*Progressing Waves Diffracted by Smooth Surfaces**

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1. Introduction. Progressing waves are series solutions of partial differential equations governing wave propagation. They are important because they can be used to solve many kinds of initial value, initial-boundary value, radiation and other problems. Their general theory has been presented by Courant–Hilbert [1], Friedlander [2], Ludwig [3] and Lewis [4]. They have been used to solve reflection problems by Friedlander [2], Lewis [4], Ahluwalia and Lewis [5] and Ahluwalia, Keller and Resende [6] and to treat diffraction from the edge of a plane screen by Ahluwalia and Lewis [5]. Since the usual representation of a progressing wave becomes infinite at a caustic, Ludwig [7] has shown how to modify it to obtain a representation which is uniformly valid in a region containing a caustic. For the special class of time harmonic progressing waves, the theory and applications are much more extensive. In particular Friedlander and Keller [8] have obtained a nonuniform representation of a surface diffracted or creeping wave, *i.e.* a wave diffracted around a smooth surface, and Lewis, Bleistein and Ludwig [9] have obtained a uniform representation of such a wave.

After summarizing the theory of progressing waves, we shall illustrate in detail the method of Ludwig [7] for obtaining a representation of such waves which is uniformly valid in a region containing a caustic. Then we shall generalize the results of Friedlander and Keller [8] and of Lewis, Bleistein and Ludwig [9] by presenting both nonuniform and uniform representations of general surface diffracted or creeping progressing waves. We shall also show how our representations reduce in the time harmonic case, to those given in [8] and [9]. These results are first obtained for the wave equation with a variable propagation speed, and then for a general system of linear first order partial differential equations, following the procedure of Ludwig [7].

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For so-called time independent progressing waves, but not for general ones, our results can be obtained by Fourier synthesis of the corresponding results for time harmonic waves. We have used the results obtained in this way to suggest the appropriate representations for the general case.

2. Progressing waves. Let us consider progressing wave solutions of the wave equation

$$(2.1) \quad Lu = 0, \quad L \equiv \Delta - c^{-2}(t, \mathbf{x}) \partial_t^2.$$

A progressing wave is a function of the form

$$(2.2) \quad u(t, \mathbf{x}) = \sum_{m=0}^{\infty} e_m[\phi(t, \mathbf{x})] Z_m(t, \mathbf{x}),$$

in which the e_m satisfy the recursion relation

$$(2.3) \quad e_m(\phi) = \frac{d}{d\phi} e_{m+1}(\phi), \quad m = 0, 1, \dots$$

The progressing wave (2.2) is a solution of (2.1) if the phase function ϕ and the amplitude functions Z_m satisfy the equations

$$(2.4) \quad (\nabla\phi)^2 - c^{-2}\phi_t^2 = 0,$$

$$(2.5) \quad 2[\nabla\phi \cdot \nabla Z_m - c^{-2}\phi_t Z_{mt}] + Z_m L\phi = -LZ_{m-1}, \quad m = 0, 1, \dots; Z_{-1} \equiv 0.$$

To simplify (2.4) and (2.5) we introduce the characteristic curves of (2.4), or bicharacteristics of (2.1), which we write in terms of a parameter γ as $t(\gamma)$, $\mathbf{x}(\gamma)$. They satisfy the characteristic ordinary differential equations

$$(2.6) \quad \begin{aligned} \frac{dt}{d\gamma} &= -2\alpha c^{-2}\phi_t, & \frac{d\mathbf{x}}{d\gamma} &= 2\alpha \nabla\phi, \\ \frac{d\phi_t}{d\gamma} &= \alpha\phi_t^2 \partial_t c^{-2}, & \frac{d\nabla\phi}{d\gamma} &= \alpha\phi_t^2 \nabla c^{-2}. \end{aligned}$$

