Solutions of the n -Dimensional Wave Equation in the Exterior of the Characteristic Cones (I)*

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Summary. The notion of the hyperboloidal means (HM) is introduced such that it is invariant with respect to the Lorentz and similarity transformations and satisfies the wave equation. In the exterior of the characteristic cones C_0 , the even and odd solutions which carry smooth initial data can be expressed in terms of HM of the matching boundary data on C_0 . This is proved by establishing two identities. They are put subsequently in a "weak" form applicable to L_2 -data. By approximation, the result is extended to L_2 -solutions: Using HM of the boundary values on C_0 one has a simple representation of L_2 -solutions as distributions in the exterior of C_0 ; the uniqueness of the solutions and their domains of dependence are obtained. The space dimension of the wave equation is assumed to be odd.

Introduction. The paper extends the result of a previous work [1] on the wave equation of three space dimensions to that of higher odd dimensional space. We consider smooth solutions and also L_2 -solutions $w(X, t)$ of

$$\square^2 w \equiv \sum_{i=1}^n w_{x_i x_i} - w_{t t} = 0,$$

where (X, t) is a point in the space-time $R_{X,t}^{n+1}$ and lies in the exterior of the double cone C_0 given by $t^2 - r^2 = 0$ with $r = |X|$, $X = (x_1, x_2, \dots, x_n) \in R_X^n$, n odd. Representations of $w(X, t)$ are obtained in terms of the data on C_0 and the domain of the dependence will consist of the intersection $C_0 \cap C(X, t)$ where $C(X, t)$ is the characteristic double cone with the vertex at (X, t) .

Let C_0^+ and C_0^- be respectively the upper ($t \geq 0$) and the lower ($t \leq 0$) part of C_0 . We shall consider them *separately* by projecting each cone onto the n -dimensional space R_Y^n ($Y = (y_1, y_2, \dots, y_n)$, $|Y| = \rho$) by the maps

$$(0.1) \quad P_{\pm} : (X_0, t_0) \in C_0^{\pm} \rightarrow Y = X_0 \in R_Y^n.$$

Data on C_0^+ (and C_0^-) may then be considered as being prescribed on R_Y^n . In

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particular, we have

$$(0.2) \quad \begin{aligned} C_0^+ \cap C(X, t) &\xrightarrow{P\pm} Hd(X, \\ C_0^- \cap C(X, t) &\xrightarrow{P\pm} Hd(X) - \end{aligned}$$

where $Hd(X, t)$ is a hyperboloid in R_Y^n given by

$$(0.3) \quad \sum_{i=1}^n x_i y_i - t\rho = \frac{1}{2}(r^2 - t^2).$$

The central notion of this paper is that of the mean value of a function $\zeta(Y)$ in R_Y^n over a hyperboloid $Hd(X, t)$. It is called the “hyperboloidal mean” and denoted by $HM[\zeta(Y); X, t]$. It is linear with respect to ζ , and satisfies the wave equation for arbitrary ζ (see Section 1).

A reasonable definition of HM should be invariant with respect to the Lorentz transformations and the similarity transformations in $R_{X,t}^{n+1}$. Let G_0 be the group of all such transformations which leave the cones $C_0^\pm$ in place, such that the interior of C_0^+ remains the interior. Let the transformation $A \in G_0$ and let A_* be the restriction of A to C_0 . Then A_* induces the following maps of R_Y^n onto itself:

$$(0.4) \quad A' = PA_*P^{-1}$$

where P stands either for P_+ or P_- . We say that $HM[\zeta; X, t]$ is invariant with respect to G_0 if for any $A \in G_0$ the equation

$$(0.5) \quad HM[\zeta(Y); X', t'] = HM(\zeta'(Y); X, t]$$

holds with $(X', t') = A(X, t)$ and $\zeta(A'Y) \equiv \zeta'(Y)$.

For a definition of HM we need first of all an invariant measure over the hyperboloids $Hd(X, t)$. If there exists such a measure, then by using it to define HM , the function $HM[\zeta; X, t]$ will satisfy the wave equation. This is proved in Section 1, Theorem 1.1. It turns out (Theorem 1.2) that there is essentially only one invariant, absolutely continuous measure over the hyperboloids $Hd(X, t)$. The measure is given explicitly by (1.19) which leads to the definition of HM by (1.19)'. The definition holds both for even and odd n . For $n = 3$ it coincides with the HM used in [1] and for $n = 5$ with that used by Wang [11]. However it differs from mean value given by Rhee in his thesis on the odd dimensional Darboux Equation [9]. The present definitions of $HM[\zeta; X, t]$ makes it possible to avoid the elaboration of the “iterated means” for establishing the identities, as was done in the papers just mentioned.

When n is odd as we assume in this paper we can find a solution $w(X, t)$ in the form of a HM such that 1) $w(X, r) = \varphi(X)$ with φ arbitrarily prescribed on C_0^+ , and 2) $w = 0$ on C_0^- (Theorem 1.3). Since a solution in the exterior of C_0 is not uniquely determined by its boundary values $w(X, \pm r)$ on C_0 , the solution $w(X, t)$ of Theorem 1.3 requires further characterization. The purpose of the theorem is to introduce the form of boundary data:

$$\delta_*^{k+1} \rho^{-k-1} w(Y, \rho),$$

where $k = (n - 3)/2$, $\delta_* = \rho^{-2k} \delta \rho^{2k}$ and $\delta = (\partial/\partial \rho) \rho^2$ (see (1.20)). It is the *HM* of the above boundary data that represents, up to a multiplicative constant, $w(X, t)$ of the theorem.

In both Sections 2 and 3 we identify solutions which can be represented by *HM* of the boundary data on C_0 on the one hand, and on the other hand by the initial data. Set $w(X, 0) = g(X)$, $w_t(X, 0) = f(X)$, then one has the classical representation of $w(X, t)$ in terms of the spherical means (*SM*) of f and g [2]. In Section 2 the initial data are assumed to be smooth (*i.e.*, C^∞ functions with compact supports), while in Section 3 the functions $f(X)$, $g(X)/r$ and $|g_X|$ are square integrable in R_X^n .

Some technical details become necessary in order to match the boundary data with the given smooth functions f and g (Theorem 2.1). Two identities are then derived, one for the odd, the other for the even solutions (Theorem 2.2). The order of growth of the boundary data is studied in some detail in order to obtain a "weak" form of the representation to be used for L_2 -solutions (Theorem 2.3). In the "weak" form the δ_* operator under *HM* is replaced by the operator T_* (see (1.26)) at the front of *HM*, (Lemma 1.6). When $w(X, r)/r$ is a L^2 -function one shows that *HM* [$\rho^{-k-1} w(Y, \rho)$; X, t] exists a.e. in the exterior of C_0 , and T_* is then applied to the above *HM* in the sense of distribution.

Section 3 is brief because the same estimates and approximations used in [1] for $n = 3$ can be carried over to the higher dimensional space (Theorem 3.1). The "weak" identity leads to the representation of an L_2 -solution as a distribution in terms of its boundary values on C_0 , the uniqueness of the solution and the domains of dependence (Theorem 3.2). An elegant proof of the uniqueness theorem has been previously given by Lax and Phillips [7].

Because the solutions of even dimensional space behave differently with respect to the domains of dependence on C_0 , we shall discuss the even case in a subsequent paper.

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1. The hyperboloidal mean.

