*Elastic-plastic Torsion of Convex Cylindrical Bars**

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1. Introduction. Let G be a bounded convex plane domain bounded by a finite number of twice continuously differentiable arcs. That is, ∂G , the boundary of G , possesses a continuously varying positive radius of curvature except at a finite number of corners. According to von Mises [1] the elastic-plastic torsion problem of a cylindrical bar with cross sections G is to find a function $\psi(x, y)$ continuous in $\bar{G}$, continuously differentiable in G and vanishing on ∂G , such that (i) $|\text{grad } \psi|$ is less than or equal to a positive constant k (the yielding constant) in G and (ii) wherever the strict inequality holds, ψ must satisfy the Poisson equation $\Delta\psi = -2\mu\theta_0$ in the strict sense. Here θ_0 is the angle of twist per unit length and μ is the (positive) shearing modulus. Although the problem has not been completely solved in the strict sense for arbitrary domain G , much has been done during the recent years [1-5, 9-14, 31].

In what follows we shall formulate it as a variational minimum problem with inequality-side constraint. To this end, we define the distance function Ψ as follows:

$$(1.1) \quad \Psi(q) = k\rho(q, \partial G), \quad q \in \bar{G},$$

where $\rho(q, \partial G)$ stands for the distance from the point q to ∂G and k is the yielding constant.

For every point s on ∂G , let $\gamma(s)$ be the point in $\bar{G}$ such that

- (i) $\rho(\gamma, s) = \rho(\gamma, \partial G)$,
- (ii) the open disk $D(\gamma, \rho(\gamma, s))$ with center at γ and radius $\rho(\gamma, s)$ is the maximum disk contained in G and it touches ∂G at s . We shall call the set of points

$$(1.2) \quad \Gamma \equiv \bigcup_{s \in \partial G} \gamma(s)$$

the ridge of the domain G . It is known [2] that Γ consists of a finite number of smooth curves and that Ψ is smooth in $G - \Gamma$. Hence, Ψ possesses finite Dirichlet norm over G .

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Let $C_0^\infty(G)$ be the class of infinitely differentiable functions with compact support in G . For every f in $C_0^\infty(G)$, we denote by $\|f\|_1$ the Dirichlet norm of f , i.e.,

$$\|f\|_1^2 = \iint_G [(\nabla f)^2 + f^2] dx dy.$$

Let $H_0^1(G)$ be the completion of $C_0^\infty(G)$ under $\|\cdot\|_1$ -norm. Then $H_0^1(G)$ is a complete Hilbert space. Also, $H_0^0(G) = H^0(G)$ consists of square integrable functions over G . Let F be the family consisting of functions u such that

- (i) u belongs to $H_0^1(G)$,
- (ii) $u \leq \Psi$ almost everywhere in G .

Our problem is to find a function ψ such that it minimizes the integral,

$$(1.3) \quad J[u] = \iint_G [(\text{grad } u)^2 - 4\mu\theta_0 u] dx dy,$$

among all functions u in F and to show that the extremal ψ solves the elastic-plastic problem in the strict sense.

The formulation as stated differs from the usual one [12–14] in that instead of using the yield condition [5, 22] in the mean we majorize the admissible functions by the distance function Ψ defined in (1.1). In this way the admissible class has been enlarged (see §5). However, it turns out that the extremal ψ satisfies the yield condition point-wisely. Thus, by enlarging the admissible class, more information about the extremal has been derived by means of variations.

In what follows, we first establish the existence of a Lipschitz continuous extremal by approximating the convex domain G by convex polygons, §2. Next, we show how the domain G was partitioned into elastic and plastic portions according to which one of the two different physical laws holds there, §3. It is in this part of the analysis that the present inequality-side constraint plays the important role. Topologically, the unknown elastic-plastic boundary could be very complicated. Although the present results are not so complete as those obtained in [3, 4], the existence of an elastic core, parametric representation of the free boundary, simple connectivity of the elastic region and the intersection property of the plastic portion have been established, §3, 6, 7. If the elastic-plastic surface which separates the convex cylindrical bar into elastic and plastic parts is known to have finite Lebesgue area, then an application of the well-known Thomas compatibility conditions [19, 20] leads to the fact that the stress components must be continuous across the elastic-plastic surface. Accordingly, it is desirable to know whether the free boundary is rectifiable or even smooth. I even suspect that the extremal surface is concave for G being convex. Although these questions remain open, by applying the Nash-DeGiorgi-Möser-Morrey regularity theorem [15–17, 30] and that of Brezis and Stampacchia [14], the continuity of the stress components in the whole domain G is proved, §5. Finally we add a remark here: by precisely the same proof as was given in [§10, 3], one concludes that the plastic portion of $\tilde{G}$ grows as the angle of twist per unit length

increases, or equivalently the angle of twist per unit length is monotone increasing function of the applied torque and conversely. Throughout the entire work, Hopf's maximum principle offers great help. Also, the author has been benefited by many helpful discussions with his colleague S. B. Bank.

2. Existence of the Lipschitz-continuous extremal.

a. Existence and uniqueness of the extremal. Applying precisely the same argument as was shown in §2 [4, III], one can easily establish the existence of the extremal ψ which belongs to $H_0^1(G)$. However, in order to show that the extremal ψ solves the elastic-plastic torsion problem in the strict sense we rely heavily on the results obtained in [4, III].

b. Inner approximations. Let $\{G_n^*\}$ be a sequence of convex polygons such that each G_n^* is inscribed in the convex domain G and that

$$(2.1) \quad G_{n+1}^* \supset G_n^*, \quad n = 1, 2, \dots, \lim_{n \rightarrow \infty} G_n^* = G.$$

The results in [4, III] assure that for each integer n the variational minimum problem with domain being the convex polygon G_n^* has a unique smooth solution ψ_n^* such that ψ_n^* vanishes on ∂G_n^* and

$$(2.2) \quad |\text{grad } \psi_n^*| \leq k \text{ in } G_n^*,$$

where k is the same positive constant appearing in (1.1).

Now, we extend each extremal ψ_n^* to the whole domain G by setting

$$(2.3) \quad \psi_n^* \equiv 0 \text{ in } \bar{G} - G_n^*, \quad n = 1, 2, \dots.$$

From (2.2) and (2.3) and the mean-value theorem we see that each ψ_n^* , so extended to the whole domain G , is Lipschitz continuous in the closed domain $\bar{G}$ with the Lipschitz constant being equal to k . By Arzela's theorem, there exists a subsequence which we also denote by $\{\psi_n^*\}$ such that it converges uniformly to a Lipschitz continuous limit ψ^* in G with the same Lipschitz constant k . We wish to show that ψ^* so constructed is not only an admissible function but also the minimizing extremal of our variational minimum problem with domain G .

Let Ψ_n^* be the majorant function of the minimum problem with domain G_n^* . That is,

$$(2.4) \quad \Psi_n^*(q) = k\rho(q, \partial G_n^*), \quad q \in \bar{G}_n^*.$$

Further, we extend Ψ_n^* to the whole closed domain $\bar{G}$ by setting

$$(2.5) \quad \Psi_n^* \equiv 0 \text{ in } \bar{G} - G_n^*.$$

From (1.1) and (2.4) and the inclusion relations in (2.1) it follows that for every point q in $\bar{G}$

$$(2.6) \quad \psi_n^*(q) \leq \Psi_n^*(q) \leq \Psi_{n+1}^*(q) \leq \Psi(q), \quad n = 1, 2, \dots.$$

By letting $n \rightarrow \infty$ in the inequalities above, we conclude that for every point q in $\bar{G}$, $\psi^*(q) \leq \Psi(q)$. This proves that the function ψ^* so constructed is an admissible one for the variational minimum problem with domain G .