In these equations ϕ_t and $\nabla\phi$ are to be considered functions of γ which are to be found, c and its derivatives are to be evaluated at $t(\gamma)$, $\mathbf{x}(\gamma)$, and α is an arbitrary factor which determines the nature of the parameter γ . Since $d\phi/d\gamma = \nabla\phi \cdot d\mathbf{x}/d\gamma + \phi_t dt/d\gamma$, we find from (2.6) and (2.4) that

$$(2.7) \quad \frac{d\phi}{d\gamma} = 0.$$

We can also use (2.6) to write (2.5) in the form

$$(2.8) \quad \frac{dZ_m}{d\gamma} + Z_m \alpha L\phi = -\alpha LZ_{m-1}, \quad m = 0, 1, \dots$$

Thus (2.4) and (2.5) have been converted into ordinary differential equations along the bicharacteristics.

To solve (2.7) and (2.8) we introduce a three parameter family of bicharacteristics

$$(2.9) \quad t = t(\gamma, \gamma), \quad \mathbf{x} = \mathbf{x}(\gamma, \gamma).$$

Here $\gamma = (\gamma^1, \gamma^2, \gamma^3)$ is a set of three labelling parameters which are constant on a bicharacteristic. Sometimes we shall omit them for simplicity if no confusion can result. The four variables γ, γ are called *ray coordinates*. Now from (2.7) we see that ϕ is constant on a bicharacteristic so we have

$$(2.10) \quad \phi = \phi(\gamma).$$

From (2.8) we find

$$(2.11) \quad Z_m(\gamma, \gamma) = \exp \left\{ - \int_{\tilde{\gamma}}^{\gamma} \alpha L \phi \, d\gamma' \right\} Z_m(\tilde{\gamma}, \gamma) - \int_{\tilde{\gamma}}^{\gamma} \exp \left\{ - \int_{\gamma'}^{\gamma} \alpha L \phi \, d\gamma'' \right\} \cdot \alpha(\gamma', \gamma) L Z_{m-1}(\gamma', \gamma) \, d\gamma', \quad m = 0, 1, \dots$$

The result (2.11) can be rewritten in the form [5]

$$(2.12) \quad Z_m(\gamma, \gamma) = \left| \frac{J(\tilde{\gamma}, \gamma) \alpha(\gamma, \gamma)}{J(\gamma, \gamma) \alpha(\tilde{\gamma}, \gamma)} \right|^{1/2} Z_m(\tilde{\gamma}, \gamma) - \int_{\tilde{\gamma}}^{\gamma} \left| \frac{J(\gamma', \gamma) \alpha(\gamma, \gamma)}{J(\gamma, \gamma) \alpha(\gamma', \gamma)} \right|^{1/2} \cdot \alpha(\gamma', \gamma) L Z_{m-1}(\gamma', \gamma) \, d\gamma', \quad m = 0, 1, \dots$$

Here $J(\gamma, \gamma)$ is the Jacobian of the transformation (2.9) from ray coordinates to $t, \mathbf{x}$, given by

$$(2.13) \quad J(\gamma, \gamma) = \frac{\partial(t, \mathbf{x})}{\partial(\gamma, \gamma)}.$$

In (2.12) $Z_m(\gamma, \gamma)$ is expressed in terms of its value $Z_m(\tilde{\gamma}, \gamma)$ at the point $\tilde{\gamma}$ on the same bicharacteristic $\gamma = \text{constant}$. The initial point $\tilde{\gamma}$ may depend upon γ , $\tilde{\gamma} = \tilde{\gamma}(\gamma)$.

We have now shown how to determine a progressing wave solution (2.2) of the wave equation (2.1) in parametric form. The procedure is as follows: We first obtain a three parameter family of bicharacteristics (2.9) by solving (2.6). Then ϕ is given by (2.10), the Z_m are given recursively by (2.12), and the e_m are given recursively by integrating (2.3). In this way all the quantities in (2.2) can be found in terms of the parameters γ and γ . The initial values of ϕ and the Z_m at one point on each bicharacteristic are arbitrary, as are e_0 , the integration constants in the e_m and the initial conditions determining the bicharacteristics themselves. All these arbitrary quantities can be adjusted to make u solve particular problems.