1.1. Lorentz transformations group G in $R_{X,t}^{n+1}$ and the group G' of induced maps in R_X^n . A linear transformation in $R_{X,t}^{n+1}$, $A : (X, t) \rightarrow (X', t')$ is a Lorentz transformation if

$$(1.1) \quad t'^2 - \sum_{i=1}^n x_i'^2 = t^2 - \sum_{i=1}^n x_i^2,$$

hence C_0 is mapped into itself, and if, in addition, points inside C_0^+ (C_0^-) remain inside. Suppose in the $(x_1, x_2, \dots, x_n, t)$ coordinate system with t as the $(n + 1)$ -th axis, the transformation A is given by the matrix

$$(1.2) \quad (A) = (a_{kl}), \quad k, l = 1, 2, \dots, (n + 1).$$

Then the induced map A' of $A : Y \rightarrow Y'$ in R_Y^n is given by

$$(1.3) \quad y'_i = \sum_{j=1}^n a_{ij} y_j \pm a_{i,n+1} \rho, \quad i = 1, 2, \dots, n,$$

with $\rho = |Y|$. Thus A' is in general not a linear transformation in Y . It follows from the properties of Lorentz transformations that $a_{n+1,n+1} \geq 1$ [8, p. 22], and hence by setting $\rho' = |Y'|$ we have

$$(1.4) \quad \pm \rho' = \sum_{j=1}^n a_{n+1,j} y_j \pm a_{n+1,n+1} \rho.$$

If we substitute in (1.3) and (1.4), (Y, Y') by (X, X') and $\pm(\rho, \rho')$ by (t, t') , then we get back transformation A .

The hyperboloid $Hd(X, t)$ is represented in R_Y^n by

$$(1.5) \quad \sum_{i=1}^n x_i y_i - t \rho = \frac{1}{2}(r^2 - t^2).$$

If $Y \in Hd(X, t)$ and $Y' = A'Y$, then for $(X', t') = A(X, t)$ we have $Y' \in Hd(X', t')$. This follows from (1.1) and that the scalar product on the left hand side of (1.5) is invariant. Thus

$$Hd(X, t) \xrightarrow{A} Hd(A(X, t)).$$

From G we need only its generators: the rotations D_{ij} , $i < j$, and the hyperbolic skews E_i , $i, j = 1, 2, \dots, n$. They are given respectively as follows:

$$\begin{aligned} D_{ij}(\theta) : (X, t) &\rightarrow (X'(\theta), t'(\theta)) \\ x'_i(\theta) &= x_i \cos \theta + x_j \sin \theta, \\ x'_j(\theta) &= -x_i \sin \theta + x_j \cos \theta, \\ x'_l(\theta) &= x_l, \quad l \neq i, j, \quad t'(\theta) = t; \end{aligned}$$

and

$$\begin{aligned} E_i(\lambda) : (X, t) &\rightarrow (X'(\lambda), t'(\lambda)) \\ x'_i(\lambda) &= x_i \cosh \lambda + t \sinh \lambda, \\ t'(\lambda) &= x_i \sinh \lambda + t \cosh \lambda, \\ x'_j(\lambda) &= x_j, \quad j \neq i. \end{aligned}$$

The induced maps D'_{ij} and E'_i are derived from D_{ij} and E_i respectively as (1.3) from (1.2). We mention only

$$\begin{aligned} E'_i(\lambda) : Y &\rightarrow Y'(\lambda) \\ y'_i(\lambda) &= y_i \cosh \lambda \pm \rho \sinh \lambda, \\ y'_j(\lambda) &= y_j, \quad j \neq i. \end{aligned}$$

To the Lorentz transformations we now add the similarity transformations

$$D(\mu) : (X, t) \rightarrow (X' = \mu X, t' = \mu t), \quad \mu > 0,$$

and the induced maps

$$D'(\mu) : Y \rightarrow Y' = \mu Y.$$

Lemma 1.1. *Let $\Phi(X, t)$ and $\varphi(Y)$ be differentiable functions. Then,*

(a) *for $(X'(\theta), t'(\theta)) = D_{i,i}(\theta)(X, t)$ and $Y'(\theta) = D'_{i,i}(\theta)Y$,*

$$(1.6) \quad \lim_{\theta \rightarrow 0} \frac{1}{\theta} (\Phi(X'(\theta), t'(\theta)) - \Phi(X, t)) = \left(x_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_i} \right) \Phi(X, t),$$

$$\lim_{\theta \rightarrow 0} \frac{1}{\theta} (\varphi(Y'(\theta)) - \varphi(Y)) = \left(y_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial y_i} \right) \varphi(Y);$$

(b) *for $(X'(\lambda), t'(\lambda)) = E_i(\lambda)(X, t)$ and $Y'(\lambda) = E'_i(\lambda)Y$,*

$$(1.7) \quad \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} (\Phi(X'(\lambda), t'(\lambda)) - \Phi(X, t)) = \left(t \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial t} \right) \Phi(X, t),$$

$$\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} (\varphi(Y'(\lambda)) - \varphi(Y)) = \pm \rho \frac{\partial}{\partial y_i} \varphi(Y);$$

(c) *for $X' = (1 + \epsilon)X$, $t' = (1 + \epsilon)t$ and $Y' = (1 + \epsilon)Y$, $\epsilon > 0$,*

$$(1.8) \quad \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (\Phi((1 + \epsilon)X, (1 + \epsilon)t) - \Phi(X, t)) = \left(r \frac{\partial}{\partial r} + t \frac{\partial}{\partial t} \right) \Phi(X, t),$$

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (\varphi((1 + \epsilon)Y) - \varphi(Y)) = \rho \frac{\partial}{\partial \rho} \varphi(Y).$$

1.2. *A decomposition of the wave operator.*

Notation. *Let the operators be denoted as follows:*

$$d_{i,i} = x_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_i}, \quad d'_{i,i} = y_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial y_i},$$

$$e_i = t \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial t}, \quad e'_i = \pm \rho \frac{\partial}{\partial y_i}, \quad d = r \frac{\partial}{\partial r} + t \frac{\partial}{\partial t}$$

and $d' = \rho \partial / \partial \rho$, $i < j$, $i, j = 1, 2, \dots, n$. Furthermore Δ_x denotes the Laplacian in R_x^n , and for $X = r\alpha$, α unit vector, Δ_α denotes the Laplacian over the unit sphere so that

$$r^2 \Delta_x = r^2 \frac{\partial^2}{\partial r^2} + (n-1)r \frac{\partial}{\partial r} + \Delta_\alpha.$$

Similarly for $Y = \rho\gamma$ we have Δ_Y and Δ_γ .

Lemma 1.2. *The wave operator can be decomposed as follows:*

$$(1.9) \quad (r^2 - t^2) \square^2 = \sum_{i < j} d_{ij}^2 - \sum_{i=1}^n e_i^2 + (n-1)d + d^2.$$

Proof. By direct verification using $\Delta_a^2 = \sum_{i < j} d_{ij}^2$.

Theorem 1.1. Suppose $HM[\zeta(Y); X, t]$ is linear with respect to ζ and has the invariance property

$$(1.10) \quad HM[\zeta(Y); A(X, t)] = HM[\zeta(A'Y); X, t],$$

where $A \in G_0$, A' the induced map of A . Suppose ζ is of compact support and of class C^2 , then in the exterior of C_0 ,

$$\square^2 HM[\zeta(Y); X, t] = 0.$$

Proof. (a) Let A_τ be a continuous one parametric family of transformations with parameter τ tending to zero and $A_0 = \text{identity}$, and let the induced maps be A'_τ . From (1.10) we find,

$$(1.11) \quad \frac{1}{\tau} (HM[\zeta(Y); A_\tau(X, t)] - HM[\zeta(Y); X, t]) \\ = HM \left[\frac{1}{\tau} (\zeta(A'_\tau Y) - \zeta(Y)); X, t \right].$$

By letting $\tau \rightarrow 0$ we shall obtain the limits on both sides of (1.11), when they exist.

(b) Replace A_τ in (1.11) successively by $D_{ii}(\theta)$, $E_i(\lambda)$ and $D(1 + \epsilon)$, and let the parameters tend to zero in each case. As the result we find

$$(1.12) \quad \begin{aligned} d_{ii} HM[\zeta(Y); X, t] &= HM[d'_{ii} \zeta(Y); X, t], \\ e_i HM[\zeta(Y); X, t] &= HM[e'_i \zeta(Y); X, t], \\ d HM[\zeta(Y); X, t] &= HM[d' \zeta(Y); X, t]. \end{aligned}$$

Starting from (1.12) we repeat the process and obtain the effect of d_{ii}^2 , e_i^2 and d^2 on $HM[\zeta(Y); X, t]$. It follows from Lemma 1.2 that

$$(r^2 - t^2) \square^2 HM[\zeta(Y); X, t] = HM[L'_\tau \zeta(Y); X, t],$$

where

$$\begin{aligned} L'_\tau &\equiv \sum_{i < j} (d'_{ij})^2 - \sum_{i=1}^n (e'_i)^2 + (n-1)d' + (d')^2 \\ &= \Delta_\tau - \rho^2 \Delta_\tau + \rho^2 \frac{\partial^2}{\partial \rho^2} + (n-1)\rho \frac{\partial}{\partial \rho} \equiv 0. \end{aligned}$$

Thus Theorem 1.1 is proved.

