV). *The functions $A_\alpha(x, t, u, \dots, D^m u)$ are measurable in (x, t) on $\Omega \times E^1$, continuous in $(u, \dots, D^m u)$. We assume there exists a positive constant M such that*

$$|A_\alpha(x, t, u, \dots, D^m u)| \leq M \sum_{|\alpha| \leq m} |D^\alpha u|$$

for all (x, t) in $\Omega \times E^1$.

Corresponding to the above representation of $A(t)$ we define the nonlinear Dirichlet form:

$$a(u, v; w) = \sum_{|\alpha| \leq m} (A_\alpha(\cdot, t, u, \dots, D^{m-1} u, D^m v), D^\alpha w)$$

for all u, v, w in $W^{m,2}(\Omega)$.

Let V be a closed subspace of $W^{m,2}(\Omega)$ with $W_0^{m,2}(\Omega) \subset V \subset W^{m,2}(\Omega)$. Denote by V^* , the dual of V .

Definition. For $f \in L^2(-\infty, +\infty; V^*)$, $u_0 \in L^2(\Omega)$; u is said to be a solution of

$$D_t u + A(t)u = f \quad \text{on } \Omega \times [0, \infty),$$

$$u(x, 0) = u_0 \quad \text{on } \Omega$$

if (i) $u \in L^2(0, \infty; V)$, $D_t u \in L^2(0, \infty; V^*)$,

$$(ii) -\int_0^\infty (u, D_t \varphi) dt + \int_0^\infty a(u, u; \varphi) dt = \int_0^\infty ((f, \varphi)) dt + (u_0, \varphi(0))$$

for all $\varphi \in C_c([0, \infty); V)$ with $D_t \varphi \in L^2(0, \infty; L^2(\Omega))$.

Theorem 2. Suppose that Assumption (IV) is satisfied. Suppose further that

- (i) $a(u, v; v - w) - a(u, w; v - w) \geq 0$ for all u, v, w in V ,
- (ii) there exists a positive continuous function $c(r)$ with $c(r) \rightarrow +\infty$ as $r \rightarrow +\infty$ such that

$$\int_0^T a(u, u; u) dt \geq c(\|u\|_{L^2(0, T; V)}) \|u\|_{L^2(0, T; V)}.$$

Then for each $f \in L^2(-\infty, +\infty; V^*)$ and for each $u_0 \in L^2(\Omega)$, there exists a solution u of

$$D_t u + A(t)u = f \quad \text{on } \Omega \times [0, \infty),$$

$$u(x, 0) = u_0 \quad \text{on } \Omega.$$

Moreover $D_t^\gamma u \in L^2(0, \infty; L^2(\Omega))$ for any γ with $0 < \gamma < \frac{1}{4}$.

Proof. Let u, v, w be in $L^2(0, \infty; V)$, then

$$\begin{aligned} & \int_0^\infty a(u, v; w) dt \\ &= \sum_{|\alpha| \leq m} \int_0^\infty (A_\alpha(\cdot, t, u, \dots, D^{m-1}u, D^m v), D^\alpha w) dt = \int_0^\infty ((A(u, v), w)). \end{aligned}$$

We note that V is dense in $H = L^2(\Omega)$. It follows from Proposition 4.2 of [2], p. 60 that Assumption (I) is verified. To prove Theorem 2 it suffices then to apply Lemma 1 and Theorem 1. Q.E.D.

From now on, Ω will be a bounded open subset of R^n with $n \leq 4$. u is an n -vector function on Ω or on $\Omega \times E^1$.

We consider elliptic differential operators on Ω of the form

$$A(u) = \sum_{|\alpha|, |\beta| \leq m} D^\beta (a_{\alpha\beta}(x, t, u, \dots, D^{m-1}u) D^\alpha u),$$

where for each α, β ; $a_{\alpha\beta}(x, t, u, \dots, D^{m-1}u)$ is an n -vector function.

Assumption (V). The functions $a_{\alpha\beta}(x, t, \xi_1, \dots, \xi_{m-1})$ are measurable in (x, t) on $\Omega \times E^1$ and continuous in $(\xi_1, \dots, \xi_{m-1})$. We assume there exists a constant M such that

$$|a_{\alpha\beta}(x, t, \xi_1, \dots, \xi_{m-1})| \leq M,$$

for all x in Ω and all $(\xi_1, \dots, \xi_{m-1})$.

Let V be the completion in the $\|\cdot\|_{m,2}$ norm of the set of all infinitely differentiable functions φ with compact support in Ω and such that $\operatorname{div} \varphi = 0$. V is a Hilbert space and for $n \leq 4$ there exists a continuous natural injection mapping of V into $L^4(\Omega)$. V^* denotes the dual of V .

Let $a(u, v; w) = \sum_{|\alpha|, |\beta| \leq m} (a_{\alpha\beta}(\cdot, t, u, \dots, D^{m-1}u) D^\alpha v, D^\beta w)$,

$$((B(u, v), w)) = \sum_{i,k=1}^n \int_{\Omega} u_k D_k v_i w_i dx; \quad u, v, w \text{ in } V.$$

$(\cdot)$ denotes the L^2 inner product and $((\cdot))$ denotes the pairing between V and V^* .