From the definition in (1.1) for Ψ , the first and the second inequality in (2.6) and the inclusion relations in (2.1) we see that for every integer n , ψ_n^* is an admissible function of the minimum problem with domain G_{n+1}^* . Consequently, for every integer n ,

$$\begin{aligned}
 (2.7) \quad J[\psi_n^*, G_n^*] &\equiv \iint_{G_n^*} [(\nabla \psi_n^*)^2 - 4\mu\theta_0\psi_n^*] dx dy \\
 &= \iint_{G_{n+1}^*} [(\nabla \psi_n^*)^2 - 4\mu\theta_0\psi_n^*] dx dy \\
 &\geq \iint_{G_{n+1}^*} [(\nabla \psi_{n+1}^*)^2 - 4\mu\theta_0\psi_{n+1}^*] dx dy \\
 &\equiv J[\psi_{n+1}^*, G_{n+1}^*] \geq J[\psi, G],
 \end{aligned}$$

where ψ is the minimizing extremal of the minimum problem with domain G , the second equality sign follows from (2.3) and (2.1) and the two inequality signs follow from the fact that ψ_n^* is an admissible function in G_{n+1}^* and ψ_{n+1}^* is the minimizing extremal in G_{n+1}^* and is an admissible function in G .

The inequality (2.7) proves that the numerical sequence, $\{J[\psi_n^*, G_n^*]\}$, is monotone non-increasing and is bounded from below by $J[\psi, G]$. Thus, it is convergent and

$$(2.8) \quad \lim_{n \rightarrow \infty} J[\psi_n^*, G_n^*] = J[\psi^*, G] \equiv d^* \geq d \equiv J[\psi, G].$$

We wish to show that d^* is actually equal to d so that ψ and ψ^* become the same element in $H_0^1(G)$. To this end we consider

c. Outer approximations. Let $\{G_n^{**}\}$ be a sequence of convex polygons such that

- (i) $G_n^{**} \supset G \supset G_n^*$, $n = 1, 2, \dots$,
- (ii) the ridge of G_n^{**} is properly contained in the ridge of G_n^{**} , i.e., G_n^* and G_n^{**} are similar with parallel corresponding sides,
- (iii) at least one side of G_n^{**} just touches ∂G . It follows immediately from (i)–(iii) and (2.1) that

$$(2.9) \quad G_{n+1}^{**} \subset G_n^{**}, \quad n = 1, 2, \dots, \quad \lim_{n \rightarrow \infty} G_n^{**} = G.$$

For each convex polygon G_n^{**} , let ψ_n^{**} be the unique minimizing extremal of the minimum problem with domain G_n^{**} . Correspondingly, let Ψ_n^{**} be the majorant function defined by

$$(2.10) \quad \Psi_n^{**}(q) = k\rho(q, \partial G_n^{**}), \quad q \text{ in } \bar{G}_n^{**}, \quad n = 1, 2, \dots$$

For every integer n , we can extend the functions, ψ , Ψ , ψ_n^{**} and Ψ_n^{**} to the closed domain $\bar{G}_n^{**}$ by setting them equal to zero in $\bar{G}_n^{**} - G$ and in $\bar{G}_n^{**} - G_{n+1}^{**}$ respectively. Clearly, these four functions so extended to $\bar{G}_n^{**}$ all belong to $H_0^1(G_n^{**})$. Further, it is easy to check that

$$(2.11) \quad \psi(q) \leq \Psi(q) \leq \Psi_n^{**}(q) \geq \Psi_{n+1}^{**}(q) \geq \psi_{n+1}^{**}(q),$$

where the first inequality in (2.11) holds almost everywhere in $\bar{G}_n^{**}$ and all other inequalities hold everywhere. Hence, for every integer n , the extended extremals ψ and ψ_{n+1}^{**} are both admissible functions of the minimum problem with domain G_n^{**} . Accordingly,

$$(2.12) \quad \begin{aligned} J[\psi_n^{**}, G_n^{**}] &\equiv \iint_{G_n^{**}} [(\nabla \psi_n^{**})^2 - 4\mu\theta_0\psi_n^{**}] dx dy \\ &\leq \iint_{G_n^{**}} [(\nabla \psi)^2 - 4\mu\theta_0\psi] dx dy \\ &= \iint_G [(\nabla \psi)^2 - 4\mu\theta_0\psi] dx dy \equiv J[\psi, G], \end{aligned}$$

where the inequality sign follows from the fact that ψ is an admissible function in G_n^{**} and the second equality sign follows from the fact that $\psi \equiv 0$ in $\bar{G}_n^{**} - G$. Similarly, we have

$$(2.13) \quad \begin{aligned} J[\psi_{n+1}^{**}, G_{n+1}^{**}] &\equiv \iint_{G_{n+1}^{**}} [(\nabla \psi_{n+1}^{**})^2 - 4\mu\theta_0\psi_{n+1}^{**}] dx dy \\ &= \iint_{G_n^{**}} [(\nabla \psi_{n+1}^{**})^2 - 4\mu\theta_0\psi_{n+1}^{**}] dx dy \\ &\geq \iint_{G_n^{**}} [(\nabla \psi_n^{**})^2 - 4\mu\theta_0\psi_n^{**}] dx dy \equiv J[\psi_n^{**}, G_n^{**}], \end{aligned}$$

where the second equality sign follows from the vanishing of ψ_{n+1}^{**} in $\bar{G}_n^{**} - G_{n+1}^{**}$ and the inequality sign follows from the fact that ψ_{n+1}^{**} is an admissible function and ψ_n^{**} is the minimizing extremal in G_n^{**} . Now, the inequalities in (2.12) and (2.13) assure that the numerical sequence, $\{J[\psi_n^{**}, G_n^{**}]\}$, is monotone non-decreasing and is bounded from above by $J[\psi, G]$. Hence, it is convergent and

$$(2.14) \quad \lim_{n \rightarrow \infty} J[\psi_n^{**}, G_n^{**}] \equiv d^{**} \leq d \equiv J[\psi, G].$$

Comparing the two inequalities in (2.14) and (2.8) we see that

$$(2.15) \quad d^{**} \leq d \leq d^*.$$

d. Lipschitz continuity of the extremal. We are now in a position to show that the function ψ^* defined by means of the inner approximations is actually the minimizing extremal of the minimum problem with domain G . To show this it suffices, in view of (2.15), to show that

$$(2.16) \quad d^{**} = d^*.$$

To establish the equality (2.16) we shall modify the function ψ_n^{**} so as to define an admissible function for the minimum problem with domain G_n^* . For each integer n , let

$$(2.17) \quad \epsilon_n = \rho(\partial G_n^*, \partial G_n^{**}).$$

Then $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Let H_n^* be the open set on which $\psi_n^{**}(x, y) > \epsilon_n$. Define

$$(2.18) \quad \varphi_n(x, y) = \begin{cases} \psi_n^{**}(x, y) - k\epsilon_n & \text{for } (x, y) \text{ in } H_n^*, \\ 0 & \text{for } (x, y) \text{ in } \bar{G}_n^* - H_n^*. \end{cases}$$