1.3. The invariant measure. We shall need the parametric representation of $Hd(X, t)$ and we keep the same notation as that used in [1]. Given in R^n_x a

point $X = r\alpha \neq 0$, consider all unit vectors $\alpha' \perp \alpha$ which form the sphere $\omega_{\alpha'}$ of $(n-2)$ -dimensions. For any Y we may set $Y = \rho\gamma$ with $\gamma = p\alpha + p'\alpha'$, $p' = (1 - p^2)^{1/2}$, $|p| \leq 1$. By substituting γ into (1.5) we find that a point Y lies on $Hd(X, t)$ if and only if

$$(1.13) \quad p = p(t, r, \rho) = (2t\rho + r^2 - t^2)/2r\rho.$$

Thus $Hd(X, t)$ is represented by $(n-1)$ parameters: the vectors α' varying over $\omega_{\alpha'}$ and the parameter ρ in the range of $(r+t)/2 \leq \rho < \infty$. (Note that $Hd(X, 0)$ is the hyperplane in R^n which is perpendicular to OX at the midpoint $Y = X/2$.) We shall obtain the invariant measure in the form of: a weight factor $\cdot d\omega_{\alpha'} d\rho$.

Lemma 1.3. *The volume measure $\rho^{-1} dy_1 dy_2 \cdots dy_n$ is invariant with respect to the transformation in G' .*

Proof. G' can be generated by D'_{i_1} and E'_{i_1} . The rotations D'_{i_1} leave both $dy_1 dy_2 \cdots dy_n$ and ρ invariant. From E'_{i_1} , for instance, one finds $dy_2 dy_3 \cdots dy_n = dy'_2 dy'_3 \cdots dy'_n$ and

$$\begin{aligned} dy'_1 &= \text{Cosh } \lambda dy_1 \pm \text{Sinh } \lambda d\rho \\ &= \text{Cosh } \lambda dy_1 \pm \text{Sinh } \lambda \cdot \rho^{-1} \sum_{i=1}^n y_i dy_i. \end{aligned}$$

Hence by (1.4),

$$\begin{aligned} dy'_1 \cdots dy'_n &= (\text{Cosh } \lambda \pm \text{Sinh } \lambda \cdot y_1 \rho^{-1}) dy_1 \cdots dy_n \\ &= (\rho \text{Cosh } \lambda \pm y_1 \text{Sinh } \lambda) \rho^{-1} dy_1 \cdots dy_n \\ &= \rho' \rho^{-1} dy_1 \cdots dy_n. \end{aligned}$$

The lemma is proved.

Lemma 1.4. *Let (X, t) be a fixed point in the exterior of C_0 , and let $G'_1(X, t)$ be the subgroup of all transformations in G' which map $Hd(X, t)$ into itself. Then the measure on $Hd(X, t)$ given by*

$$(1.14) \quad dm(X, t; Y) = r^{-1}(1 - p^2)^k \rho^{2k} d\omega_{\alpha'} d\rho, \quad Y \in Hd(X, t),$$

is invariant with respect to $G'_1(X, t)$ where in (1.14), as also in the following, $k = (n-3)/2$.

Proof. $G'_1(X, t)$ consists of all the induced maps of those Lorentz transformations which leave the point (X, t) fixed. Such transformations in G must then leave every point $(\mu X, \mu t)$ fixed for any value of μ . Hence every $Hd(\mu x, \mu t)$ is mapped by $G'_1(X, t)$ into itself.

We now consider $Hd(X, t)$ as being imbedded in the family of hyperboloids $Hd(\mu x, \mu t)$ by varying μ in $1 - \epsilon \leq \mu \leq 1 + \epsilon$ for some small $\epsilon > 0$. They provide a simple covering of a neighborhood of $Hd(X, t)$ in R^n .

The invariant volume element of Lemma 1.3 can be expressed as follows: First on $Hd(\mu X, \mu t)$ we have by (1.5):

$$(1.15) \quad \mu = 2 \left(\sum_{i=1}^n y_i x_i - t\rho \right) (r^2 - t^2)^{-1}.$$

Secondly at a point Y on $Hd(X, t)$ the volume element

$$(1.16) \quad dy_1 dy_2 \cdots dy_n = |\text{grad}_Y \mu|_{\mu=1}^{-1} dS d\mu,$$

where dS is the surface element. By a simple computation we find from (1.15) that

$$|\text{grad}_Y \mu|_{\mu=1} = 2(\rho(\rho - t))^{1/2} (r^2 - t^2)^{-1/2} \rho^{-1}.$$

On the other hand one has by elementary geometry, using the above parameter representation

$$dS = r^{-1}(\rho(\rho - t))^{1/2} (r^2 - t^2)^{1/2} \rho^{2k} (1 - p^2)^k d\omega_\alpha d\rho.$$

It follows from (1.16) that

$$(1.17) \quad \rho^{-1} dy_1 \cdots dy_n = \frac{1}{2} r^{-1} (1 - p^2)^k \rho^{2k} d\omega_\alpha d\rho d\mu (r^2 - t^2).$$

Each of the last two factors $d\mu$ and $(r^2 - t^2)$ is invariant with respect to $G'_1(X, t)$. The lemma is proved.

Lemma 1.5. Let $A \in G$, $A(X_1, t_1) = (X_2, t_2)$. Suppose

$$Y_1 \in Hd(X_1, t_1) \xrightarrow{A'} Y_2 \in Hd(X_2, t_2),$$

then

$$(1.18) \quad dm(X_1, t_1; Y_1) = dm(X_2, t_2; Y_2).$$

Proof. We need only prove this for the case $(X_1, t_1) \neq (X_2, t_2)$. When $(X_1, t_1) \rightarrow (X_2, t_2)$ by A , so also $(\mu X_1, \mu t_1) \rightarrow (\mu X_2, \mu t_2)$ by A for any μ . We may therefore repeat the same argument as that used in the previous lemma. Now we have $(r_1^2 - t_1^2) = (r_2^2 - t_2^2)$ since $A \in G$, and furthermore $Hd(\mu X_1, \mu t_1) \xrightarrow{A'} Hd(\mu X_2, \mu t_2)$, so $d\mu$ is unchanged by A' . Hence from (1.17) we conclude (1.18).

So far only transformations in G and G' have been considered. We now take up the similarity transformations. We define, using (1.14),

$$(1.19) \quad dM(X, t; Y) = \frac{1}{\omega_{n-1}} (r^2 - t^2)^{-k} dm(X, t; Y), \quad Y \in Hd(X, t).$$

Evidently $dM(X, t; Y)$ satisfies the invariance requirements with respect to G_0, G'_0 .

Definition of the hyperboloidal mean:

$$(1.19)' \quad HM[\zeta(Y); X, t] = \int \zeta(Y) dM(X, t; Y),$$

where Y runs over $Hd(X, t)$, i.e., $|\alpha'| = 1$ and $(r + t)/2 \leq \rho < \infty$.