If several bicharacteristics of the family (2.9) pass through a given point, each one yields a value of the progressing wave at that point. Since (2.1) is a linear equation, the solution u is the sum of these values. Whenever a region of $t, \mathbf{x}$ space is multiply covered by bicharacteristics, this region is usually bounded in part by a *caustic*. This is a locus on which the Jacobian J vanishes,

and on it Z_m given by (2.12) becomes infinite. Therefore the representation (2.2) of a progressing wave solution of the wave equation is not valid on or near a caustic.

To illustrate the representation (2.2), let us set

$$(2.14) \quad e_m(\phi) = (ik)^{-m} e^{ik\phi},$$

which satisfies (2.3), and suppose that

$$(2.15) \quad c(t, \mathbf{x}) = c_0/n(\mathbf{x}).$$

Here k and c_0 are constants, and we define $\omega = kc_0$. Then we set

$$(2.16) \quad \phi(t, \mathbf{x}) = \hat{\theta}(\mathbf{x}) - c_0 t.$$

If $Z_m(t, \mathbf{x})$ is independent of t , (2.2) becomes the time harmonic progressing wave

$$(2.17) \quad u(t, \mathbf{x}) = \exp \{-i\omega t + ik\hat{\theta}(\mathbf{x})\} \sum_{m=0}^{\infty} (ik)^{-m} Z_m(\mathbf{x}).$$

Fourier synthesis of the time harmonic waves (2.17) yields a time independent progressive wave, which is a wave of the form (2.2) with ϕ given by (2.16) and Z_m independent of t .

3. Uniform representation of waves with caustics. We shall now use the method of Ludwig [7] to obtain a uniform representation of a progressing wave, which is valid on and near a caustic as well as away from it. To do so we introduce the more complicated representation

$$(3.1) \quad u(t, \mathbf{x}) = \sum_{m=0}^{\infty} \{g_m(t, \mathbf{x}) \partial_{\theta} U_m[\theta(t, \mathbf{x}), \rho(t, \mathbf{x})] \} \\ - h_m(t, \mathbf{x}) \partial_{\rho} U_m[\theta(t, \mathbf{x}), \rho(t, \mathbf{x})].$$

Here the U_m are assumed to satisfy the recursion relation

$$(3.2) \quad \partial_{\theta} U_{m+1} = U_m, \quad m = 0, 1, \dots,$$

and the Tricomi equation

$$(3.3) \quad \partial_{\rho}^2 U_m = \rho \partial_{\theta}^2 U_m, \quad m = 0, 1, \dots$$

The functions g_m , h_m , θ and ρ are to be determined in such a way as to make (3.1) a solution of the wave equation.

We now substitute (3.1) into (2.1) and make use of (3.2) and (3.3). Then we find that (2.1) is satisfied if the following equations hold:

$$(3.4) \quad \rho[(\nabla \rho)^2 - c^{-2} \rho_i^2] + (\nabla \theta)^2 - c^{-2} \theta_i^2 = 0,$$

$$(3.5) \quad \nabla \rho \cdot \nabla \theta - c^{-2} \rho_i \theta_i = 0,$$

$$\begin{aligned}
 (3.6) \quad & 2[\nabla \theta \cdot \nabla g_m - c^{-2} \theta_t g_{mt}] - 2\rho[\nabla \rho \cdot \nabla h_m - c^{-2} \rho_t h_{mt}] \\
 & + g_m L \theta - h_m [(\nabla \rho)^2 - c^{-2} \rho_t^2 + \rho L \rho] + L g_{m-1} = 0, \\
 & m = 0, 1, \dots,
 \end{aligned}$$

$$\begin{aligned}
 (3.7) \quad & 2[\nabla \rho \cdot \nabla g_m - c^{-2} \rho_t g_{mt}] - 2[\nabla \theta \cdot \nabla h_m - c^{-2} \theta_t h_{mt}] \\
 & + g_m L \rho - h_m L \theta - L h_{m-1} = 0, \quad m = 0, 1, \dots.
 \end{aligned}$$

