Cauchy Problem in S'

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1. Introduction. The *Cauchy problem* is to “solve” a given system of partial differential equations,

$$(1.1) \quad \frac{\partial u_j(x, t)}{\partial t} = \sum_{k=1}^N P_{jk}(D) u_k(x, t), \quad j = 1, 2, \dots, N, \quad t > 0,$$

subject to the initial conditions,

$$(1.2) \quad \lim_{t \downarrow 0} u_j(x, t) = u_j^0(x), \quad j = 1, 2, \dots, N.$$

t is a real variable, $u_j^0(x)$ are given functions of x , and $p_{jk}(D)$ ($j, k = 1, \dots, N$) is a linear partial differential operator of order $\leq p$ with constant coefficients, *i.e.*,

$$p_{jk}(D) = \sum_{|\alpha| \leq p} p_{\alpha_1, \dots, \alpha_n}^{jk} D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n, \quad D_{x_i} \equiv -i \frac{\partial}{\partial x_i}.$$

The $u_j(x, t)$ are functions of (x, t) , $x = (x_1, x_2, \dots, x_n) \in X = R_s \times T^{n-s}$, the Cartesian product of s -dimensional Euclidean space and the $n - s$ torus $T^{n-s} \equiv R_{n-s}/Z^{n-s}$, ($Z \equiv$ additive group of integers).

The system (1.1), (1.2) may be rewritten in vector form,

$$(1.3) \quad \frac{\partial u(x, t)}{\partial t} = P(D)u(x, t), \quad t > 0,$$

$$(1.4) \quad \lim_{t \downarrow 0} u(x, t) = u_0(x).$$

Despite the recent trend, *cf.* [11], of treating space and time coordinates equally, the time coordinate will be given a preferred role. The equations are such that if a “state” of a given system is specified at some initial time $t = 0$ then a “solution” exists for $t > 0$ and is uniquely determined by the equations together with side conditions, which usually take the form either of boundary conditions or of growth conditions at infinity. The state of the system at any time t is given by a finite set of functions, denoted by $u(t)$, which depend *only* on the space coordinates. As time progresses, the state of the system changes

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according to laws given by the differential equations and the functional values are replaced by new ones giving the later state of the system. One may think of these functions, $u(t)$, as represented by a point in a phase-space Ω , *i.e.*, as a point in a function space. Each point in the phase-space represents a possible state of the system and the states, $u(t)$, of the system as t increases are represented by a one-parameter curve in phase-space.

Physical intuition requires that the Cauchy problem (1.3), (1.4) be well-set in Ω . That is, given $u_0 \in \Omega$ a solution, $u(x, t) = u(x, t, u_0) \in \Omega$, exists for all $t > 0$, is unique, and depends continuously on $u_0(x)$. This concept is somewhat ambiguous, as the original Cauchy problem does not specify a phase space, Ω , or a topology on Ω . In recent years, the Cauchy problem (1.3), (1.4) has been considered to be well-set if and only if it is *regular* in the sense of Petrowski, *i.e.*, if and only if

$$\Lambda(P) = \sup_{q \in R_n, j} \operatorname{Re} \{ \lambda_j [P(q)] \} < +\infty,$$

where the λ_j denote the eigenvalues of P . This conclusion has been supported by the work of several prominent mathematicians, including Birkhoff, Petrowsky, Garding, Hormander, Gelfand, Shilov and Hadamard who obtained similar conditions in the early 1900's, *cf.* [2], [6]; [11]; [5], pp. 330-1.

We will show that if (1.3), (1.4) is regular, then the Cauchy problem is well-posed in S' and the solution is generated by the action of the C_0 -semigroup " $\{e^{tP}\}$ " on S' and conversely if " $\{e^{tP}\}$ " is a C_0 -semigroup in S' then (1.3), (1.4) is regular. We remark that our results should be contrasted with those of [2] in which it was shown that every regular Cauchy problem is well-posed in a Banach space, specially constructed for that problem.