Theorem 1.2. *The hyperboloidal means of ζ , $HM[\zeta(Y); X, t]$, satisfy (1.10) of Theorem 1.1.*

Corollary 1.2. *When ζ and its first and second derivatives $\rho(\partial/\partial\rho)$, $\rho^2(\partial^2/\partial\rho^2)$ and Δ_γ are continuous and of the order $O\rho^{-(n-1)}$ for large ρ , $Hd[\zeta; X, t]$ is a solution of the wave equation.*

Proof. We have for $A(X, t) = (X', t')$ and $A'(Y) = Y'$:

$$\begin{aligned} HM[\zeta(A'Y); X, t] &= \int \zeta(A'Y) dM(X, t, Y) \\ &= \int \zeta(Y') dM(X', t', Y') \\ &= HM[\zeta(Y); A(X, t)]. \end{aligned}$$

Thus (1.10) is proved. When $\zeta(Y) = O\rho^{-(n-1)}$ for large ρ the integral $HM[\zeta(Y); X, t]$ converges absolutely.

Notation. Let δ and δ_* be the following differential operators in the radial direction in R_γ^n :

$$(1.20) \quad \delta = \frac{\partial}{\partial\rho} \rho^2, \quad \delta_* = \rho^{-2k} \delta \rho^{2k}.$$

From now on the space dimension n must be odd so that k is a non-negative integer.

Theorem 1.3. *Let $\varphi(Y)$ be a smooth function in R_γ^n such that for large ρ*

$$(1.21) \quad \left(\frac{\partial}{\partial\rho}\right)^{k+1-i} (\rho^{2k+1} \varphi(Y)) = O\rho^{-k-2+i}$$

for $j = 0, 1, \dots, (k + 1)$. Then the function $w(X, t)$ given by

$$(1.22) \quad w(X, t) = (-1)^{k+1} \frac{1}{k!} HM[\delta_*^{k+1} \rho^{-k-1} \varphi(Y); X, t],$$

is a solution of the wave equation satisfying the boundary conditions

$$(1.23) \quad w(X, r) = \varphi(X), \quad w(X, -r) = 0.$$

Remark. We do not insist on having the weakest assumption on the growth of φ and its derivatives. The theorem will not be needed later on, its purpose here is to introduce the form of the boundary data which will be discussed in detail in Section 2.

Proof. First, we have

$$(1.24) \quad (1 - p^2) = (r^2 - t^2)(2\rho - r - t)(2\rho + r - t)(2r\rho)^{-2}$$

so that the common factor $(r^2 - t^2)^k$ in $(1 - p^2)^k$ and in dM cancels out. After this we may substitute t by r in order to find $w(X, r)$ from (1.22). Secondly, one can show that

$$(1.25) \quad \delta^{k+1} \rho^{k-1} \varphi(Y) = \rho^k \left(\frac{\partial}{\partial \rho} \right)^{k+1} \rho^{2k+1} \varphi(Y)$$

holds for any sufficiently differentiable φ . Indeed (1.25) holds for $\varphi(Y) = \rho^s \psi$ where ψ is independent of ρ and $s \geq 0$. Hence the operators on the two sides of (1.25), each applied to φ , are identical. Altogether we obtain

$$w(X, r) = \frac{(-1)^{k+1} r^{-2k-1}}{\omega_{n-1} k!} \int \int_r^\infty (\rho - r)^k \left(\frac{\partial}{\partial \rho} \right)^{k+1} \rho^{2k+1} \varphi(Y) d\rho d\omega_\alpha.$$

Apply integration by parts k times and use (1.21). The result gives the first equation in (1.23). To obtain the second condition one proceeds in the same way, noting that the lower limit of the integral representing $w(X, -r)$, is equal to zero.

Notation. Let T_* denote the operator in $R_{X,t}^{n+1}$:

$$(1.26) \quad T_* = r^{-k} T r^{k+1}, \quad T = \frac{r^2 + t^2}{2r} \frac{\partial}{\partial t} + t \frac{\partial}{\partial r}.$$

For $n = 3$, $T_* = Tr$ as was used in [1]. The following lemma generalizes Lemma 3.1 of the previous paper and is proved in essentially the same way. It will be needed only later on in 2.6.

Lemma 1.6. *Let us assume that*

$$\eta(Y), \delta_* \eta(Y) = O\rho^{-(n-1)},$$

hold for large ρ . We have the following:

(a) *if $\eta(Y) = o\rho^{-(n-1)}$, then*

$$HM [\delta_* \eta; X, \pm t] = \pm T_* HM [\eta; X, \pm t];$$

(b) *if $\eta(\rho\gamma) + \eta(\rho\gamma') = o\rho^{-(n-1)}$ for every γ where*

$$\gamma = p(t, r, \rho)\alpha + p'(t, r, \rho)\alpha' = \gamma(\rho, r, t, \alpha, \alpha'),$$

and the corresponding γ' is given by

$$\gamma' = p(-t, r, \rho)\alpha - p'(-t, r, \rho)\alpha' = \gamma(\rho, r, -t, \alpha, -\alpha'),$$

then

$$HM [\delta_* \eta; X, t] + HM [\delta_* \eta; X, -t] = T_*(HM [\eta; X, t] - HM [\eta; X, -t]);$$

(c) *if $\eta(\rho\gamma) - \eta(\rho\gamma') = o\rho^{-(n-1)}$ for every (γ, γ') as in (b), then*

$$HM [\delta_* \eta; X, t] - HM [\delta_* \eta; X, -t] = T_*(HM [\eta; X, t] + HM [\eta; X, -t]).$$

Remark. γ' is defined in such a way so that $\gamma' \rightarrow -\gamma$ when $\rho \rightarrow \infty$.

Proof. Since $T(r^2 - t^2)r^{-1} = 0$ we find

$$(1.27) \quad T_* HM[\eta; X, t] = \frac{1}{\omega_{n-1}} (r^2 - t^2)^{-k} T \iint_{(r+t)/2}^{\infty} (1 - p^2)^k \rho^{2k} \eta(Y) d\rho d\omega_\alpha.$$

Observe that the variables r and t occur in the integrand only inside the argument of $p = p(t, r, \rho)$ (see (1.13)), and furthermore, [1, p. 662],

$$Tp = -r^{-1} \rho^2 \frac{\partial p}{\partial \rho}.$$

Hence by noting $Y = \rho\gamma$, $\gamma = p\alpha + (1 - p^2)^{1/2}\alpha'$, we obtain

$$\begin{aligned} T \cdot (1 - p^2)^k \rho^{2k} \eta(Y) &= -r^{-1} \rho^{2k+2} \frac{\partial}{\partial p} ((1 - p^2)^k \eta(\rho\gamma)) \cdot \frac{\partial p}{\partial \rho} \\ &= -r^{-1} \frac{d}{d\rho} \cdot \rho^{2k+2} (1 - p^2)^k \eta(Y) + r^{-1} (1 - p^2)^k \rho^{2k} \delta_* \eta, \end{aligned}$$

where in $\delta_* \eta(\rho\gamma)$ we have p and hence γ fixed.

At $\rho = (r + t)/2$, $p = 1$ by (1.13). Each of the conditions (a), (b) and (c) assures that there is no contribution at $\rho = \infty$ (where $p = t/r$) when the part or the parts involving the total differential $d/d\rho$ are evaluated. Finally we note that T_* , just as T , is odd in t , so that $-T_* HM[\eta; X, -t]$ is obtained from $T_* HM[\eta; X, t]$ by replacing in the latter t by $-t$.

Remark on T_* . More explicitly

$$T_* = \frac{1}{2}(r^2 + t^2) \frac{\partial}{\partial t} + tr \frac{\partial}{\partial r} + (1 + k)t.$$

One verifies easily that

$$(1.28) \quad \square^2 T_* = T_* \square^2 + 2t \square^2,$$

so that $T_* w(X, t)$ is a solution of the wave equation whenever $w(X, t)$ is one.