We consider the initial-value problem:

$$(3.1) \quad \begin{cases} D_t u + A(u) + u \nabla u = f - \operatorname{grad} p & \text{on } \Omega \times [0, \infty), \\ \operatorname{div} u = 0; \quad D^\alpha u = 0 & \text{on } \partial\Omega \times [0, \infty) \text{ for } |\alpha| \leq m-1, \\ u(x, 0) = u_0 & \text{on } \Omega. \end{cases}$$

$u_0 \in V_0$, the closure of V in the $L^2(\Omega)$ norm.

Definition. For $f \in L^2(-\infty, +\infty; V^*)$, $u_0 \in V_0$; u is said to be a solution of (3.1) if

$$(i) \quad u \in L^2(0, \infty; V),$$

$$(ii) \quad \int_0^\infty a(u, u; \varphi) dt$$

$$+ \int_0^\infty ((B(u, u), \varphi)) dt - \int_0^\infty ((f, \varphi)) dt - \int_0^\infty (u, D_t \varphi) dt = (u_0, \varphi(0))$$

for all $\varphi \in C_c([0, \infty); V)$ and such that $D_t \varphi \in L^2(0, \infty; L^2(\Omega))$.

Theorem 3. Let $A(u)$ be a system of partial differential operators on Ω as above and satisfying Assumption (V). Suppose further that (i) $a(u, v; v - w) - a(u, w; v - w) \geq 0$ for all u, v, w in V ,

$$(ii) \quad \int_0^T a(u, u; u) dt \geq c(\|u\|_{L^2(0,T;V)}) \|u\|_{L^2(0,T;V)},$$

where $c(r) \rightarrow +\infty$ as $r \rightarrow +\infty$.

Then for each $f \in L^2(-\infty, +\infty; V^*)$ and each $u_0 \in V_0$, there exists a solution u of (3.1). Moreover $D_t^\gamma u \in L^2(0, \infty; L^2(\Omega))$ for any γ with $0 < \gamma < \frac{1}{4}$ and $u \in L^\infty(0, T; L^2(\Omega))$ for any finite T .

Proof. In view of Proposition 4.2 of [2], p. 60, it is obvious that Assumptions (I), (II) are satisfied. It remains to check Assumption (III). Suppose that $u_n \rightarrow u$ weakly in $L^2(0, T; V)$, $D^\gamma u_n \rightarrow D^\gamma u$ weakly in $L^2(0, T; L^2(\Omega))$ for some γ with $0 < \gamma \leq 1$. Since the natural injection mapping of V into $W^{m-1,2}(\Omega)$ is compact, it follows again from Proposition 4.2 of [2] that $u_n \rightarrow u$ in $L^2(0, T; W^{m-1,2}(\Omega))$. By an elementary argument, using the Lebesgue Convergence Theorem we get $a(u_n, u_n; \varphi) \rightarrow a(u, u; \varphi)$ in $L^2(0, T)$ for any φ in $L^2(0, T; V)$. We may apply Theorem 1 and Theorem 3 is proved. Q.E.D.

Finally, let $V = W_0^{m,2}(\Omega)$ with $4m \geq n$ and let

$$\begin{aligned} & ((B(u, v), w)) \\ &= \sum_{i,k=1}^n \int_{\Omega} D_i^{m-1} u_i \cdot D_i v_k \cdot w_k \, dx - \int_{\Omega} D_i^{m-1} u_i \cdot D_i w_k \cdot v_k \, dx; \quad u, v, w \text{ in } V. \end{aligned}$$

Since $4m \geq n$, there exists a continuous natural injection mapping of V into $L^4(\Omega)$. The above expression is therefore well-defined.

Let $A(u)$ be as in Theorem 3 and consider the initial-value problem:

$$(3.2) \quad \begin{cases} D_i u + A(u) + 2 \sum_{i=1}^n D_i^{m-1} u_i \cdot D_i u + \sum_{i=1}^n D_i^m u_i \cdot u = f & \text{on } \Omega \times [0, \infty), \\ D^\alpha u(x, t) = 0 & \text{on } \partial\Omega \times [0, \infty) \text{ for } |\alpha| < m, \\ u(x, 0) = u_0 & \text{on } \Omega. \end{cases}$$

Definition. Let $f \in L^2(-\infty, +\infty; V^*)$ and u_0 be in $L^2(\Omega)$. Then u is said to be a solution of (3.2) if

$$(i) \quad u \in L^2(0, \infty; V),$$

$$\begin{aligned} (ii) \quad & \int_0^\infty a(u, u; \varphi) \, dt \\ & + \int_0^\infty ((B(u, u), \varphi)) \, dt - \int_0^\infty ((f, \varphi)) \, dt - \int_0^\infty (u, D_i \varphi) \, dt = (u_0, \varphi(0)) \end{aligned}$$

for all $\varphi \in C_c([0, \infty); V)$ and such that $D_i \varphi \in L^2(0, \infty; L^2(\Omega))$.

Theorem 4. Let $A(u)$ be a system of differential operators on Ω satisfying the hypotheses of Theorem 3. Then for each f in $L^2(-\infty, +\infty; V)$ and each u_0 in $L^2(\Omega)$, there exists a solution of (3.2). Moreover $D_i^\gamma u \in L^2(0, \infty; L^2(\Omega))$ for any γ with $0 < \gamma < \frac{1}{4}$ and $u \in L^\infty(0, T; L^2(\Omega))$ for any finite T .

Proof. The proof is similar to that of Theorem 3.

Acknowledgment. The writer wishes to thank the referee for his suggestion.

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Date communicated: SEPTEMBER 12, 1969