To solve these equations we introduce the new function ϕ^+ , ϕ^- , G_m^+ and G_m^- defined by

$$(3.8) \quad \phi^\pm = \theta \pm \frac{2}{3} \rho^{3/2},$$

$$(3.9) \quad G_m^\pm = g_m \pm \rho^{1/2} h_m.$$

In terms of these new functions, the old functions θ , ρ , g_m and h_m are given by

$$(3.10) \quad \theta = \frac{1}{2}(\phi^+ + \phi^-), \quad \rho = [\frac{3}{4}(\phi^+ - \phi^-)]^{2/3},$$

$$(3.11) \quad g_m = \frac{1}{2}(G_m^+ + G_m^-), \quad h_m = \frac{1}{2}[\frac{3}{4}(\phi^+ - \phi^-)]^{-1/3}(G_m^+ - G_m^-).$$

Upon using (3.10) in (3.4) and (3.5), we find that ϕ^+ and ϕ^- satisfy the equation

$$(3.12) \quad (\nabla \phi^\pm)^2 - c^{-2} (\phi_t^\pm)^2 = 0.$$

Then by using (3.10) and (3.11) in (3.6) and (3.7), we find that the $G_m^\pm$ satisfy the equations

$$\begin{aligned}
 (3.13) \quad & 2[\nabla \phi^\pm \cdot \nabla G_m^\pm - c^{-2} \phi_t^\pm G_{mt}^\pm] \\
 & + \left[L \phi^\pm - \frac{\rho}{2} (\nabla \phi^\pm \cdot \nabla \rho - c^{-2} \phi_t^\pm \rho_t) \right] G_m^\pm = -L g_{m-1} \pm \rho^{1/2} L h_{m-1}, \\
 & m = 0, 1, \dots.
 \end{aligned}$$

Since (3.12) for ϕ^+ and for ϕ^- is the same as (2.4), it can be solved for ϕ^+ and for ϕ^- by introducing the two sets of bicharacteristics which are solutions of (2.6)

$$(3.14) \quad t = t^\pm(\gamma, \gamma), \quad \mathbf{x} = \mathbf{x}^\pm(\gamma, \gamma),$$

with $\alpha = \alpha^\pm$. Then (3.12) becomes $d\phi^\pm/d\gamma = 0$ and its solutions are

$$(3.15) \quad \phi^\pm = \phi^\pm(\gamma).$$

Similarly (3.13) is of the form (2.5) and it can be solved in the same way as (2.5) was with the result

$$\begin{aligned}
 (3.16) \quad G_m^\pm(\gamma, \gamma) = & \left| \frac{J^\pm(\tilde{\gamma}, \gamma) \alpha^\pm(\gamma, \gamma)}{J^\pm(\gamma, \gamma) \alpha^\pm(\tilde{\gamma}, \gamma)} \right|^{1/2} \left[\frac{\rho(\gamma, \gamma)}{\rho(\tilde{\gamma}, \gamma)} \right]^{1/4} G_m^\pm(\tilde{\gamma}, \gamma) \\
 & - \int_{\tilde{\gamma}}^{\gamma} \left| \frac{J^\pm(\gamma', \gamma) \alpha^\pm(\gamma, \gamma)}{J^\pm(\gamma, \gamma) \alpha^\pm(\gamma', \gamma)} \right|^{1/2} \left[\frac{\rho(\gamma, \gamma)}{\rho(\gamma', \gamma)} \right]^{1/4} \\
 & \cdot \alpha^\pm(\gamma', \gamma) [L g_{m-1} \mp \rho^{1/2} L h_{m-1}] d\gamma', \quad m = 0, 1, \dots.
 \end{aligned}$$

Then θ , ρ , g_m and h_m can be obtained by using (3.15) and (3.16) in (3.10) and (3.11). The values of t and $\mathbf{x}$ on each set of bicharacteristics are given by (3.14). The U_m are given uniquely in terms of U_0 by integration of (3.2) provided they vanish at $\theta = \infty$. Thus all the functions entering into (3.1) can be found in parametric form. The various arbitrary quantities which they contain can be chosen to make (3.1) solve particular problems.