2. Two identities.

2.1. SM^* and VM^* The spherical means of a function $f(Y)$ in R_r^* are denoted by

$$SM[f; X, s] = \frac{1}{\omega_n} \int_{|\beta|=1} f(X + s\beta) d\omega_\beta,$$

when $Y = X$ is the center, s the radius of the sphere, and vector β is integrated over the unit sphere in R_r^* . We shall need more frequently the means

$$SM^* f(X) = SM[f; X, r], \quad r = |X|.$$

Later on it will be convenient to have the means over the solid sphere in the following form (see Lemma 2.6)

$$VM[f; X, s] = \frac{1}{s^{n-1}} \int_0^s SM[f; X, \tau] \tau^{n-1} d\tau,$$

and

$$VM^* f(X) = VM[f; X, r].$$

Lemma 2.1. *Suppose $f(Y) = 0$ for $|Y| > R$, then for large r $|SM^*f(X)| \leq \frac{1}{2} \text{Max } |f| R^{n-1} r^{-(n-1)}$ and $|VM^*f(X)| = Or^{-(n-1)}$.*

Both SM^* and VM^* have the following invariance property: Let A be a rotation or a similarity transformation each of which leaves the origin fixed. Then by setting for a moment $f(AY) = f_A(Y)$ we find

$$SM^*f(AX) = SM^*f_A(X), \quad \left(\frac{1}{r} VM^*f\right)(AX) = \frac{1}{r} VM^*f_A(X).$$

From this follows

Lemma 2.2. *Let $d_m(r) = r^m(\partial/\partial r)r^{-m+1}$, m integer, then by setting $d_m f(Y) = d_m(\rho)f(Y)$ we have*

$$\begin{aligned} d_m(r) \cdot SM^*f &= SM^* d_m f, & \Delta_\alpha SM^*f &= SM^* \Delta_\gamma f, \\ d_m(r) \cdot VM^*f &= VM^* d_{m-1} f, & \Delta_\alpha VM^*f &= VM^* \Delta_\gamma f. \end{aligned}$$

Proof. It suffices for the proof to verify it for $m = 1$, $d_1(r) = r(\partial/\partial r)$.

2.2. Even and odd solutions carrying smooth initial data. Let $u(X, t)$ and $v(X, t)$ be respectively the odd and even solutions satisfying the initial conditions

$$(2.1) \quad u(X, 0) = 0, \quad \frac{\partial}{\partial t} u(X, 0) = f(X),$$

and

$$(2.2) \quad v(X, 0) = g(X), \quad \frac{\partial}{\partial t} v(X, 0) = 0.$$

We assume that f and g are C^∞ functions with compact supports in R_x^n . The following are the explicit representations of u and v in terms of the SM of f and g [2, p. 682].

$$(2.3) \quad u(X, t) = \frac{1}{N} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^k t^{n-2} SM[f; X, t],$$

and

$$(2.4) \quad v(X, t) = \frac{1}{N} \frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^k t^{n-2} SM[g; X, t],$$

where $N = (n-2)(n-4) \cdots 1$ and as always $k = (n-3)/2$.

We are guided by Theorem 1.3 to consider the following boundary data:

$$(2.5) \quad \xi(X) = \frac{1}{k!} \delta_*^{k+1} r^{-k-1} v(X, r),$$

and

$$(2.6) \quad \psi(X) = \frac{1}{k!} \delta_*^k r^{-k-1} u(X, r).$$

Theorem 2.1. *The following holds:*

$$(a) \quad \xi(Y) = \frac{1}{2} \frac{1}{(n-2)!} (\rho \Delta_r^{k+1} \rho^{n-2}) SM^* g(Y),$$

and

$$(b) \quad \psi(Y) = \frac{1}{(n-2)!} (\rho^{-1} \Delta_r^k \rho^{n-2}) SM^* f(Y)$$

In (a) and (b) above we have a matching of the boundary data (ψ, ξ) with the initial data (f, g) . The theorem will be proved in 2.3.

Lemma 2.3. *Let*

$$z_0 = t^{n-2} SM[g; X, t], \quad z_1 = \frac{1}{t} \frac{\partial}{\partial t} z_0, \dots, z_k = \frac{1}{t} \frac{\partial}{\partial t} z_{k-1}.$$

Then for $j = 0, 1, \dots, k$, the function $z_j(X, t)$ satisfies the differential equation

$$(2.7) \quad L_j z_j \equiv \square^2 z_j + (2k - 2j) \frac{1}{t} \frac{\partial}{\partial t} z_j = 0.$$

Proof. $t^{2-n} z_0$ is a SM , it satisfies the Darboux Equation. By substituting $t^{2-n} z_0$ into the Darboux Equation we find $L_0 z_0 = 0$. To continue the proof of the lemma we apply $(1/t)(\partial/\partial t)$ to (2.7) and find

$$\frac{1}{t} \frac{\partial}{\partial t} L_j = L_{j+1} \frac{1}{t} \frac{\partial}{\partial t}.$$

Thus (2.7) is proved for all j by induction.

Lemma 2.4. *Let*

$$(2.8) \quad \Phi_j(X) = r z_j(X, r), \quad j = 0, 1, \dots, k,$$

$$\Phi_{k+1}(X) = \left[\frac{\partial}{\partial t} z_k(X, t) \right]_{t=r} \quad (= N \cdot v(X, r)).$$

Then the following holds:

$$(2.9) \quad \frac{\partial}{\partial r} r^j \Phi_j(X) = \frac{1}{2} r^j \Delta_x r^{-1} \Phi_{j-1}, \quad j = 1, 2, \dots, k+1.$$

Proof. We decompose the operator $r^{2i}L_{i-1}$ in the following way:

$$\begin{aligned} r^{2i}L_{i-1} &= \left(t^i \frac{\partial}{\partial t} + r^i \frac{\partial}{\partial r}\right) \left(-t^i \frac{\partial}{\partial t} + r^i \frac{\partial}{\partial r}\right) + r^{2i-2} \Delta_\alpha \\ &\quad + (2k + 2 - j) \left(t^{2i-1} \frac{\partial}{\partial t} + r^{2i-1} \frac{\partial}{\partial r}\right) \\ &\quad - (r^{2i} - t^{2i}) \left(\frac{\partial^2}{\partial t^2} - (2k + 2 - 2j, \frac{1}{t} \frac{\partial}{\partial t})\right) \end{aligned}$$

for $j = 1, 2, \dots, (k + 1)$. If we set for a moment as a notation

$$\left(\frac{\partial}{\partial r} - \frac{\partial}{\partial t}\right) z_i(X, t) = z'_i(X, t),$$

then we may write

$$z'_i(X, t) = -2 \frac{\partial}{\partial t} z_i(X, t) + \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial t}\right) z_i(X, t),$$

and hence

$$(2.10) \quad z'_i(X, r) = -2\Phi_{i+1}(X) + \frac{\partial}{\partial r} r^{-1}\Phi_i(X).$$

From $r^{2i}L_{i-1}z_{i-1}(X, t) = 0$ we obtain its limit relation when $t \rightarrow r$, by using the above decomposition of $r^{2i}L_{i-1}$. This leads to

$$r^i \frac{\partial}{\partial r} r^i z'_{i-1}(X, r) + r^{2i-2} \Delta_\alpha r^{-1}\Phi_{i-1} + (2k + 2 - j) r^{2i-1} \frac{\partial}{\partial r} r^{-1}\Phi_{i-1} = 0.$$

Replacing z'_{i-1} by (2.10) we find (2.9).

2.3. Proof of Theorem 2.1. Instead of dealing with $\xi(Y)$ of (2.5) directly, we define $\xi_0(Y), \xi_1(Y), \dots, \xi_{k+1}(Y)$, by setting

$$(2.11) \quad \xi_i(Y) = \delta_*^{k+1-i} \rho^{-k-1} \Phi_{k+1}(Y)$$

for each j , so that $\xi_{i-1} = \delta_* \xi_i$ and $\xi_0(Y) \equiv k!N\xi(Y)$. Also the $\xi_i(Y)$ will be useful later on, see 2.5.

Lemma 2.5. *We have*

$$(2.12) \quad \xi_i(Y) = \left(\frac{1}{2}\right)^{k+1-i} (\rho^{-i+1} \Delta_Y^{k+1-i} \rho^{n-2-i}) \rho^{i-(n-1)} \Phi_i(Y)$$

for $j = 0, 1, \dots, (k + 1)$.