Then $\varphi_n(x, y)$ is continuous in $\bar{G}_n^*$, piecewise smooth in G_n^* and vanishes on ∂G_n^* . Hence, it belongs to $H_0^1(G_n^*)$. Furthermore for every point (x, y) in $\bar{G}_n^* - H_n^*$,

$$\varphi_n(x, y) - \Psi_n^*(x, y) = -\Psi_n^*(x, y) \leq 0,$$

and for (x, y) in H_n^* ,

$$\begin{aligned} \varphi_n(x, y) - \Psi_n^*(x, y) &= \psi_n^{**}(x, y) - k\epsilon_n - [\Psi_n^{**}(x, y) - k\epsilon_n] \\ &= \psi_n^{**}(x, y) - \Psi_n^{**}(x, y) \leq 0. \end{aligned}$$

This proves that the function $\varphi_n(x, y)$ defined in (2.18) is admissible for the minimum problem with domain G_n^* . Consequently, for every integer n ,

$$\begin{aligned} J[\varphi_n, G_n^*] &\equiv \iint_{G_n^*} [(\nabla \varphi_n)^2 - 4\mu\theta_0\varphi_n] dx dy \\ &\geq \iint_{G_n^*} [(\nabla \psi_n^{**})^2 - 4\mu\theta_0\psi_n^{**}] dx dy \equiv J[\psi_n^{**}, G_n^*] \geq d^*. \end{aligned}$$

It follows immediately that

$$(2.19) \quad \liminf_{n \rightarrow \infty} J[\varphi_n, G_n^*] \geq d^*.$$

On the other hand, we have from (2.18) that

$$\begin{aligned} (2.20) \quad J[\varphi_n, G_n^*] &\equiv \iint_{G_n^*} [(\nabla \varphi_n)^2 - 4\mu\theta_0\varphi_n] dx dy \\ &= \iint_{H_n^*} [(\nabla \psi_n^{**})^2 - 4\mu\theta_0(\psi_n^{**} - k\epsilon_n)] dx dy \\ &= \iint_{H_n^*} [(\nabla \psi_n^{**})^2 - 4\mu\theta_0\psi_n^{**}] dx dy + O(\epsilon_n) \\ &= \iint_{G_n^*} [(\nabla \psi_n^{**})^2 - 4\mu\theta_0\psi_n^{**}] dx dy \\ &\quad + \iint_{G_n^* - H_n^*} 4\mu\theta_0\psi_n^{**} dx dy + O(\epsilon_n) \\ &= \iint_{G_n^*} [(\nabla \psi_n^{**})^2 - 4\mu\theta_0\psi_n^{**}] dx dy + O(\epsilon_n) \end{aligned}$$

$$\begin{aligned}
&= \iint_{G_n^{**}} [(\nabla \psi_n^{**})^2 - 4\mu\theta_0\psi_n^{**}] dx dy \\
&\quad + \iint_{G_n^{**}-G_n^*} 4\mu\theta_0\psi_n^{**} dx dy + O(\epsilon_n) \\
&= \iint_{G_n^{**}} [(\nabla \psi_n^{**})^2 - 4\mu\theta_0\psi_n^{**}] dx dy + O(\epsilon_n) \\
&\leq d^{**} + O(\epsilon_n),
\end{aligned}$$

where the 5th and the 8th equality signs follow from the fact that $0 \leq \psi_n^{**} \leq \epsilon_n$ in $G_n^* - H_n^*$ and in $G_n^{**} - G_n^*$ and the last inequality sign follows from the fact that $\{J[\psi_n^{**}, G_n^{**}]\}$ is a monotone non-decreasing numerical sequence and the definition of d^{**} in (2.14). As an immediate consequence of the estimate in (2.20) we have

$$(2.21) \quad \limsup_{n \rightarrow \infty} J[\varphi_n, G_n^*] \leq d^{**}.$$

Finally, it follows from (2.21) and (2.19) that

$$(2.22) \quad d^{**} \geq \limsup_{n \rightarrow \infty} J[\varphi_n, G_n^*] \geq \liminf_{n \rightarrow \infty} J[\varphi_n, G_n^*] \geq d^*.$$

Thus, our equality (2.16) follows directly from (2.15) and (2.22), which means that ψ and ψ^* are the same element in $H_0^1(G)$. The Lipschitz continuity of the minimizing extremal for the minimum problem with domain G is now completely established.

3. Natural partition of cross sections.

a. Elastic-plastic partition of G . Our problem involves two different physical laws, namely Hook's law of elasticity and von-Mises-Prandtl's law of plasticity [5, 22]. Accordingly, we divide the convex cylindrical bar into two parts according to which one of these two laws holds. Indeed, knowing the continuity of the extremal, we can now define the elastic region E and the plastic portion P of $\tilde{G}$ as follows:

$$(3.1) \quad E \equiv \{q \mid q \in \tilde{G}, \psi(q) < \Psi(q)\}, P \equiv \{q \mid q \in \tilde{G}, \psi(q) = \Psi(q)\}.$$

Clearly, E and P constitute a non-overlapping covering of $\tilde{G}$. Also, we shall see, in a moment, that the definition (3.1) for E and P agrees with the standard ones [5, 22]. Indeed, if the interior of P is non-empty then

$$(3.2) \quad |\text{grad } \psi| = |\text{grad } \Psi| = k$$

there. Therefore, our definition for P agrees with the usual one [5, 22] except on some portion of ∂G . Moreover, if we denote by $Y_0(E)$ the cylindrical bar with cross section E and denote by $Y(E)$ the bar deformed from $Y_0(E)$ by twisting it uniformly along the geometrical axis of $Y_0(E)$ with an angle θ_0 per unit length,

then we have already shown that all the constitutive equations in the theory of plasticity as well as all the field equations [5, 22] are satisfied in the exterior of $Y(E)$. This justifies our definition for P .

b. Validity of Hook's law in E . Since $\psi < \Psi$ in the region E , the inequality-side constraint will no longer be a constraint for all small localized variations. Using this fact and introducing suitable localized variations, one can transform the variational condition into an integral representation for the extremal ψ , which then shows that *the extremal ψ is actually analytic in the elastic region E and that it satisfies the Poisson's equation, $\Delta\psi = -2\mu\theta_0$, there* [§6, 3]. This proves that the material occupying the region $Y(E)$ does obey Hook's laws in the *strict* sense. Of course, all the field equations are also satisfied there in view of the very method for deriving the Poisson equation.

c. Non-existence of yield points in E . We proceed to prove that the material occupying the region $Y(E)$ not only obeys Hook's law but also stays below the yield point in the sense of Tresca or von Mises [5, 22]. Indeed, since the extremal is analytic in E , we can perform computations to check that

$$(3.3) \quad \Delta(\text{grad } \psi)^2 = 2[(\psi_{xx})^2 + 2(\psi_{xy})^2 + (\psi_{yy})^2] \geq 0 \text{ in } E.$$

Thus, $|\text{grad } \psi|^2$ is subharmonic in the elastic region E . Consequently, the strong maximum principle [6, 7] assures that it can not achieve its interior maximum in E unless it is a constant there. But if $|\text{grad } \psi|^2$ were a constant in E then the extremal ψ would be a function linear in x and y . It would then follow that $\Delta\psi = 0$ in E . This contradicts that it satisfies the Poisson equation, $\Delta\psi = -2\mu\theta_0 < 0$, there. On the other hand, the Lipschitz constant for the extremal ψ is not greater than k in $\bar{G}$ which implies that

$$\limsup_{q \rightarrow s} |\text{grad } \psi(q)|^2 \leq k^2$$

where the point q in E approaches a point s on ∂E . Hence, we conclude from the maximum principle that

$$(3.4) \quad |\text{grad } \psi| < k \text{ in } E,$$

which means that *the material occupying the region $Y(E)$ is, indeed, below the yield point*. Thus, our definition for the elastic region E is justified and it agrees with the usual definition for E .

d. An "intersection property" of P . Having established the smooth property of the extremal ψ in the elastic region E , we can now show that *each inward normal of ∂G intersects the plastic closed portion P in a single line segment or in a single point, wherever the inward normal of ∂G is uniquely defined*. To show this it suffices to show that if q is a point in $P \cap G$ and if qq' is the line segment such that

$$\rho(q, q') = \rho(q, \partial G), \quad q' \in \partial G,$$

then the segment qq' is completely contained in P .