We shall now show how these arbitrary quantities can be adjusted to make each term in (3.1) regular at a smooth caustic hypersurface S , so that (3.1) will be valid on and near S . First we consider the three parameter family of bicharacteristics $t(\gamma, \mathbf{r})$, $\mathbf{x}(\gamma, \mathbf{r})$ which are tangent to S at $\gamma = \hat{\gamma}(\mathbf{r})$. We denote by $t^-(\gamma, \mathbf{r})$, $\mathbf{x}^-(\gamma, \mathbf{r})$ and by $t^+(\gamma, \mathbf{r})$, $\mathbf{x}^+(\gamma, \mathbf{r})$ the portions of these bicharacteristics with $\gamma \leq \hat{\gamma}$ and $\gamma \geq \hat{\gamma}$ respectively. Thus contact with S separates the bicharacteristics into two families. Then $J^\pm(\hat{\gamma}, \mathbf{r}) = 0$ and the first term on the right side of (3.16) appears to become infinite at the caustic $\gamma = \hat{\gamma}$. To avoid this singularity we require that

$$(3.17) \quad \rho(t, \mathbf{x}) = 0 \quad \text{on} \quad S.$$

It can be proved that when (3.17) is satisfied, and S satisfies a certain generalized convexity condition like that in [9], the ratio $\rho^{1/4}(\gamma, \mathbf{r})J^{-1/2}(\gamma, \mathbf{r})$ is regular at $\gamma = \hat{\gamma}$ if this is a regular point of S . The proof is essentially the same as that given by Lewis, Bleistein and Ludwig [9] for time harmonic waves, so we shall omit it.

To show that when (3.17) holds, the $G_m^\pm$, g_m and h_m are regular at S , we proceed by induction. Thus we assume that g_{m-1} and h_{m-1} are regular. This is true for $m = 0$ since $g_{-1} = h_{-1} \equiv 0$. It follows from (3.16) and the regularity of $\rho^{1/4}(J^\pm)^{-1/2}$ that $G_m^\pm$ is regular. Then from the first equation in (3.11), we see that g_m is regular. To show that h_m is regular we write the second equation in (3.11) as $h_m = \frac{1}{2}\rho^{-1/2}(G_m^+ - G_m^-)$. Since $\rho = 0$ on S , h_m can be regular there only if we require

$$(3.18) \quad G_m^+(\hat{\gamma}, \mathbf{r}) = G_m^-(\hat{\gamma}, \mathbf{r}), \quad m = 0, 1, \dots$$

When (3.18) holds it can be proved that h_m is regular at S , the proof again being essentially the same as that in [9]. This completes the induction and proves that each term in (3.1) is regular at S . It is to be noted from (3.8) or (3.10) that the condition (3.17) is equivalent to

$$(3.19) \quad \phi^+ = \phi^- \quad \text{on} \quad S.$$

This shows that the two functions ϕ^+ and ϕ^- are the two values of the function $\phi(t, \mathbf{x})$ defined on the original bicharacteristics.

We shall now show that at points not on the caustic S , the uniform representation (3.1) can be converted into the sum of two progressing waves, each of the form (2.2), provided that U_m is given by

$$(3.20) \quad U_m = \int_{-\infty}^{\infty} f_{m+1}(\theta - \xi\rho + \xi^3/3) d\xi.$$

This U_m satisfies (3.3), and it satisfies (3.2) if $f'_{m+1} = f_m$, as we assume. Now we shall express the integrals representing $\partial_\theta U_m$ and $\partial_\rho U_m$ as series and then rearrange the terms in (3.1) to obtain the desired result. The integrals can be expanded by using the following lemma due to Ludwig [7, Appendix B].