2. Spaces of distributions. In an attempt to construct a general theory, the theory of distributions is employed in this paper. See [5], [6], or [11] for a discussion of distributions. Let $C_0^\infty(R_n)$ be the locally convex topological vector space, LCTVS, of all complex-valued, infinitely-differentiable functions on R_n with compact support, provided with the topology induced by the following definition of convergence to 0: $\{\phi_i\} \in C_0^\infty(R_n)$ converges to 0 if and only if there is a fixed compact subset of R_n containing the supports of all ϕ_i and $D^\alpha \phi_j \rightarrow 0$ uniformly when $j \rightarrow \infty$, for every multi-index α . A continuous linear functional on $C_0^\infty(R_n)$ is called a *distribution*, *cf.* [11], p. 5. The space of all distributions in R_n is commonly denoted by $D'(R_n)$.

At first sight, one might think that $D'(R_n) = D'(R_n) \times \cdots \times D'(R_n)$, (N -times), the N -fold Cartesian product of the space of distributions, might suffice as an appropriate phase-space. However, because of the Tychonov counterexample, *cf.* [5], Remark 2, p. 183, one knows *a priori* that it would be impossible to prove a uniqueness theorem in $D'(R_n)$. Likewise, one might try the W -spaces of Gelfand and Shilov. However, it is impossible to construct appropriate semigroups in such spaces.

To overcome these difficulties, we introduce the space of "tempered distributions." Let $S(R_n)$ denote the LCTVS of all functions $\phi \in C^\infty(R_n)$ such that

$$(2.1) \quad \|\phi\|_m = \sup_{|\alpha| \leq m} \sup_{x \in R_n} (1 + |x|)^m |D^\alpha \phi(x)| < +\infty,$$

for all m , provided with the topology defined by the norms in the left-hand side of (2.1). The completion of S with respect to the m -th norm is denoted by S_m . $S_1 \supset S_2 \supset \dots \supset S_m \supset \dots \supset S$ and $S = \bigcap_{m=1}^\infty S_m$, cf. [5], p. 27. A continuous linear functional on $S(R_n)$ is called a *tempered distribution*, cf. [11], p. 18. It is well known that the Fourier transform may be extended to a homeomorphism of $S'(R_n)$ onto itself, cf. [11], p. 20.

$$S'(R_n) = S'(R_n) \times \dots \times S'(R_n), \quad (N\text{-times}),$$

i.e., the N -fold Cartesian product of the space of tempered distributions will be taken as the phase-space. The topology of Ω is the product topology induced by the topology of $S'(R_n)$. A function α on R_n is called a *multiplier* in $S(R_n)$, if and only if $\alpha\phi \in S$ for any $\phi \in S$ and the mapping $\phi \rightarrow \alpha\phi$ from S into S is continuous, cf. [5], p. 28. A necessary and sufficient condition for α to be a multiplier in S is that $\alpha \in C^\infty$ and all derivatives of α be bounded by a polynomial, cf. [5], p. 96. Later it will be important to know that $[e^{tP(q)}]^*$ is a multiplier in S , the N -fold Cartesian product of S , where $[]^*$ denotes the adjoint mapping defined by the duality between S and S' , cf. [4].

To this end, an inequality of Gelfand and Shilov is recalled, cf. [5], p. 169. If A is any $N \times N$ matrix with complex entries, $H(A)$ is the convex hull of eigenvalues, and $f(Z)$ is an entire function, then

$$(2.2) \quad \|f(A)\| \leq \sum_{j=0}^{N-1} 2^j \|A\|^j \max_{Z \in H(A)} |f(Z)|,$$

for any matrix-norm. Letting $f(Z) = e^Z$ and $Z = tP(q)$, one obtains

$$(2.3) \quad \|e^{tP(q)}\| \leq C_0(t)(1 + |q|)^{p(N-1)} e^{t\Lambda(q)}.$$

If $P(q)$ is from a regular Cauchy problem, then $\|e^{tP(q)}\| \leq C_1(t)(1 + |q|)^{p(N-1)}$. It is easy to see that for any multi-index α , $\|D^\alpha e^{tP(q)}\| \leq Q_\alpha(|q|)$ where Q_α is a polynomial having coefficients depending only on t . Thus, one obtains the following