Proof. We prove the above lemma by induction, starting from $j = k + 1$ with j decreasing until $j = 0$. For $j = k + 1$, (2.12) holds by the definition of ξ_{k+1} in (2.11). Suppose we have (2.12) for some j , and we want to examine

$$(2.13) \quad \xi_{i-1} = \delta_* \xi_i = \rho^{-2k} \frac{\partial}{\partial \rho} \rho^{2k-2} \xi_i(Y),$$

by replacing ξ_i on the right hand side by (2.12). Here we need a procedure which will allow us to commute $\partial/\partial\rho$ with Δ_r . Of course $\partial/\partial\rho$ commutes with Δ_γ . In general we note the following rule which will be used repeatedly later on. Let the weight of an operator $\rho^a(\partial/\partial\rho)^b\rho^c$ where a, b and c are integers, be defined by $(a + c - b)$. An operator with weight equal to zero is said to be homogeneous. Any homogeneous operator can be written in the form of a sum $\sum a_m(\rho\partial/\partial\rho)^m$ and hence any two homogeneous operators commute. Note that Δ_r has weight equal to -2 , δ_* has weight 1 and Δ_γ is homogeneous. By commutation of

$$\left(\rho^{-2k-1}\frac{\partial}{\partial\rho}\rho^{2k+2}\right), \quad (\rho^{-i+1}\Delta^{k+1-i}\rho^{n-2-i})$$

we find from (2.12) and (2.13) that

$$\xi_{i-1} = (\tfrac{1}{2})^{k+1-i}(\rho^{-i+2}\Delta^{k+1-i}\rho^{-i})\frac{\partial}{\partial\rho}\rho^i\Phi_i.$$

An application of (2.9) leads to the required result. Thus (2.12) is proved for all j . When we reach $j = 0$ we obtain (a) of Theorem 2.1 since $\Phi_0 = \rho^{n-1}SM^*g$.

The same result can be obtained by starting from ξ_0 and by decreasing the exponent of δ_* :

$$(2.14) \quad \xi_0 = \delta_*^k\delta_*\rho^{-k-1}\Phi_{k+1} = \tfrac{1}{2}(\rho\Delta\rho)\delta_*^k(\rho^{-k-2}\Phi_k),$$

where we have commuted Δ with δ_*^k . This process can be continued, now starting with the second factor $\delta_*^k\rho^{-k-2}\Phi_k$, etc.

For the proof of (b) Theorem 2.1 we replace g by f in the definition of Lemma 2.3 and keep the notation for z_i and Φ_i , $j = 1, 2, \dots, k$. This time we have $\Phi_k(X) = Nr u(X, r)$, so that by (2.6),

$$\psi(X) = \frac{1}{N}\frac{1}{k!}\delta_*^k r^{-k-2}\Phi_k(X).$$

By comparing the right hand side above with the second factor of (2.14) we see that (b) is obtained from (a) by cancelling out the factor $(\rho\Delta\rho)/2$. Hence (b) of Theorem 2.1 is proved.

Corollary 2.1. *We have*

$$(a)' \quad \xi(Y) = \frac{1}{2}\frac{1}{(n-2)!}SM^*(\rho\Delta^{k+1}\rho^{n-2})g,$$

$$(b)' \quad \psi(Y) = \frac{1}{(n-2)!}SM^*(\rho^{-1}\Delta^k\rho^{n-2})f.$$

Proof. By Lemma 2.2, using the fact that the operators shifted behind SM^* are homogeneous.

2.4. The identities. Since both f and g are C^∞ and have compact supports,

it follows from Lemma 2.1 and (a)', (b)' of Corollary 2.1, that ψ and ξ satisfy the growth condition of Theorem 1.2. Hence the following functions

$$(2.15) \quad U(X, t) = HM[\psi; X, t] + HM[\psi; X, -t],$$

and

$$(2.16) \quad V(X, t) = HM[\xi; X, t] + HM[\xi; X, -t],$$

are solutions of the wave equation.

Theorem 2.2. *In the exterior of C_0 one has for the even and odd solutions carrying smooth initial data the following representations:*

$$(2.17) \quad u(X, t) = (-1)^{k+1} T_* U(X, t),$$

$$(2.18) \quad v(X, t) = (-1)^{k+1} V(X, t),$$

where u and v are given by (2.3) and (2.4), while U and V by (2.15) and (2.16).

Proof. Clearly it suffices to match the initial conditions of u and U , and those of v and V . (a) Clearly $V_t(X, 0) = 0$, and

$$\lim_{t \rightarrow 0} T_* U(X, t) = \lim_{t \rightarrow 0} \frac{1}{2} r^2 \frac{\partial}{\partial t} U(x, t) = 0.$$

(b) The requirement of $V(X, 0) = g(X)$ leads to the following: Set $HM_0 \xi = HM[\xi; X, 0]$, then by the invariance properties of HM_0 and SM^* we find

$$2HM_0 \xi(X) = \frac{1}{(n-2)!} (r \Delta^{k+1} r^{n-2}) HM_0 SM^* g.$$

To verify the initial condition amounts to the proof of the identity

$$(2.19) \quad \frac{1}{(n-2)!} (r \Delta^{k+1} r^{n-2}) HM_0 SM^* g = (-1)^{k+1} g(X).$$

Now, one is led to the same identity (with g replaced by f) when one wishes to verify the condition $U_t(X, 0) = f(X)$. Indeed we find,

$$\lim_{t \rightarrow 0} \frac{\partial}{\partial t} T_* HM[\psi; X, t] = \left(\frac{r^2}{2} \Delta_x + r \frac{\partial}{\partial r} + (k+1) \right) HM_0 \psi,$$

where the operator $\partial^2/\partial t^2$ on the left has been changed to Δ_x in the right, since it is applied to a HM . This leads to

$$\lim_{t \rightarrow 0} \frac{\partial}{\partial t} T_* HM[\psi; X, t] = \frac{1}{2} (r \Delta_x r) HM_0 \psi,$$

and from (b)' of Corollary 2.1 we find (2.19) for f .

Altogether we find that the matching of the initial conditions is accomplished by verifying (2.19). This identity has been previously used by Rhee [9]; it is easily proved by using a theorem of iterated means [5]. Note that $HM_0 \xi$ differs from the Radon transform of ξ only in a weight factor of the plane integral. For

comparision with (2.19), we note that the inversion formula of the Radon transform for odd dimensional space [5, p. 13] reads in our notation as follows:

$$(2.20) \quad \frac{1}{(n-2)!} (r\Delta^{1+k}r^{n-2})SM^*HM_0g = (-1)^{k+1}g(X).$$

2.5. *The growth of the boundary data ξ_i .* ξ_0 or ξ has been represented in Corollary 2.1 by a SM^* , hence $\xi_0(X) = Or^{-(n-1)}$ for large X by Lemma 2.1. We shall need estimates for ξ_i for the purpose of applying Lemma 1.6, in which the sharper estimate $or^{-(n-1)}$ is required. In the following discussion we consider ξ_i as being represented by (2.12). First we go back to $x_i(X, t)$ and $\Phi_i(X)$ of Lemma 2.4.

Lemma 2.6. *We have*

$$\frac{\partial}{\partial t} SM[g; X, t] = VM[\Delta g; X, t],$$

$$\frac{\partial}{\partial t} VM[g; X, t] = SM[g; X, t] - (n-1)t^{-1}VM[g; X, t].$$

In the above g may be replaced by $\Delta^i g$. By induction it follows:

$$(2.21) \quad \left(\frac{\partial}{\partial t}\right)^{2m} SM[g; X, t] \\ = SM[\Delta^m g; X, t] + \sum_{i=0}^{m-1} \dots t^{-2i-1} VM[\Delta^{m-i} g; X, t] \\ + \sum_{i=1}^{m-1} \dots t^{-2i} SM[\Delta^{m-i} g; X, t],$$

$$(2.22) \quad \left(\frac{\partial}{\partial t}\right)^{2m+1} SM[g; X, t] \\ = VM[\Delta^{1+m} g; X, t] + \sum_{i=0}^{m-1} \dots t^{-2i-1} SM[\Delta^{m-i} g; X, t] \\ + \sum_{i=1}^m \dots t^{-2i} VM[\Delta^{m-i+1} g; X, t],$$

where $\dots$ denote integer coefficients.