Lemma. *Let f_0 have compact support, let $f'_m = f_{m-1}$ with $f_m \rightarrow 0$ as $|\xi| \rightarrow \infty$, let $|\psi(\rho, \xi)| \rightarrow \infty$ uniformly in ρ as $|\xi| \rightarrow \infty$ and let $|\psi_\xi|^2 + |\psi_{\xi\xi}|^2$ be bounded away from zero. Then we have formally*

$$(3.21) \quad \int_{-\infty}^{\infty} f_m[\theta + \psi(\rho, \xi)]q(\xi) d\xi = \sum_{j=1}^p \sum_{\alpha=0}^{\infty} I_{\pm}^{1/2} f_{m+\alpha}[\theta + \psi_i(\rho)]a_i^\alpha(\rho).$$

Here p is the number of roots $\xi_i(\rho)$ of the equation

$$\psi_\xi[\rho, \xi_i(\rho)] = 0, \quad j = 1, \dots, p,$$

$\psi_i(\rho)$ and $a_i^0(\rho)$ are defined by

$$(3.22) \quad \psi_i(\rho) = \psi[\rho, \xi_i(\rho)],$$

$$(3.23) \quad a_i^0(\rho) = (2\pi)^{1/2} |\psi_{\xi\xi}[\rho, \xi_i(\rho)]|^{-1/2} q(\xi_i),$$

and the sign $+$ or $-$ is the sign of $-\psi_{\xi\xi}[\rho, \xi_i(\rho)]$. The fractional integral operators $I_{\pm}^{1/2}$ are defined by

$$(3.24) \quad I_+^{1/2} f(r) = \pi^{-1/2} \int_{-\infty}^r f(s)(r-s)^{-1/2} ds,$$

$$(3.25) \quad I_-^{1/2} f(r) = -\pi^{-1/2} \int_r^{\infty} f(s)(r-s)^{-1/2} ds.$$

The $a_i^\alpha(\rho)$ with $\alpha > 0$ are also expressible in terms of ψ and q and their derivatives at ξ_i .

In applying this lemma to the integrals representing $\partial_\theta U_m$ and $\partial_\rho U_m$ we have $q(\xi) = 1$ in the first integral, $q(\xi) = \xi$ in the second integral, and in both integrals $\psi = -\rho\xi + \xi^3/3$. Then $\psi_\xi = 0$ at $\xi_+ = -\rho^{1/2}$ and $\xi_- = \rho^{1/2}$, so $p = 2$ and we have used the labels $\pm$ instead of $j = 1, 2$. Thus $\psi_+(\rho) = \psi[\rho, \rho^{1/2}] = 2\rho^{3/2}/3$, $\psi_-(\rho) = \psi[\rho, -\rho^{1/2}] = -2\rho^{3/2}/3$, $\psi_{\xi\xi}[\rho, -\rho^{1/2}] = -2\rho^{1/2}$ and $\psi_{\xi\xi}[\rho, \rho^{1/2}] = 2\rho^{1/2}$. We now define $e_m^\pm$ by

$$(3.26) \quad e_m^\pm = I_{\pm}^{1/2} f_m.$$

Then by using (3.20), (3.21) and (3.26) in (3.1) we obtain

$$(3.27) \quad u(t, \mathbf{x}) = \sum_{m=0}^{\infty} \left[g_m \sum_{\alpha=0}^{\infty} \{ e_{m+\alpha}^- [\theta + \psi_-(\rho)] a_-^\alpha(\rho) + e_{m+\alpha}^+ [\theta + \psi_+(\rho)] a_+^\alpha(\rho) \} \right. \\ \left. + h_m \sum_{\alpha=0}^{\infty} \{ e_{m+\alpha}^- [\theta + \psi_-(\rho)] b_-^\alpha(\rho) + e_{m+\alpha}^+ [\theta + \psi_+(\rho)] b_+^\alpha(\rho) \} \right].$$