Lemma 2.1. *If $\Lambda(P) < +\infty$, then $e^{tP(q)}$ is a multiplier in S' .*

3. Abstract Cauchy Problems; C_0 -semigroups. In this section, the definitions of C_0 -semigroup and infinitesimal generator, given by Hille and Phillips in [9] and [10] for a Banach space, are generalized to a LCTVS. The related concepts of an Abstract Cauchy Problem, abbreviated ACP, over a LCTVS, Φ , and a weak solution of an ACP over Φ are defined. Then the connection between C_0 -semigroups on Φ' , the LCTVS of continuous linear functionals on Φ , and ACP's over Φ is explored. The reader is referred to [15] for alternate development of these concepts in a LCTVS.

Definition 3.1. A one-parameter set, R , of continuous linear transformations, $\{T_t\}$, $t > 0$, of a LCTVS Ψ into itself is said to be a (one-parameter) C_0 -semigroup if and only if $T_\alpha, T_\beta \in R$ imply $T_\alpha T_\beta = T_{\alpha+\beta} \in R$ and $\lim_{t \downarrow 0} T_t[\psi] = \psi$ in the topology of Ψ for every $\psi \in \Psi$.

Because we are interested in spaces of distributions, we may give a more stringent definition of "infinitesimal generator" than is done for Banach spaces, cf. [10].

Definition 3.2. Let $\{T_t\}$ be a C_0 -semigroup on Φ' . When $\lim_{t \downarrow 0} (1/t)(T_t[\phi'] - \phi') = A[\phi']$ exists as a continuous linear operator on Φ' , one calls A the infinitesimal generator of $\{T_t\}$.

If $\Phi' = S'$, then sufficient conditions under which an operator, A , in S' is the infinitesimal generator of a unique C_0 -semigroup are given by Cor. 4.1 and Thm. 4.2.

Definition 3.3. Let A be any continuous linear operator in Φ' . The *Abstract Cauchy Problem*, (ACP), for A over Φ is to find for every $y_0 \in \Phi'$ a function $y(t) = y(t, y_0) \in \Phi'$ for all $t > 0$, such that

$$(3.1) \quad \frac{dy(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{y(t + \Delta t) - y(t)}{\Delta t} = Ay(t), \quad t > 0,$$

$$(3.2) \quad \lim_{t \downarrow 0} y(t) = y_0,$$

where both limits are taken in the strong topology. Such a function $y(t)$ is called a weak solution of (3.1), (3.2) over Φ .

The following result is a direct consequence of the above definitions.

Lemma 3.1. If the continuous linear operator, A , in Φ' is the infinitesimal generator of a C_0 -semigroup, $\{T_t\}$, in Φ' , then $T_t[y_0]$ is a weak solution of the ACP, (3.1), (3.2) for A over Φ .

Proof. By the definition of infinitesimal generator,

$$\frac{T_{t+\Delta t}[y_0] - T_t[y_0]}{\Delta t} = \frac{(T_{\Delta t} - I)}{\Delta t} [T_t[y_0]] = A(T_t[y_0]).$$

Since $\{T_t\}$ is a C_0 -semigroup, one has $\lim_{t \downarrow 0} T_t[y_0] = y_0$.

Q.E.D.

The following result is a version of an "abstract uniqueness" theorem of Gelfand and Shilov, cf. [5], p. 177. Its proof, which follows [5], p. 177, is presented in detail as the theorem will be of central importance for results to be obtained later in this paper.