We now consider the function z_i of Lemma 2.3, and we have

$$(2.23) \quad t^{i-(n-1)} \cdot tz_i(X, t) = t^{i-(n-1)} \frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t}\right)^{i-1} t^{n-2} SM[g; X, t] \\ = \left(\frac{\partial}{\partial t}\right)^i SM[g; X, t] + \dots,$$

where all terms with a negative power of t as factor are not given explicitly. Such terms are of the form:

$$\text{const. } t^{i-i} \left(\frac{\partial}{\partial t} \right)^i SM[g; X, t], \quad 0 \leq i < j.$$

We replace each $(\partial/\partial t)^i SM[g; X, t]$, $0 \leq i \leq j$ by an expansion using either (2.21) or (2.22), and as the result we obtain an expansion on the right hand side of (2.23). In this expansion we set $t = r$ and find in this way the function $r^{j-(n-1)} \Phi_j(X)$. We have

Lemma 2.7. (a) *When j is even,*

$$(2.24) \quad r^{j-(n-1)} \Phi_j(X) = SM^* \Delta^{i/2} g + \cdots ;$$

(b) *when j is odd,*

$$(2.25) \quad r^{j-(n-1)} \Phi_j(X) = VM^* \Delta^{(1+i)/2} g + \cdots .$$

The terms which are not explicitly given in the above are of two kinds:

$$\text{const. } r^{2s-i} SM^* \Delta^s g; \quad \text{const. } r^{2s-i+1} VM^* \Delta^{1+s} g,$$

with $s = 0, 1, \dots, (\frac{1}{2}j - 1)$ in the even case, and in the odd case, $s = 0, 1, \dots, (\frac{1}{2}j - \frac{1}{2})$ in SM^* , while $s = 0, 1, \dots, (\frac{1}{2}j - \frac{3}{2})$ in VM^* .

Finally we obtain an expansion of ξ_i by using (2.12) and the above lemma. If we set for a moment

$$(2.26) \quad \Lambda_i = \left(\frac{1}{2} \right)^{k+1-i} r^{-i+1} \Delta^{k+1-i} r^{n-2-i},$$

then to obtain ξ_i , we should apply the operator Λ_i to each term on the right hand sides of (2.24) and (2.25). Now Λ_i is a homogeneous operator. When it is written as a sum of products of homogeneous operators $(r(\partial/\partial r))^i$ and Δ_α we see that each operator of the first order or Δ_α will affect a power of r only by a multiplicative constant, or such an operator can be shifted behind SM^* or VM^* by Lemma 2.2.

Lemma 2.8. *For $j = 0, 1, \dots, (k+1)$, we have for large r*

$$(2.27) \quad \xi_i(X) = Or^{-(n-1)}.$$

Furthermore,

$$(2.28) \quad \xi_i(X) = SM^* \Lambda'_i g + Or^{-n} \quad \text{for } j \text{ even},$$

$$(2.29) \quad \xi_i(X) = VM^* \Lambda'_i g + Or^{-n} \quad \text{for } j \text{ odd},$$

where

$$\Lambda'_i = \Lambda_i \Delta^{i/2}, \quad \Lambda'_i = r^{-1} \Lambda_i r \Delta^{(1+i)/2},$$

with Λ_i given by (2.26).

Proof. By Lemma 2.7 the first term in (2.24) and that in (2.25) are the only terms which have no factor r^{-1} , hence we obtain Or^{-n} in (2.28) and (2.29). Λ'_i and Λ'_i are the result of shifting Λ_i behind SM^* and VM^* , Lemma 2.2. Obviously (2.27) follows from (2.28), (2.29), and Lemma 2.1.

2.6. *A weak form of the identities.* The following lemma is a generalization of Theorem 3.1 in [1, p. 663].

Lemma 2.9. *With ξ_{i-1} and ξ_i given by (2.12) and $\xi_{i-1} = \delta_* \xi_i$, the equation*
 (2.30)
$$HM[\xi_{i-1}; X, t] + (-1)^{i-1} HM[\xi_{i-1}; X, -t]$$

$$= T_*(HM[\xi_i; X, t] + (-1)^i HM[\xi_i; X, -t])$$

holds for (X, t) in the exterior of C_0 .

Proof. Apply Lemma 1.6 with $\eta = \xi_i$. We shall verify the assumptions on η by using Lemma 2.8. Each term in ξ_i except the first one in (2.28) or (2.29) satisfies condition (a) of Lemma 1.6, because of the estimates $O\rho^{-n}$. There remains to verify condition (b) when j is odd, and condition (c) when j is even.

Now, when j is odd, we prove first that

$$\rho^{n-1}(VM^*\Delta'_i g(\rho\gamma) + VM^*\Delta'_i g(\rho\gamma')) \rightarrow 0$$

as $\rho \rightarrow \infty$, where γ is fixed and $\gamma' = -\gamma$. The above sum of VM^* converges to the volume integral of $\Delta'_i g$ over the whole space. Since $\Delta'_i g$ is easily seen to be a divergence expression, and g has compact support, so the integral goes to zero when $r \rightarrow \infty$ (to obtain the divergence, shift the last Δ in Δ'_i to the front).

For the case that j is even we show first

$$SM^*\Delta'_i g(\rho\gamma) - SM^*\Delta'_i g(\rho\gamma') = o\rho^{-(n-1)},$$

when $\rho \rightarrow \infty$ and $\gamma' = -\gamma$. Both spheres in the above SM^* converge to the same hyperplane ($\perp \gamma$ and passing through the origin). We obtain the above estimate by using Lemma 2.1. Indeed we may identify the points of the two spheres by letting Q_1 of the one sphere correspond to its mirror image Q_2 of the other sphere, the limit hyperplane is taken as the plane of symmetry. Since g has compact support, it follows

$$\text{Max}_{(Q_1, Q_2)} |\Delta'_i g(Q_1) - \Delta'_i g(Q_2)| \rightarrow 0$$

as $t \rightarrow \infty$. Substitute the above Max for Max $|f|$ of Lemma 2.1 and the estimate is established.

Now when $\rho \neq \infty$, γ' of Lemma 1.6 is not equal to $-\gamma$. But $\gamma' \rightarrow -\gamma_0$ and $\gamma \rightarrow \gamma_0$ as $\rho \rightarrow \infty$, since $p(t, r, \rho) \rightarrow t/r$ which is odd in t . By a slight modification of the above arguments the conditions (b) and (c) are verified.

Theorem 2.3. *Let $w(X, t)$ be a solution of the wave equation with smooth initial data. Let its boundary values on C_0 be*

$$(2.31) \quad \begin{aligned} \zeta_1(X) &= w(X, r) = u(X, r) + v(X, r), \\ \zeta_2(X) &= w(X, -r) = -u(X, r) + v(X, r). \end{aligned}$$

Then in the exterior of C_0 we have the following representation of $w(X, t)$:

$$(2.32) \quad \begin{aligned} &(-1)^{k+1} w(X, t) \\ &= \frac{1}{k!} T_*^{k+1} (HM[\rho^{-k-1} \zeta_1(Y); X, t] + (-1)^{k+1} HM[\rho^{-k-1} \zeta_2(Y); X, t]). \end{aligned}$$

Proof. We may verify separately for even and odd solutions, $w(X, t) = v(X, t)$ and $w(X, t) = u(X, t)$. For $v(X, t)$ one starts from the identity (2.16) and (2.18), with $\xi = (1/N)(1/k!)\xi_0$. Now apply Lemma 2.9 (2.30) first with $j = 1$, then $j = 2$, etc. until $j = k + 1$. But $\xi_{k+1}(Y) = \rho^{-k-1}\Phi_{k+1}(Y) = N\rho^{-k-1}v(X, r)$ by (2.8). Thus (2.32) is established for $w = v$. The case of $w = u$ is similar, starting here from the identity (2.15) and (2.17), and one uses Lemma 2.9 k times. We find (2.32) by addition, $w = u + v$.