Here we have used $b_\pm$ to denote the coefficients $a_i(\rho)$ for the second integral,

in which $q(\xi) = \xi$. We now rearrange the double sums in (3.27) to obtain

$$(3.28) \quad u(t, \mathbf{x}) = \sum_{m=0}^{\infty} [e_m^-(\phi^-)Z_m^- + e_m^+(\phi^+)Z_m^+].$$

Here $\phi^\pm$ and $Z_m^\pm$ are defined by

$$(3.29) \quad \phi^\pm = \theta + \psi_\pm(\rho) = \theta \mp 2\rho^{3/2}/3,$$

$$(3.30) \quad Z_m^\pm = \sum_{j=0}^m (g_j a_\pm^{m-j} + h_j b_\pm^{m-j}).$$

The result (3.28) is exactly a sum of two progressing waves of the form (2.2), as we wanted to show. In view of the definition (3.26) and the recursion relation (3.2) satisfied by f_m , it follows that the $e_m^\pm$ satisfy the recursion relation (2.3). From (3.29) we see that $\phi^\pm$ is the same as that given by (3.8), so it satisfies (2.4). It can also be shown that $Z_m^\pm$ satisfy (2.5). In particular from (3.30), (3.23) and (3.9) we have

$$(3.31) \quad Z_0^\pm = g_0 a_\pm^0 + h_0 b_\pm^0 = \pi^{1/2} \rho^{-1/4} (g_0 \pm \rho^{1/2} h_0) = \pi^{1/2} \rho^{-1/4} G_0^\pm.$$

By using (3.16) with $m = 0$ in (3.31), we find that $Z_0^\pm$ given by (3.31) is the same as $Z_0^\pm$ given by (2.12), which verifies the statement for $m = 0$.

Let us specialize (3.20) by letting $2\pi k^{-1/3} f_m$ be given by the right side of (2.14), c by (2.15), θ by (2.16) and let ρ , g_m and h_m be independent of t . Then (3.1) becomes

$$(3.32) \quad u(t, \mathbf{x}) = \exp \{-i\omega t + ik\hat{\theta}(\mathbf{x})\} Ai[-k^{2/3}\rho(\mathbf{x})] \\ \cdot \sum_{m=0}^{\infty} (ik)^{-m} g_m(\mathbf{x}) + ik^{-1/3} Ai'[-k^{2/3}\rho(\mathbf{x})] \sum_{m=0}^{\infty} (ik)^{-m} h_m(\mathbf{x}).$$

Here Ai is the Airy function and Ai' is its derivative. The result (3.32) is just the uniform asymptotic expansion of a time harmonic progressing wave, given by Ludwig [7] and Kravtsov [10], [11].

In concluding this section we observe from (3.24)–(3.26) that the waveforms e_m^+ and e_m^- are related to each other by the Hilbert transformation

$$(3.33) \quad e_m^+(r) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e_m^-(s)}{s - r} ds.$$

This change in waveform corresponds to the phase shift of $-\pi/2$ experienced by a time harmonic wave upon passing through a caustic.

4. Waves diffracted by a smooth surface (creeping waves). It is well known that waves can travel along curved surfaces. Such waves are produced by diffraction when a wave is incident upon a curved surface, and they are called surface diffracted waves or creeping waves. In the time harmonic case their asymptotic form has been determined by Keller [12], and their asymptotic expansion has been found by Friedlander and Keller [8]. We shall now obtain

a representation for general time dependent creeping waves which reduces to that of Friedlander and Keller in the time harmonic case.