Theorem 3.1. Let A be a continuous linear operator in Φ' , where Φ is a LCTVS. The adjoint, A^* , with respect to the pairing between Φ and Φ' , is a continuous linear operator in Φ . If, for any $t_0 > 0$ and any $\phi \in \Phi$, there exists a function $\phi(t)$ defined for $t \in (0, t_0]$ such that $\phi_0(t) \in \Phi$ and

$$(3.3) \quad \frac{d\phi(t)}{dt} = -A^*\phi(t), \quad (0 < t \leq t_0),$$

$$(3.4) \quad \phi(t_0) = \phi_0,$$

then there exists at most one weak solution of the ACP for A over Φ , i.e., for every $y_0 \in \Phi'$ there exists at most one function $y(t)$ defined for $t \in (0, \infty)$ such that $y(t) \in \Phi'$ and

$$(3.5) \quad \frac{dy(t)}{dt} = Ay(t), \quad t > 0,$$

$$(3.6) \quad \lim_{t \rightarrow 0} y(t) = y_0.$$

Proof. By definition,

$$\frac{d\phi(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\phi(t + \Delta t) - \phi(t)}{\Delta t}$$

in the topology of Φ and

$$\frac{dy(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{y(t + \Delta t) - y(t)}{\Delta t}$$

in the topology of Φ' . It suffices to prove that if $y(t)$ is any weak solution over Φ and $y_0 = 0$, then for any $t_0 > 0$, $y(t_0) = 0$. Consider the function $a(t) = \langle y(t), \phi(t) \rangle$, (the brackets denote the pairing of Φ and Φ'), in the interval $(0, t_0]$, where $\phi(t)$ is a solution of (3.3), (3.4). Then

$$\frac{a(t + \Delta t) - a(t)}{\Delta t} = \left\langle \frac{y(t + \Delta t) - y(t)}{\Delta t}, \phi(t + \Delta t) \right\rangle + \left\langle y(t), \frac{\phi(t + \Delta t) - \phi(t)}{\Delta t} \right\rangle.$$

Since $\phi(t + \Delta t) \rightarrow \phi(t)$ in Φ and $\{(y(t + \Delta t) - y(t))/\Delta t\}$ is strongly convergent in Φ' as $\Delta t \rightarrow 0$,

$$\left\langle \frac{y(t + \Delta t) - y(t)}{\Delta t}, \phi(t + \Delta t) \right\rangle \rightarrow \left\langle \frac{dy(t)}{dt}, \phi(t) \right\rangle \quad \text{as } t \rightarrow 0.$$

Taking $\Delta t \rightarrow 0$, one obtains

$$\frac{da(t)}{dt} = \left\langle \frac{dy(t)}{dt}, \phi(t) \right\rangle + \left\langle y(t), \frac{d\phi(t)}{dt} \right\rangle \Rightarrow \frac{da(t)}{dt} = 0.$$

Since $a(t) \rightarrow 0$ as $t \rightarrow 0$, one has $a(t_0) = a(t) \equiv \text{constant} = 0$. This shows that $\langle y(t_0), \phi_0 \rangle = 0$ for all $\phi_0 \in \Phi$. But this implies $y(t_0) = 0$. Q.E.D.

4. M-Abstract Cauchy Problems in S' . When one takes the Fourier transform of equations (1.3), (1.4) one obtains

$$(4.1) \quad \frac{\partial \hat{u}(q, t)}{\partial t} = P(q) \hat{u}(q, t), \quad t > 0,$$

$$(4.2) \quad \lim_{t \downarrow 0} \hat{u}(q, t) = \hat{u}_0(q).$$

A new generalization of problems (4.1), (4.2) is given by

Definition 4.1. Let $M(x)$ be an $N \times N$ matrix all of whose entries are multipliers in $S(R_n)$, i.e., $M^*(x)$ is a multiplier in $S(R_n)$. The *M-Abstract Cauchy Problem in S'* is to find a function $y(t)$ such that for $t \in (0, \infty)$, $y(t) \in S'$ and

$$(4.3) \quad \frac{\partial y(t)}{\partial t} = \lim_{\Delta t \rightarrow 0} \frac{y(t + \Delta t) - y(t)}{\Delta t} = M(x)y(t), \quad t > 0,$$

$$(4.4) \quad \lim_{t \downarrow 0} y(t) = y_0,$$

where y_0 is a given element of S' . The limits in (4.3) and (4.4) may be taken in either the weak or strong topology, as it is well known that in S' they are equivalent, cf. [5], Thm. 21, p. 15.