Suppose that the last assertion is false. Then there is, at least, a point O on the segment qq' such that $\psi(O) < \Psi(O)$. Clearly, the point O is in the open set E . Hence, it follows from the continuity of ψ and Ψ in $\bar{G}$ that $\Psi > \psi$ in some complete neighborhood of O . Let q_1q_2 be the largest open connected segment which contains O and which lies in the intersection of E and the segment qq' . That is,

$$\psi(q_1) = \Psi(q_1), \quad \psi(q_2) = \Psi(q_2),$$

and for every interior point O' of the segment qq' , $\psi(O') < \Psi(O')$. Now, the extremal ψ is continuous on the closed segment qq' and is smooth in its interior. Accordingly, if

$$\Psi(q_2) > \Psi(O) > \Psi(q_1),$$

then Rolle's theorem assures the existence of a point q^* on the open segment Oq_2 such that

$$(3.5) \quad \begin{aligned} \psi(q_2) - \psi(O) &= [\partial\psi(q^*)/\partial n]_{\rho}(O, q_2) \\ &> \Psi(q_2) - \Psi(O) = k\rho(O, q_2), \end{aligned}$$

where $\partial\psi(q^*)/\partial n$ stands for the derivative of ψ along the segment Oq_2 evaluated at the interior point q^* of the segment Oq_2 . Since the inequality (3.5) shows that

$$\partial\psi(q^*)/\partial n > k$$

and since the point q^* lies in E , we have reached a contradiction to the inequality (3.4). Our assertion is thereby proved.

We shall derive more information about E and P as well as the free boundary ℓ .

$$(3.6) \quad \ell \equiv \partial P \cap G.$$

4. Certain properties of the ridge. In this section we shall state some results about the ridge Γ of G . These results will lead to certain informations about the relative position of E in G and about the minimum size of E .

a. Parametric representation. For simplicity, we shall use the same letter s to denote a point on ∂G and the arc length of ∂G at the point s . As was shown in [2], the ridge Γ partitions G into exactly n simply connected subdomains, G_j , $j = 1, 2, \dots, n$, such that each corner s_j of G becomes a corner of two adjacent subdomains G_j and G_{j+1} . The subdomain G_j is bounded by the arc S_jS_{j+1} of ∂G and a subarc $C_{j,j+1}$ of Γ . Let γ be a point on $C_{j,j+1}$ and s a point on ∂G such that

$$\rho(\gamma, s) = \rho(\gamma, \partial G) = \xi(s).$$

It was shown in [2] that the arc $C_{j,j+1}$ has the parametric representation

$$(4.1) \quad x = f(s) + \xi(s)n_x(s), \quad y = g(s) + \xi(s)n_y(s)$$

for $s_j \leq s \leq s_{j+1}$ where $(f(s), g(s))$ and (x, y) are respectively Cartesian co-

ordinates of the point s on ∂G and the point γ on $C_{i,j+1}$ and where n_x and n_y are the x - and the y -components of the unit inward normal to ∂G at the point s .

b. Piecewise smoothness of Γ . We state two general theorems in [2].

Theorem 1. *For each $j, j = 1, 2, \dots, n$, the function $\xi(s)$ in (4.1) is continuous for $s_i \leq s \leq s_{i+1}$.*

Theorem 2. *If the curvature $K(s)$ of ∂G is in general monotone between two adjacent corners of G , then each of the arcs $C_{i,j+1}$, $j = 1, 2, \dots, n$, possesses continuously turning tangents except at a finite number of points on Γ . If the tangent at s exists then its components are given by*

$$\begin{aligned}\frac{dx}{ds} &= f'(s) + \frac{d\xi}{ds} n_x(s) - \xi(s) K(s) n_y(s), \\ \frac{dy}{ds} &= g'(s) + \frac{d\xi}{ds} n_y(s) + \xi(s) K(s) n_x(s),\end{aligned}$$

with $\xi(s)$ being continuous there.

For later application, we also state the following results, [2]: if a point γ on Γ is such that there are two and only two points s and s^* on ∂G with the property that

$$\rho(\gamma, s) = \rho(\gamma, s^*) = \rho(\gamma, \partial G)$$

then the tangent to Γ at γ is uniquely defined and it bisects the angle between the segments γs and γs^* .

5. Hölder continuity of stress components. In this section we establish Hölder continuity of $\text{grad } \psi$ in G so as to assure that the stress components are everywhere continuous. In particular, they are continuous across the elastic-plastic boundary. To this end we first consider

a. An alternative formulation of the problem. Let K be the closed bounded convex set of $H_0^1(G)$ such that

$$(5.1) \quad K = \{ \varphi \mid \varphi \in H_0^1(G), |\text{grad } \varphi| \leq k \text{ a.e. in } G \}.$$

An alternative formulation of elastic-plastic torsion problem [12–14, 31] is to find a function ψ such that it minimizes the integral,

$$(5.2) \quad J[\varphi] \equiv \iint_G \{ (\nabla \varphi)^2 - 4\mu \theta_0 \varphi \} dx dy,$$

among all φ in K . Recent advance in monotonicity method [14, 15, 23–27] not only answers the basic existence question but also leads to certain regularity results [14] for problems of this type. But, for the present simple case, the uniqueness and existence problem can be solved in a more elementary way as was shown in [§2, 4, III].

Since the extremal ψ of problem (1.3) has been shown to be Lipschitz continuous in $\bar{G}$ with a Lipschitz constant equal to k , it belongs to the set K defined in (5.1). On the other hand, it is easy to see that for every φ in K , $\varphi \leq \Psi$ in $\bar{G}$ with Ψ being the majorant function defined in (1.1). Accordingly, the set K is contained in the set F defined by (i) and (ii). Therefore, the unique solution ψ of problem (1.3) is actually the unique solution of problem (5.2).

b. Smoothness of the extremal. Since the minimizing extremal ψ of problem (1.3) turns out to be the unique extremal of problem (5.2), to establish the smoothness of ψ it suffices to establish that for the extremal of (5.2). Although this is a non-trivial problem, in the present case the proof can be made more elementary by the application of DeGiorgi's Theorem [15-17]. Following the idea of Brezis and Stampacchia [14], we consider for all $\epsilon > 0$ the solution ψ_ϵ of the Dirichlet problem:

$$(5.3) \quad \epsilon \Delta \psi_\epsilon - \psi_\epsilon = -\psi \text{ in } G, \quad \psi_\epsilon = 0 \text{ on } \partial G.$$

For all $\epsilon > 0$, a unique solution ψ_ϵ to (5.3) exists. Since ψ is the unique extremal which is known to be Lipschitz continuous in $\bar{G}$, ψ_ϵ has the well-known nice properties. Moreover, it follows from the non-negativeness of ψ in $\bar{G}$ that $\psi_\epsilon \geq 0$ in $\bar{G}$. For if this were not the case, then ψ_ϵ would achieve its interior negative minimum at some point q in G . But then we would arrive at a contradiction by evaluating the first equation in (5.3) at the point q .