3. L_2 -solutions.

3.1. Square integrable initial data. We denote by $\|f\|$ the norm of f ,

$$\|f\|^2 = \int_{R^n} |f(X)|^2 dX.$$

Definition. The initial data f and g of a generalized solution $w(X, t)$ of the wave equation is said to be square integrable if the following hold: f and g are locally square integrable in R^n and

$$(3.1) \quad \|f\|, \quad \| |g_X| \| < \infty,$$

$$(3.2) \quad \|g(X)/r\| < \infty.$$

To explain the use of the norm of $g(X)/r$ we give the following two lemmas.

Lemma 3.1. Let $\|g_r\| < \infty$ and suppose

$$(3.3) \quad g(X) = \int_r^\infty \frac{\partial g}{\partial r}(r\alpha) dr$$

a.e., then $\|g(X)/r\| < \infty$.

Proof. We have for $k = (n - 3)/2$, by assumption,

$$(3.4) \quad \left(\frac{g(X)}{r} \right)^2 r^{n-1} = \left(r^k \int_r^\infty \frac{\partial}{\partial s} g(s\alpha) ds \right)^2 \leq \left(\int_r^\infty s^k \left| \frac{\partial}{\partial s} g(s\alpha) \right| ds \right)^2.$$

By using an inequality of Hardy [6, p. 244] we find

$$\begin{aligned} \left\| \frac{g(X)}{r} \right\|^2 &= \int d\omega_\alpha \int_0^\infty \left| \frac{g(X)}{r} \right|^2 r^{n-1} dr \\ &\leq 4 \int d\omega_\alpha \int_0^\infty s^{n-1} \left| \frac{\partial}{\partial s} g(s\alpha) \right|^2 ds = 4 \|g_r\|^2. \end{aligned}$$

Lemma 3.2. Let g be locally L_2 , and $\|g_X\| < \infty$. Furthermore let

$$\left\| \frac{g(X)}{r} \right\|_1^2 = \int d\omega_\alpha \int_1^\infty \left| \frac{g(X)}{r} \right|^2 r^{n-1} dr < \infty.$$

Then g can be approximated by a sequence of C^∞ functions with compact supports, g_n , such that as $n \rightarrow \infty$,

- 1) $g_n - g \rightarrow 0$ in local L_2 -norm,
- 2) $\|g_{nx} - g_x\| \rightarrow 0$,
- 3) $\|g_n/r - g/r\|_1 \rightarrow 0$,
- 4) $\|g/r\| \leq 2\|g_x\|$.

Proof. Same as Lemma 5.1 in [1, p. 669]. Note for each g_n , (3.3) holds.

3.2. Boundary values of L_2 -solutions.

Definition. A generalized solution [10], $w(X, t)$, of the wave equation in the (X, t) -space is called an L_2 -solution, if it has the following properties: 1) w and its generalized derivatives of the first order are locally square integrable, and 2) in any hyperplane, $\|w/r\|$, $\|w_x\|$ and $\|w_t\|$ are finite.

A solution $w(X, t)$ is an L_2 -solution if its initial data is square integrable [4], [10].

Theorem 3.1. The boundary values of an L_2 -solution $\xi_1(Y)$ and $\xi_2(Y)$ respectively on C_0^+ and C_0^- satisfy the following conditions:

$$(3.5) \quad \|\xi_x\| < \infty,$$

$$(3.6) \quad \|\xi(X)/r\| \leq 4 \|\xi_x\|,$$

where ξ stands for ξ_1 or ξ_2 .

Proof. (3.5) follows from the energy inequality,

$$\left\| \left| \frac{\partial \xi}{\partial X} \right| \right\|^2 \leq \|f\|^2 + \|g_x\|^2,$$

see [1, p. 668. To conclude (3.6) we note that when f_n and g_n are smooth functions, the corresponding boundary values

$$(3.7) \quad u_n(X, r), v_n(X, r) = Or^{-k-1} \text{ for large } r,$$

by Lemma 2.7 (see (2.4), (2.8), and (2.24)). Hence, for $\xi_n(X) = \pm u_n(X, r) + v_n(X, r)$,

$$\xi_n(X) = \int_r^\infty \frac{d\xi_n}{dr} dr,$$

and we have $\|\xi_n/r\|^2 \leq 4\|\xi_{nx}\|^2$. In the limit we obtain (3.6). Note that when $\xi = u_n$ or v_n , $\|\xi/r\| < \infty$ follows also directly from (3.7).

3.3. Representation of L_2 -solutions.

Lemma 3.3. Let ξ satisfy (3.5) and (3.6), then $HM[\rho^{-k-1}\xi(Y); X, t]$ exists for any $X \neq 0$ and t a.e. in $-r < t < r$. The same holds true for its first partial derivative.

Proof. Let $X = X_0 \neq 0$ be fixed. Consider the family of hyperboloids $Hd(X_0, t)$ with parameter t , $-r_0 < t < r_0$, it covers the space R_X^n simply, with

the exception of two half-lines. The volume element in R_Y^* can be expressed in terms of dt and the invariant measure $dm(X_0, t, Y)$ of (1.14):

$$(3.8) \quad dY = \rho(\rho - t) dm(X_0, t, Y) dt.$$

Hence

$$(3.9) \quad \int_{t'}^{t''} HM[\rho^{-k-1}\zeta(Y); X_0, t] dt \\ = \frac{1}{\omega_{n-1}} \frac{1}{r_0} \int_{B_0} (r_0^2 - t^2)^{-k} \rho^{-1} |Y - X_0|^{-1} \rho^{-k-1} \zeta(Y) dY,$$

where B_0 is the domain in R_Y^n which is covered by the hyperboloids $Hd(X_0, t)$ with $-r_0 < t' \leq t \leq t'' < r_0$. In the integrand above we made use of $\rho - t = |Y - X_0|$ which when squared on both sides, is the same as (0.3). Now $(r_0^2 - t^2)$ is bounded away from zero. The absolute integrability of the right hand side of (3.9) follows from the Schwartz Inequality applied to the product of $\rho^{-k-1} \cdot |Y - X_0|^{-1}$ and $\rho^{-1}\zeta(Y)$. From this one concludes the existence of $HM[\rho^{-k-1}\zeta(Y); X_0, t]$ for t a.e. in $[-r_0, r_0]$.

Since the differentiation can be carried out under the integral sign by the invariance property of HM (see (1.2)), we can use (3.5) to conclude the existence of the integral in the same way as above.

Theorem 3.2. *An L_2 -solution of the wave equation $w(X, t)$ can be represented as a distribution in $R_{X,t}^{n+1}$ in the exterior of C_0 , by the following:*

$$(3.10) \quad (-1)^{k+1}w(X, t) = \frac{1}{k!} T_*^{k+1}(HM[\rho^{-k-1}\zeta_1(Y); X, t] \\ + (-1)^{k+1}HM[\rho^{-k-1}\zeta_2(Y); X, t]),$$

where $\zeta_1 = w(X, r)$ and $\zeta_2 = w(X, -r)$. An L_2 -solution is uniquely determined by its boundary values ζ_1 and ζ_2 , and the domain of dependence consists of any neighborhood of the intersection of C_0 with the characteristic cones having (X, t) as the vertex.

Proof. (3.10) has exactly the same form as (2.32) except that we interpret T_* as a distribution derivative of HM in the exterior of C_0 . When the L_2 -data is approximated by smooth functions f_n, g_n we obtain the corresponding $HM[\rho^{-k-1}\zeta_n(Y); X, t]$. From the analysis of Lemma 3.3 we have $\|(\zeta_n - \zeta)/r\|^2 \leq 4(\|f_n - f\|^2 + \|g_n - g\|^2)$. Hence

$$\int_{t'}^{t''} HM[\rho^{-k-1}\zeta_n(Y); X, t] dt \rightarrow \int_{t'}^{t''} HM[\rho^{-k-1}\zeta(Y); X, t] dt$$

uniformly for X in any compact set in the exterior of C_0 , when (t', t'') are properly adjusted. We can then proceed in the same way as in [1] Lemma 4.1, (4.5) to conclude (3.10) and the uniqueness.

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