One may extend the definition of regularity.

Definition 4.2. An *M-ACP* is said to be *regular* if and only if

$$\Lambda(M) = \sup_{x \in R_n} \sup_{1 \leq i \leq N} \operatorname{Re} \{\lambda_i(x)\} < +\infty,$$

where $\lambda_i(x)$ are the eigenvalues of $M(x)$.

It will be shown that if $\Lambda(M) < +\infty$, then the *M-ACP* in S' has a unique solution which is generated by the C_0 -semigroup $\{e^{tM(x)}\}$.

Theorem 4.1. If $M(x)$ is regular, then there exists at most one weak solution of the *M-ACP* over S .

Proof. Since the multiplier $M^*(x)$ is a continuous linear operator in S , we may apply Theorem 3.1. Hence, it suffices to prove an existence theorem for the adjoint problem of (4.3), (4.4), namely,

$$(4.5) \quad \frac{\partial \phi(x, t)}{\partial t} = -M^*(x)\phi(x, t), \quad 0 < t \leq t_0,$$

$$(4.6) \quad \phi(x, t_0) = \phi_0(x).$$

A solution of this problem is given by

$$(4.7) \quad \phi(x, t) = e^{-(t-t_0)M^*(x)}\phi_0(x).$$

In fact, by arguments similar to those used to prove Lemma 2.1, one can see that $e^{-(t-t_0)M^*(x)}$ is a multiplier in S . This implies that $\phi(x, t)$, as defined by (4.7) belongs to S . Next, one can show that $\phi(x, t)$ is a solution of (4.5), (4.6) in S .

$$(4.8) \quad \frac{\phi(x, t + \Delta t) - \phi(x, t)}{\Delta t} = \frac{1}{\Delta t} \{e^{-(t+\Delta t-t_0)M^*(x)} - e^{-(t-t_0)M^*(x)}\}\phi_0.$$

As $\Delta t \rightarrow 0$, the right-hand side converges properly, (a sequence $\{\phi_m\}$ is said to

converge properly if and only if for any multi-index, α , $\{D^\alpha \phi_m\}$ converges uniformly on bounded sets of R_n), to

$$(4.9) \quad -M^*(x)e^{-(t-t_0)M^*(x)}\phi_0(x) = -M^*(x)\phi(x, t).$$

Moreover, the right-hand side of (4.8) is bounded in S if, say, $(\Delta t)_0 \geq |\Delta t|$. For, its value is

$$\frac{1}{\Delta t} \int_t^{t+\Delta t} \frac{\partial}{\partial \xi} [e^{-(\xi-t_0)M^*(x)}]\phi_0(x) d\xi = -\frac{1}{\Delta t} \int_t^{t+\Delta t} (\xi - t_0)M^*(x)e^{-(\xi-t_0)M^*(x)}\phi_0(x) d\xi,$$

and the integrand and all its derivatives are uniformly bounded with respect to ξ ; since, for example, $e^{-(\xi-t_0)M^*(x)}\phi_0(x) : [t, t + \Delta t] \rightarrow S$ is a bounded operator, multiplication by $M^*(x)$ is a bounded operator in S , and the image of a compact set under a bounded operator is compact $\Rightarrow$ the image of the set $(\xi - t_0)\phi_0(x)$ is bounded in S . Since S satisfies condition (P) of [5], proper convergence and uniform boundedness are equivalent to convergence, cf. [5, p. 31]. It follows that the right side of (4.8) is convergent in S to the right-hand side of (4.9). Hence, (4.5) is satisfied in S . (4.6) follows, since $e^{t_0 A_1} = I$ (the identity matrix). Uniqueness now follows from Theorem 3.1. Q.E.D.