Let s be a point on ∂G and q a point in $\bar{G}$. Choose a coordinate system (x, y) with origin at s and with the x -axis being a line of support of G at s such that G lies in the half-plane $y \geq 0$. Relative to the (x, y) -system, we have

$$(5.4) \quad \begin{aligned} \epsilon \Delta(\psi_\epsilon - ky) - (\psi_\epsilon - ky) &= -(\psi - ky) \text{ in } G, \\ \psi_\epsilon(x, y) - ky &\leq 0 \text{ on } \partial G, \end{aligned}$$

where k is the same constant appearing in (1.1) and (5.1). Since $\psi - ky \leq 0$ in $\bar{G}$, it follows for the same reason as was just shown above that $\psi_\epsilon - ky \leq 0$ in $\bar{G}$. In particular, we have

$$(5.5) \quad 0 \leq \psi_\epsilon(q) - \psi_\epsilon(s) = \psi_\epsilon(q) \leq ky(q) \leq k\rho(q, s),$$

provided $s \in \partial G$ and q in $\bar{G}$.

Let q_1, q_2 be any two points in $\bar{G}$ and let $h = \rho(q_1, q_2)$. Denote by $G^h(G^{-h})$ the translation of G along the vector $\mathbf{q}_1\mathbf{q}_2$ ($\mathbf{q}_2\mathbf{q}_1$). Denote by ψ^h and ψ_ϵ^h the functions defined by

$$\psi^h(q) = \psi(q^{-h}), \quad \psi_\epsilon^h(q) = \psi_\epsilon(q^{-h}) \text{ for } q \text{ in } G^h.$$

Then, the function $\psi_\epsilon - \psi_\epsilon^h - kh$ satisfies the equation

$$(5.6) \quad \epsilon \Delta(\psi_\epsilon - \psi_\epsilon^h - kh) - (\psi_\epsilon - \psi_\epsilon^h - kh) = -(\psi - \psi^h - kh)$$

in $G^h \cap G$. Also, it follows from (5.5) that

$$\psi_\epsilon - \psi_\epsilon^h - kh \leq 0 \text{ on } \partial(G^h \cap G).$$

Since $\psi - \psi^h - kh \leq 0$ in $\bar{G}$, we conclude from (5.5) and (5.6) and the maximum principle [6] $\psi_\epsilon - \psi_\epsilon^h - kh \leq 0$ in $\bar{G}$. In particular,

$$(5.7) \quad 0 \leq \psi_\epsilon(q_2) - \psi_\epsilon^h(q_2) = \psi_\epsilon(q_2) - \psi_\epsilon(q_1) \leq kh = k\rho(q_1, q_2).$$

This proves that for every $\epsilon > 0$, ψ_ϵ belongs to the set K , because (5.7) holds for every pair of points q_1, q_2 in $\bar{G}$ with $\psi_\epsilon(q_2) \geq \psi_\epsilon(q_1)$.

Next, we show that for all $\epsilon > 0$, $\Delta\psi_\epsilon$ stays in a bounded set of $L_2(G)$, the set of functions square summable over G . Indeed, for ψ being a solution of (5.2) we have the variational condition

$$\iint_G [\nabla\psi \cdot \nabla(\psi_\epsilon - \psi) - 2\mu\theta_0(\psi_\epsilon - \psi)] dx dy \geq 0.$$

Accordingly, for all $\epsilon > 0$,

$$(5.8) \quad \iint_G (2\mu\theta_0 + \Delta\psi_\epsilon)(\psi_\epsilon - \psi) dx dy \leq -\iint_G [\nabla(\psi_\epsilon - \psi)]^2 dx dy \leq 0.$$

By combining (5.3) and (5.8) we obtain

$$\begin{aligned} \iint_G [(2\mu\theta_0 + \Delta\psi_\epsilon)(\psi_\epsilon - \psi)] dx dy \\ = \epsilon 2\mu\theta_0 \iint_G \Delta\psi_\epsilon dx dy + \epsilon \iint_G (\Delta\psi_\epsilon)^2 dx dy \leq 0. \end{aligned}$$

That is,

$$(5.9) \quad \|\Delta\psi_\epsilon\|_0^2 \leq -2\mu\theta_0 \iint_G \Delta\psi_\epsilon dx dy \leq 2\mu\theta_0 |G| \|\Delta\psi_\epsilon\|_0,$$

where $|G|$ is the area of G and where $\|f\|_n$ stands for the n -fold Dirichlet norm of f . Upon cancelling out the common positive factor from (5.9), it gives that

$$(5.10) \quad \|\Delta\psi_\epsilon\|_0 \leq 2\mu\theta_0 |G|.$$

From the weak compactness of $L_2(G)$ and (5.10), we see that $\{\Delta\psi_\epsilon\}$ converges weakly in $L_2(G)$ as $\epsilon \rightarrow 0$.

Consider ψ_ϵ . We have from (5.3) and (5.10) that

$$(5.11) \quad \|\psi_\epsilon - \psi\|_0 \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

Furthermore, since both ψ and $\{\psi_\epsilon\}$ are contained in $K \subset H_0^1(G)$ and since every element in $H_0^1(G)$ is identified by a unique element in $L_2(G)$, it follows from (5.11) that $\|\psi_\epsilon - \psi\|_1 \rightarrow 0$ as $\epsilon \rightarrow 0$. Hence, for every A in $H_0^{-1}(G)$, the set of linear functionals on $H_0^1(G)$,

$$(5.12) \quad A(\psi_\epsilon - \psi) \equiv (A, \psi_\epsilon - \psi) \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

Now, we are able to show that

$$\lim_{\epsilon \rightarrow 0} \Delta\psi_\epsilon = \Delta\psi \text{ weakly on } L_2(G).$$

For every u, v in $H_0^1(G)$, Δu belongs to $H_0^{-1}(G)$ and we have

$$(\Delta u, v) \equiv - \iint_G \nabla u \cdot \nabla v \, dx \, dy.$$

In particular, for $u \in H_0^1(G)$, $W \equiv (1 - t)\psi + tu$ with $0 \leq t \leq 1$ belongs to $H_0^1(G)$. Hence

$$(5.13) \quad (\Delta(\psi_\epsilon - w), \psi_\epsilon - \psi) = - \iint_G [\nabla(\psi_\epsilon - w)]^2 \, dx \, dy \leq 0.$$

Now, (5.13) can be written as

$$(5.14) \quad (\Delta(\psi_\epsilon - w), \psi_\epsilon - w) = (\Delta\psi_\epsilon, \psi_\epsilon - \psi) - t(\Delta(\psi - u), (\psi_\epsilon - \psi) + t(\psi - u)) - t(\Delta\psi_\epsilon, \psi - u) - (\Delta\psi, \psi_\epsilon - \psi + t(\psi - u)) \leq 0.$$

As $\epsilon \rightarrow 0$, we have from (5.2), (5.11) and (5.12) that

$$\begin{aligned} (\Delta\psi_\epsilon, \psi_\epsilon - \psi) &= \iint_G (\Delta\psi_\epsilon)(\psi_\epsilon - \psi) \, dx \, dy = \epsilon^2 \|\Delta\psi_\epsilon\|_0^2 \rightarrow 0, \\ (\Delta(\psi - u), \psi_\epsilon - \psi) &\rightarrow 0, \quad (\Delta\psi, \psi_\epsilon - \psi) \rightarrow 0. \end{aligned}$$