Corollary 4.1. *If $M(x)$ is regular, then it is the infinitesimal generator of at most one C_0 -semigroup in S' .*

Theorem 4.2. *If $M(x)$ is regular, then it is the infinitesimal generator of the C_0 -semigroup $\{e^{tM^*(x)}\}$ in S' .*

Proof. As in the proof of Theorem 4.1, $e^{tM^*(x)}$ is a multiplier in S . Thus $e^{tM^*(x)}y_0(x) \in S'$ if $y_0(x) \in S'$, i.e., $\{e^{tM^*(x)}\}$ is a semigroup on S' . From its form it is clear that it is a C_0 -semigroup. In fact, if $\psi(x)$ is any element $\in S$, then $\langle e^{tM^*(x)}y_0(x), \psi(x) \rangle = \langle y_0(x), e^{tM^*(x)}\psi(x) \rangle \rightarrow \langle y_0(x), \psi(x) \rangle$ as $t \rightarrow 0$, since $\{e^{tM^*(x)}\psi(x) : 0 < t \leq T\}$ is a bounded set in S and $e^{tM^*(x)}\psi(x) \rightarrow \psi(x)$ as $t \rightarrow 0$ properly implies also in S .

It is claimed that $M(x)$ is the infinitesimal generator of $\{e^{tM^*(x)}\}$. If $\psi(x) \in S$, then

$$\begin{aligned} \left\langle \frac{e^{\Delta t M^*(x)}y_0(x) - y_0(x)}{\Delta t}, \psi(x) \right\rangle &= \left\langle y_0(x), \frac{e^{\Delta t M^*(x)}\psi(x) - \psi(x)}{\Delta t} \right\rangle \\ &\rightarrow \langle y_0(x), M^*(x)\psi(x) \rangle = \langle M(x)y_0(x), \psi(x) \rangle \quad \text{as } \Delta t \rightarrow 0. \end{aligned}$$

To justify the passage to the limit, one must argue the same way as in the proof of Theorem 4.1. Thus, it has been shown that the

$$\lim_{\Delta t \rightarrow 0} \frac{e^{\Delta t M^*(x)}y_0(x) - y_0(x)}{\Delta t}$$

exists in the weak topology of S' and equals $M(x)y_0(x)$. Q.E.D.

As a corollary of Theorems 4.1 and 4.2 we have

Theorem 4.3. *If $M(x)$ is regular, $M(x)$ is the infinitesimal generator of a unique C_0 -semigroup, $\{e^{tM(x)}\}$, in S' , the M -ACP has a unique weak solution over S , and the weak solution is generated by the action of the C_0 -semigroup $\{e^{tM(x)}\}$ on S' .*

Corollary 4.2. *If $M(x)$ is regular, the M -ACP in S' is well-set.*

Proof. Each element $e^{tM(x)}$ of the C_0 -semigroup is a bounded linear operator in S' . Thus, the solution depends continuously on the initial condition. Q.E.D.

5. Weak solutions. In this section, the preceding results are applied to the Cauchy problem (1.3), (1.4) in S' , to prove that any regular Cauchy problem is well-set in S' . Moreover, a characterization of the solution as the orbit under the action of the C_0 -semigroup " $\{e^{tP(D)}\}$ " in S' is obtained. The converse, i.e., that if " $\{e^{tP(D)}\}$ " is a C_0 -semigroup in S' , then the Cauchy problem is regular is shown.

Definition 5.1. Given $u_0 \in S'$, a function $u(t) = u(t, u_0)$ of $(0, \infty) \subset R_1$ into S' is said to be a *weak solution* of (1.3), (1.4) over S if and only if for $t > 0$

$$\begin{aligned} \frac{d}{dt} \langle u(t), \phi(x) \rangle &= \lim_{\Delta t \rightarrow 0} \left\langle \frac{u(t + \Delta t) - u(t)}{\Delta t}, \phi(x) \right\rangle \\ &= \langle P(D)u(t), \phi(x) \rangle \quad \text{and} \quad \lim_{t \downarrow 0} \langle u(t) - u_0(x), \phi(x) \rangle = 0 \end{aligned}$$

for all $\phi(x) \in S$, cf. [5], p. 167.

Theorem 5.1. *If the $P(D)$ of (1.3), (1.4) is regular, then there exists at most one weak solution of (1.3), (1.4) over S .*

Proof. Taking the Fourier transform of (1.3), (1.4) one obtains

$$(5.1) \quad \frac{\partial \hat{u}(q, t)}{\partial t} = P(q)\hat{u}(q, t), \quad t > 0,$$

$$(5.2) \quad \lim_{t \downarrow 0} \hat{u}(q, t) = \hat{u}_0(q).$$

The theorem follows by applying Theorem 4.1 to (5.1), (5.2) and the fact that the Fourier transform is a homeomorphism of S' , cf. §2. Q.E.D.

Corollary 5.1. *If $P(D)$ is regular, it is the infinitesimal generator of at most one C_0 -semigroup in S' .*

Theorem 5.2. *If the $P(D)$ of (1.3), (1.4) is regular, $P(D)$ is the infinitesimal generator of the C_0 -semigroup $\{G(x, t)\}$ in S' , where $G(x, t) = F^{-1}[e^{tP(q)}]$, (F^{-1} denotes the inverse Fourier transform).*

Proof. The theorem follows by applying Theorem 4.2 to (5.1), (5.2). Q.E.D.

As a corollary of Theorems 5.1 and 5.2 we have

Theorem 5.3. If $P(D)$ is regular, $P(D)$ is the infinitesimal generator of a unique C_0 -semigroup, $\{G(x, t)\}$, in S' , where $G(x, t) = F^{-1}[e^{tP(a)}]$ (F^{-1} denotes the inverse Fourier transform), the Cauchy problem (1.3), (1.4) has a unique weak solution over S , and the weak solution is generated by the action of C_0 -semigroup $\{G(x, t)\}$ on S' .

Corollary 5.2. If $P(D)$ is regular, the Cauchy problem (1.3), (1.4) is well-set in S' .

The converse of Theorem 5.3 is now proved.

Theorem 5.4. If the entire functions of $tP(q)$, $\{e^{tP(a)}\}$, form a (one-parameter) C_0 -semigroup in S' , then $P(D)$ is regular and the Cauchy problem (1.3), (1.4) has a unique solution given by $u(x, t) = F^{-1}\{e^{tP(a)}\} * u_0$.

Proof. For $\{e^{tP(a)}\}$ to be a C_0 -semigroup in S' , $e^{tP(a)}$ must be a multiplier in S . Hence, every entry of the matrix $e^{tP(a)}$ for $0 < t \leq T$ must be bounded in absolute value by a polynomial in $|q|$. If $A = [a_{ij}]$ is any $N \times N$ matrix with complex entries, then $[\rho(A)]^2 \leq \|A\|^2 \leq \sum_{i,k} |a_{ik}|^2$, where $\rho(A)$ denotes the spectral radius. Hence

$$(5.3) \quad [\rho(e^{tP(a)})]^2 \leq \|e^{tP(a)}\|^2 \leq C(1 + |q|)^R,$$

where R is some positive integer, C is some positive constant, and $\rho(e^{tP(a)}) = e^{t\Lambda(P(a))}$. Taking the log of both sides of (5.3), one obtains $2t\Lambda(P^*(q)) \leq \log C + R \log(1 + |q|)$. But it is well known that $\Lambda(P^*(r)) = \max_{|q|=r} \Lambda(P^*(q)) = \max_{|q|=r} \max_{1 \leq i \leq N} \operatorname{Re} \{\lambda_i^*(P(q))\} = Kr^h + \text{lower terms}$, $K \neq 0$, cf. [5], p. 219. Thus, $\Lambda(P) < +\infty$, i.e., $P(D)$ is regular. The remainder of the Theorem follows directly from Theorem 5.3. Q.E.D.