Thus, by letting $\epsilon \rightarrow 0$ in (5.14), it yields

$$(5.15) \quad -t^2(\Delta(\psi - u), \psi - u) + t \lim_{\epsilon \rightarrow 0} (\Delta\psi_\epsilon, \psi - u) - t(\Delta\psi, \psi - u) \leq 0,$$

where an appropriate sequence of $\{\Delta\psi_\epsilon\}$ has been chosen so that the limit exists for all u in $H_0^1(G)$. Dividing (5.15) by t and then letting $t \rightarrow 0$, we get

$$(5.16) \quad (\Delta\psi, \psi - u) \geq \lim_{\epsilon \rightarrow 0} (\Delta\psi_\epsilon, \psi - u) \quad \text{for all } u \text{ in } H_0^1(G).$$

This is so if and only if

$$(5.17) \quad (\Delta\psi, v) = \lim_{\epsilon \rightarrow 0} (\Delta\psi_\epsilon, v) \quad \text{for all } v \text{ in } H_0^1(G).$$

Since $\Delta\psi_\epsilon$ converges weakly in $L_2(G)$ and since $H_0^1(G)$ is $\|\cdot\|_0$ -dense in $L_2(G)$, we can extend, in view of (5.17), $\Delta\psi$ as a continuous linear functional on $L_2(G)$ by defining

$$(\Delta\psi, f) = \lim_{\epsilon \rightarrow 0} (\Delta\psi_\epsilon, f) \quad \text{for all } f \text{ in } L_2(G).$$

Since $\Delta\psi$ can be extended to be a continuous linear functional on $L_2(G)$ which is reflexive, it follows that $\Delta\psi$ belongs to $L_2(G)$.

Since the trivial relation $\Delta\psi = \Delta\psi$ is meaningful a.e. in G and since ψ belongs to $H_0^1(G)$, it follows that ψ has the representation

$$\psi(x, y) = \iint_G G(x, y; \xi, \eta) \Delta\psi(\xi, \eta) \, d\xi \, d\eta,$$

with G being the Green's function. It follows from the above representation and

Calderon-Zygmund Theorem [28] that ψ belongs to $H_0^1 \cap H^2$. Although it is not yet known whether $\Delta\psi$ is bounded we may apply the well-known regularity theorem for weak solutions of elliptic differential equations to conclude that $\text{grad } \psi$ is Hölder in G [15–17], in view of (5.10).

6. Existence of an elastic core. By an elastic core we mean an open neighborhood of the ridge Γ of G in which the extremal ψ is strictly less than the majorant function Ψ . Of course, an elastic core is a subset of E . However, it is not obvious that the set E is not empty. On the other hand, if the existence of an elastic core has been established for all values of μ and θ_0 , then it not only assures that elastic-plastic torsion really occurs but also describes the relative position of E in G and the extent of E . As a consequence, the *a-priori* results derived in §3 become materialized. *Since an elastic core exists for all values of μ and θ_0 , it also follows that completely plastic torsion never occurs in practice.* Indeed, this is what has been conjectured in the literature.

Consider a point $\gamma \in \Gamma$. First, we consider the general case that there are, at least, two distinct points s_1 and s_2 on ∂G such that

$$\rho(\gamma, s_1) = \rho(\gamma, s_2) = \rho(\gamma, \partial G).$$

Suppose that $\psi(\gamma) = \Psi(\gamma)$. Then it follows from the “intersection” property of the plastic portion P that $\psi = \Psi$ along the segments γs_1 and γs_2 . Let n_1 and n_2 be respectively the unit vectors with the same directions as $s_1\gamma$ and $s_2\gamma$. We immediately conclude from the continuity of $\text{grad } \psi$ in G that

$$(6.1) \quad \text{grad } \psi(\gamma) \cdot n_1 = k,$$

$$(6.2) \quad \text{grad } \psi(\gamma) \cdot n_2 = k.$$

Since ψ belongs to K , we have from (6.1) that

$$k \geq |\text{grad } \psi(\gamma)| = |\text{grad } \psi(\gamma)| |n_1| \geq |\text{grad } \psi(\gamma) \cdot n_1| = k.$$

Hence, $|\text{grad } \psi(\gamma)| = k$. In order for (6.1) to hold, it is necessary that $\text{grad } \psi(\gamma) = kn_1$. But then

$$\text{grad } \psi(\gamma) \cdot n_2 = kn_1 \cdot n_2 < k,$$

because s_1 and s_2 are two distinct points on ∂G . Thus, we have reached a contradiction with (6.2) and hence $\psi(\gamma) < \Psi(\gamma)$.

Next, we consider the special case that the point $\gamma_0 \in \Gamma \cap G$ is such that there is one and only one point s_0 on ∂G with $\rho(\gamma_0, s_0) = \rho(\gamma_0, \partial G)$. Such a special point does occur, *e.g.*, the two end points of the ridge of an ellipse. Under this circumstance, $\rho(\gamma_0, s_0)$ is equal to the radius of curvature, $1/K(s_0)$, of ∂G at s_0 . Also, the curvature $K(s_0)$ of ∂G achieves its local maximum $K(s_0)$ at s_0 (see Corollary 3 and the proof of Theorem 2 in [2]).

Let τ be the distance measured from a point s on ∂G along the inward normal there. It is known [2] that the correspondence $(s, \tau) \leftrightarrow (x, y)$ defines a twice

continuously differentiable coordinate transformation over the set $G - \Gamma$. Indeed, the equations of the coordinate transformation are given by

$$(6.3) \quad x = f(s) + \tau n_x(s), \quad y = g(s) + \tau n_y(s),$$

where $0 \leq \tau < \xi(s)$ and f, g, n_x, n_y and $\xi(s)$ have the same meaning as given in (4.1). Accordingly, the Jacobian of this coordinate transformation is equal to

$$\partial(x, y)/\partial(s, \tau) = 1 - K(s)\tau, \quad 0 \leq \tau < \xi(s).$$

Since $\xi(s) \leq 1/K(s)$, we have $K(s)\tau < 1$ in $G - \Gamma$. However, at the particular point γ_0 , $K(s_0)\xi(s_0) = 1$.

Now, relative to the curvilinear coordinate system (s, τ) , $\Psi(s, \tau) = k\tau$. Consequently, Ψ is twice continuously differentiable in the open set $G - \Gamma$. In fact,

$$(6.4) \quad \Psi_x = k \frac{\partial \tau}{\partial x} = kn_x(s), \quad \Psi_y = k \frac{\partial \tau}{\partial y} = kn_y(s),$$

where n_x and n_y are the x - and y -components of the unit inward normal to ∂G at s whose extension passes the point (x, y) in $G - \Gamma$. Further differentiation of the expressions in (6.4) leads to, after using the relations in (6.3) and the Frenet formulas [8],

$$(6.5) \quad \Delta \Psi = \Psi_{xx} + \Psi_{yy} = -kK(s)/[1 - K(s)\tau] \text{ in } G - \Gamma,$$

where $K(s) > 0$ and $0 \leq \tau < \xi(s)$.

Since $K(s_0)\xi(s_0) = 1$, it follows from the expression in (6.5) for $\Delta \Psi$ and its continuity in s and τ that $\Delta \Psi$ will become negatively infinite as the point $\gamma_0 = (s_0, \xi(s_0))$ on Γ is approached from $G - \Gamma$. Let $D(q, \epsilon; \gamma_0)$ be a small disk with radius ϵ centered at a point q on the segment $\gamma_0 s_0$ such that it contains the particular point γ_0 on its boundary. Now, it simply follows from continuity that

$$\Delta \Psi = -kK(s)/[1 - K(s)\tau] < -2\mu\theta \text{ in } D(q, \epsilon; \gamma_0)$$

provided ϵ is sufficiently small. Consider the function $\psi - \Psi$. Then, for all sufficiently small ϵ ,

$$(6.6) \quad \Delta(\psi - \Psi) = -2\mu\theta + \frac{kK(s)}{1 - K(s)\tau} > 0 \quad \text{in } D(q, \epsilon; \gamma_0),$$

$$\psi - \Psi \leq 0 \quad \text{on } \partial D(q, \epsilon; \gamma_0).$$

Thus, an application of the maximum principle [6, 7] to the solution, $\psi - \Psi$, of the problem (6.6) leads to $\psi - \Psi < 0$ in $D(q, \epsilon; \gamma_0)$. In particular, $\psi(q) < \Psi(q)$. This strict inequality ensures that $\psi(\gamma_0) < \Psi(\gamma_0)$. For, otherwise, we would have a contradiction with the "intersection" property of P . The existence of an elastic core is now established.

Since the ridge Γ is connected [2], the existence of an elastic-core together with the "intersection" property of P implies that the *elastic region E is simply connected*.

7. A parametric representation of the free boundary. In this section, we shall derive certain information about the elastic-plastic boundary. According to the "intersection" property of P , for each point s on ∂G , the intersection of the plastic portion P with the inward normal of ∂G at s is a single segment sq . Since the ridge Γ is contained in E , the end point q of the segment sq must be such that $\rho(q, s) = \rho(q, \partial G)$. If $\rho(q, s) > 0$, then q lies on the free boundary

$$(7.1) \quad \ell \equiv \partial P \cap G = \partial E \cap G.$$

Now, we define the function $R(s)$ by

$$(7.2) \quad R(s) = \rho(q, s), \quad s \in \partial G,$$

with sq being the segment specified above. For the moment, we represent the free boundary ℓ formally in terms of the arc length s of ∂G as follows:

$$(7.3) \quad x = f(s) + R(s)n_x(s), \quad y = g(s) + R(s)n_y(s),$$

where (x, y) are the rectangular Cartesian coordinates of the point $q \in \ell$, $(f(s), g(s))$ the rectangular Cartesian coordinates of the point s on ∂G , n_x and n_y the x - and y -components of the inward normal of ∂G at s , and $R(s)$ is the function defined in (7.2). *Since f, g, n_x and n_y are all continuously differentiable functions of s , equations in (7.3) will, indeed, be a parametric representation of ℓ in the strict sense if we can establish that $R(s)$ is continuous.*

a. Upper semi-continuity of $R(s)$. We wish to show that for every $s_0 \in \partial G$,

$$(7.4) \quad \limsup R(s) \equiv R^+(s_0) \leq R(s_0) \text{ as } s \rightarrow s_0.$$

Indeed, for a positive numerical sequence $\{\epsilon\}$ converging to zero, there is a sequence $\{s_\epsilon\}$ of points on ∂G such that

$$(7.5) \quad \lim_{\epsilon \rightarrow 0} s_\epsilon - s_0 = 0, \quad \lim_{\epsilon \rightarrow 0} R(s_\epsilon) = R^+(s_0).$$

From the continuity of $\psi(s, \tau)$ in the curvilinear coordinates s and τ and the limit relations in (7.5), it follows that

$$(7.6) \quad \lim_{s \rightarrow 0} \psi(s_\epsilon, R(s_\epsilon)) = \psi(s_0, R^+(s_0)).$$

On the other hand, for each $\epsilon > 0$ we have from the very definition of $R(s)$ that $\psi(s_\epsilon, R(s_\epsilon)) = kR(s_\epsilon)$. Hence

$$(7.7) \quad \lim_{\epsilon \rightarrow 0} \psi(s_\epsilon, R(s_\epsilon)) = \lim_{\epsilon \rightarrow 0} kR(s_\epsilon) = kR^+(s_0).$$

By combining (7.6) and (7.7), we obtain

$$\psi(s_0, R^+(s_0)) = kR^+(s_0).$$

This means that the point $(s_0, R^+(s_0))$ is in the plastic set P . According to the definition for $R(s_0)$, the inequality (7.4) must hold. Our proof shows that the

upper semi-continuity of $R(s)$ is a consequence of not only the continuity of ψ but also the "intersection" property of P .

b. Strict upper semi-continuity of $R(s)$. Next, we show that for every point $s_0 \in \partial G$,

$$(7.8) \quad \limsup R(s) \equiv R^+(s_0) = R(s_0) \text{ as } s \rightarrow s_0.$$

Suppose that

$$(7.9) \quad \limsup_{s \downarrow s_0} R(s) = R^+(s_0) < R(s_0).$$

Let q_0 and q_0^+ be the points with curvilinear coordinates $(s_0, R(s_0))$ and $(s_0, R^+(s_0))$ respectively. Then the segment $q_0^+q_0$ is contained in ∂P . Let (x, y) be a rectangular Cartesian coordinate system with origin at s_0 and the positive y -axis containing the segment $q_0^+q_0$. Then the inequality (7.9) implies that there is a one-sided neighborhood,

$$N(q_0^+, q_0): 0 < x \leq \delta, \quad R^+(s_0) \leq y \leq R(s_0),$$

of the segment $q_0^+q_0$ such that $N(q_0^+, q_0)$ is contained in E . Hence, the extremal $\psi(x, y)$ satisfies the equation

$$\Delta\psi = -2\mu\theta \text{ in } N(q_0^+, q_0)$$

and

$$\psi(0, y) = ky, \quad \psi_x(0, y) = 0 \text{ along segment } q_0^+q_0.$$

Consequently, the function,

$$u(x, y) \equiv \psi(x, y) + \mu\theta_0 x^2 - ky$$

is harmonic in $N(q_0^+, q_0)$ and

$$u(0, y) = 0, \quad u_x(0, y) = 0 \text{ along segment } q_0^+q_0.$$

By the unique continuation theorem for harmonic functions, we conclude that u vanishes identically in $N(q_0^+, q_0)$. Thus,

$$(7.10) \quad \psi(x, y) = ky - \mu\theta_0 x^2 \text{ in } N(q_0^+, q_0).$$

For ψ being given by (7.10), we must have

$$|\text{grad } \psi|^2 = k^2 + (2\mu\theta_0 x)^2 > k^2$$

for $0 < x \leq \delta$. But this contradicts the fact that $|\text{grad } \psi| < k$ in E . Thus, the hypothesis (7.9) is wrong. By the same reasoning we also conclude that

$$\limsup_{s \uparrow s_0} R(s) = R(s_0).$$

The assertion (7.8) is now proved.

c. Lower semi-continuity of $R(s)$. To establish the continuity of the function $R(s)$, it suffices now to show that for every point s_0 on ∂G

$$(7.11) \quad \liminf R(s) \equiv R^-(s_0) = R(s_0) \text{ as } s \rightarrow s_0.$$

Again, we assume the contrary and then derive a contradiction.