6. Strong solutions. A function $u(x, t)$ is said to be a *strong solution*, (or a *classical* or *smooth solution*), of (1.3), (1.4) if and only if $u(x, t)$ satisfies the following four conditions:

- (1) $u(x, t)$ is a continuous function of (x, t) , for all $x \in R_n$, $t > 0$;
- (2) all the derivatives of $u(x, t)$ which appear in (1.3) are also continuous functions of (x, t) , for all $x \in R_n$, $t > 0$;
- (3) $u(x, t)$ satisfies (1.3), for all $x \in R_n$, $t > 0$; and
- (4) $u(x, t)$ satisfies (1.4), for all $x \in R_n$.

Any theory of Cauchy problems certainly does not have any significance unless strong solutions come under its domain. To this end, the following standard theorem is presented.

Theorem 6.1. Any strong solution $u(x, t)$ of (1.3), (1.4), which belongs to S' for all $t > 0$, is a weak solution over S .

Proof. Following [5], it suffices to prove the theorem in the case $N = 1$. For any $\epsilon > 0$, define the function

$$\rho_\epsilon(x) = \frac{k}{\epsilon^n} e^{-\epsilon^2/(x^2-x^2)} \quad \text{if } |x| \geq \epsilon,$$

$$\quad \quad \quad \text{if } |x| < \epsilon,$$

which has the following properties: $\rho_\epsilon(x) \in C_0^\infty$; $\rho_\epsilon(x) \geq 0$ in R_n ; $\rho_\epsilon(x) > 0$ if $|x| < \epsilon$, $\rho_\epsilon(x) = 0$ if $|x| > \epsilon$; and $\int_{R_n} \rho_\epsilon(x) dx = 1$, if k is determined by the condition

$$k \int_{R_n} \frac{1}{e^{|x|^2-1}} dx = 1.$$

Let

$$\alpha_j(x) = \int_{|\xi| < j+1/2} \rho_{1/4}(x - \xi) d\xi,$$

$j = 1, 2, \dots$. $\alpha_j(x)$ has the following properties: $\alpha_j \in C_0^\infty$; $\alpha_j(x) = 1$, if $|x| \leq j$; $\alpha_j(x) = 0$, if $|x| \geq j+1$; and $|D^q \alpha_j| \leq B_q$ on R_n , where the constants B_q are independent of j . Since $\alpha_j \in C_0^\infty$, it is a multiplier in S and its product with any element from S belongs to C_0^∞ . For any $\phi \in S$,

$$\int_{R_n} [P(D)u(x, t)](\alpha_j(x)\phi(x)) dx = \int_{R_n} u(x, t)[P^*(-D)\alpha_j(x)\phi(x)] dx,$$

(the brackets denote differentiation in the classical sense). The left-hand integral is finite, since $u(x, t)$ is a strong solution and $\alpha_j(x)\phi(x)$ has compact support, and the equality follows by integration by parts.

Integrating both sides with respect to t , one obtains

$$(6.1) \quad \int_{R_n} u(x, t)\alpha_j(x)\phi(x) dx - \int_{R_n} u_0(x)\alpha_j(x)\phi(x) dx$$

$$= \int_0^t \int_{R_n} u(x, t)[P^*(-D)\alpha_j(x)\phi(x)] dx dt.$$

Taking $j \rightarrow \infty$ and using Holder's Inequality, one obtains that the inner integral of the right side of (6.1) is convergent to $\int_{R_n} u(x, t)[P^*(-D)\phi(x)] dx$, uniformly with respect to $t \in [0, T]$. Similarly, one obtains the convergence of the integrals on the left side of (6.1), as $j \rightarrow \infty$. Thus, one has

$$(6.2) \quad \int_{R_n} u(x, t)\phi(x) dx - \int_{R_n} u_0(x)\phi(x) dx = \int_0^t \int_{R_n} u(x, t)[P^*(-D)\phi(x)] dx dt.$$

This implies $\langle u(x, t) - u_0(x), \phi(x) \rangle = \int_0^t \langle u(x, t), P^*(-D)\phi(x) \rangle dt$, i.e., (1.3) is satisfied over S . Taking $t \rightarrow 0$ in (6.2), one finds that (1.4) is also satisfied over S . Q.E.D.

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