Suppose that s_0 is a point on ∂G such that

$$(7.12) \quad R(s_0) - \liminf_{s \rightarrow s_0} R(s) \equiv R(s_0) - R^-(s_0) = \tau_0 > 0.$$

We assert that at the point $(s_0, R(s_0))$ there holds the inequality

$$(7.13) \quad -2\mu\theta_0 + kK(s_0)/[1 - R(s_0)K(s_0)] \leq 0.$$

For if this were not the case, then by continuity

$$2\mu\theta_0 + kK(s)/[1 - \tau K(s)] > 0$$

in some disk D centered at $(s_0, R(s_0))$. Let φ be the solution of the Dirichlet problem:

$$\Delta\varphi = -2\mu\theta \text{ in } D, \quad \varphi = \psi \text{ on } \partial D.$$

Then, the function $\varphi - \Psi$ satisfies the equations

$$\Delta(\varphi - \Psi) = -2\mu\theta_0 + kK(s)/[1 - K(s)] > 0 \text{ in } D, \quad \varphi - \Psi \leq 0 \text{ on } \partial D.$$

Hence, by Hopf's maximum principle $\varphi < \Psi$ in D . Now, extend φ to $\tilde{G}$ by setting $\varphi \equiv \psi$ in $\tilde{G} - D$. Then

$$J[\varphi, G] - J[\psi, G] = J[\varphi, D] - J[\psi, D] \leq 0,$$

and we conclude from the uniqueness of the extremal that $\varphi \equiv \psi < \Psi$ in D . But this contradicts the hypothesis in (7.12). Hence (7.13) must hold.

In what follows, we assume, for simplicity, that only the strict inequality sign in (7.13) holds. Otherwise, we replace $R(s_0)$ by a smaller number but greater than $R^-(s_0)$. Note also that for $u \equiv \psi - \Psi$,

$$(7.14) \quad \Delta u = -2\mu\theta + kK(s)/[1 - \tau K(s)] \text{ in } E - \Gamma.$$

Denote by q_0, q_0^- the points with coordinates $(s_0, R(s_0)), (s_0, R^-(s_0))$ respectively and put

$$(7.15) \quad a(x, y) = -2\mu\theta_0 + kK(s(x, y))/[1 - \tau(x, y)K(x, y)].$$

Since $a(x, y) < -2c < 0$ at the point q_0 , it follows from continuity that

$$(7.16) \quad a(x, y) < -c < 0 \text{ in } N(q_0^-, q_0)$$

where $N(q_0^-, q_0)$ is an open neighborhood of the segment $q_0^-q_0$. Since $R^+(s_0) = R(s_0)$, for every $\epsilon > 0$ there is a point s_ϵ on ∂G such that

$$(7.17) \quad 0 < |s_\epsilon - s_0| < \epsilon, \quad |R(s_\epsilon) - R(s_0)| < \epsilon.$$

Since $R(s_0) - R^-(s_0) = \tau_0 > 0$, there is a point s_η on ∂G such that

$$(7.18) \quad 0 < \eta < \epsilon, \quad 0 < |s_\eta - s_0| < \eta, \quad R(s_0) - R(s_\eta) \geq \tau_0 - \eta.$$

Now, choose a rectangular coordinate system with origin at s_η and with the positive y -axis coinciding with the inward normal of ∂G at s_η . It follows from the very definition of $R(s_\eta)$ that

$$(7.19) \quad u(0, y) < 0 \text{ for } R(s_\eta) \leq y < R(s_0) - \eta.$$

Denote by q_η , q_η^* the points with curvilinear coordinates $(s_\eta, R(s_\eta))$ and $(s_\eta, R(s_0) - \eta)$ respectively. It follows from (7.19) and the continuity of u that $u \leq 0$ in some neighborhood $N(q_\eta^*, q_\eta)$ of the segment $q_\eta^* q_\eta$. Furthermore, we can choose the number ϵ so small that (7.16) holds in $N(q_\eta^*, q_\eta)$. Hence

$$(7.20) \quad \Delta u = a(x, y) < -c < 0 \text{ in } N(q_\eta^*, q_\eta).$$

For each fixed y , $R(s_\eta) < y < R(s_0) - \eta$, let (x^-, y) and (x^+, y) be the *first* points on the left and on the right of $(0, y)$ such that

$$(7.21) \quad u(x^-, y) = 0, \quad u(x^+, y) = 0.$$

Such two points always exist, in view of the inequalities in (7.17) and the fact that $R^+(s_0) = R(s_0)$. From the inequalities in (7.19) and (7.21) we conclude that for each y^* , $R(s_\eta) < y^* < R(s_0) - \eta$, u achieves an interior negative minimum at a point (x^*, y^*) over the horizontal segment passing through $(0, y^*)$ with end points (x^-, y^*) and (x^+, y^*) . But then

$$(7.22) \quad u_x(x^*, y^*) = 0, \quad u_{xx}(x^*, y^*) \geq 0.$$

First, we consider the open set of points in $N(q_\eta^*, q_\eta)$, on which the strict inequality in (7.22) holds. Then, by the implicit function theorem, we can solve the first equation in (7.22) for

$$(7.23) \quad x^* = x^*(y^*),$$

which defines smooth arcs. These arcs may not be connected. But we can connect these smooth arcs in a smooth way through those points on which the equality sign in the inequality (7.22) holds. We denote the connected arc so defined by C_η . Clearly, C_η is issued from the point $(0, y_0)$ and travels upward to end at some point (x_η, y_η) , where $y_0 \equiv R(s_\eta)$, $y_\eta \equiv R(s_0) - \eta$.

Now, the relations in (7.22) holds along C_η and the equation (7.20) holds in some neighborhood of C_η . Hence

$$(7.24) \quad u_{vv}(x^*, y^*) < -c < 0 \text{ along } C_\eta.$$

On the other hand, by differentiating the first equation in (7.22) totally in y^* , we obtain

$$(7.25) \quad u_{xx} \left(\frac{dx}{dy} \right)^2 + u_{xy} \frac{dx}{dy} = 0 \quad \text{along } C_\eta.$$

By combining (7.24) and (7.25), we find

$$(7.26) \quad \begin{aligned} u_\nu(x^*, y^*) &= \int [u_{\nu\nu} dy + u_{\nu x} dx] \\ &\leq \int u_{\nu\nu} dy \leq -c \int dy = -c(y^* - y_0), \end{aligned}$$

where the integrals are all evaluated along C_η from the point $(0, y)$ to the point (x^*, y^*) on C_η , the first inequality follows from (7.25) and (7.22), and the second inequality follows from (7.24). Consequently,

$$(7.27) \quad \begin{aligned} u(x_\eta, y_\eta) &= \int [u_\nu dy + u_x dx] \\ &= \int u_\nu dy \leq -c \int (y - y_0) dy \\ &= -\frac{1}{2}c(y_\eta - y_0)^2, \end{aligned}$$

where the integrals are all evaluated along C_η from the point $(0, y_0)$ to the point (x_η, y_η) , the second equality sign follows from the first equation in (7.22) and the inequality sign follows from (7.26) and where $y_\eta \equiv R(s_0) - \eta$.

Finally, we let $\epsilon \rightarrow 0$ so that $\eta \rightarrow 0$. But then the point $(x_\eta, y_\eta) \rightarrow (s_0, R(s_0))$ as $\eta \rightarrow 0$. However, the inequality (7.27) shows that

$$\begin{aligned} \lim_{\eta \rightarrow 0} u(x_\eta, y_\eta) &< \lim_{\eta \rightarrow 0} -\frac{1}{2}c(y_\eta - y_0)^2 = -\frac{1}{2}c(R(s_0) - R^-(s_0)) \\ &= -\frac{1}{2}c\tau_0^2. \end{aligned}$$

This contradicts the continuity of u , because $u(s_0, R(s_0)) = 0$. The lower semi-continuity of $R(s)$ is now established.

It follows from the results in (b) and (c) that $R(s)$ is a continuous function. In particular, this ensures that the free boundary ℓ does not contain any segment perpendicular to ∂G .

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