We begin by introducing the representation

$$(4.1) \quad u(t, \mathbf{x}) = \sum_{m=0}^{\infty} e_m[\phi(t, \mathbf{x}), \chi(t, \mathbf{x})] Z_m(t, \mathbf{x}).$$

The function $e_m(\phi, \chi)$ is assumed to satisfy the following differential equation and difference-differential equation:

$$(4.2) \quad \partial_{\chi}^3 e_m + \partial_{\phi} e_m = 0,$$

$$(4.3) \quad \partial_{\chi} e_m = i e_{m-1}, \quad m = 0, 1, \dots; \quad e_{-1} \equiv 0.$$

If $e_m(\phi, -\infty) = 0$ then $e_m(\phi, \chi)$ is uniquely determined in terms of $e_0(\phi, \chi)$. We now substitute (4.1) into the wave equation (2.1) and utilize (4.2) and (4.3). Then we find that (2.1) is satisfied if ϕ satisfies (2.4) and if the following equations hold:

$$(4.4) \quad \nabla \phi \cdot \nabla \chi - c^{-2} \phi_t \chi_t = 0,$$

$$(4.5) \quad 2[\nabla \phi \cdot \nabla Z_m - c^{-2} \phi_t Z_{m,t}] + Z_m L \phi = -i[(\nabla \chi)^2 - c^{-2} (\chi_t)^2] Z_{m-1} \\ - 2[\nabla \chi \cdot \nabla Z_{m-2} - c^{-2} \chi_t Z_{m-2,t}] - Z_{m-2} L \chi + i L Z_{m-3}, \\ m = 0, 1, \dots; \quad Z_{-1} \equiv 0.$$

Since ϕ satisfies (2.4), it is constant on each of the bicharacteristics (2.9), so it is given by (2.10). Then (4.4) can be written $d\chi/d\gamma = 0$ so χ is also constant along bicharacteristics,

$$(4.6) \quad \chi = \chi(\gamma).$$

Upon comparing (4.5) with (2.5), we see that the left sides are identical. Therefore Z_m is given by (2.12) with LZ_{m-1} in the integrand replaced by the right side of (4.5). This completes the determination of the wave (4.1).

To show the relationship of (4.1) to the expansion of Friedlander and Keller, let us set

$$(4.7) \quad e_m(\phi, \chi) = k^{-m/3} \exp \{i(k\phi + k^{1/3}\chi)\}, \quad m = 0, 1, \dots$$

This function clearly satisfies (4.2) and (4.3). Suppose that c is given by (2.15), ϕ by (2.16) and that $\chi(\mathbf{x})$ and $Z_m(\mathbf{x})$ are independent of t . Then (4.1) becomes

$$(4.8) \quad u(t, \mathbf{x}) = e^{-i\omega t} \sum_{m=0}^{\infty} k^{-m/3} Z_m(\mathbf{x}) \exp \{i[k\hat{\theta}(\mathbf{x}) + k^{1/3}\chi(\mathbf{x})]\}.$$

This is just the expansion of a time harmonic surface diffracted wave given in [8], with k being the wavenumber or propagation constant in a medium with $c = c_0$.

The surface diffracted bicharacteristics associated with a surface diffracted wave are tangent to the diffracting surface, which is therefore a caustic of these

bicharacteristics. As a result the representation (4.1) is not valid at this surface because the Z_m become infinite there. Thus the representation (4.1) is only valid away from the surface.

5. Uniform representation of waves diffracted by a smooth surface (creeping waves). In the time harmonic case, Lewis, Bleistein and Ludwig [9] have obtained a uniform representation of a surface diffracted wave. We shall obtain a uniform representation for general time dependent surface diffracted waves which reduces to theirs in the time harmonic case. To obtain such a representation, which is valid both on and away from the diffracting surface, we introduce the series

$$(5.1) \quad u(t, \mathbf{x}) = \sum_{m=0}^{\infty} \left\{ g_m(t, \mathbf{x}) \int_{-\infty}^{\infty} f_m(\theta - \xi\rho + \xi^3/3, \theta_1 - \xi\rho_1) d\xi \right. \\ \left. - h_m(t, \mathbf{x}) \int_{-\infty}^{\infty} \xi f_m(\theta - \xi\rho + \xi^3/3, \theta_1 - \xi\rho_1) d\xi \right\}.$$

The functions f_m are assumed to satisfy (4.2) and (4.3), which imply

$$(5.2) \quad \frac{d}{d\xi} f_{m+3}(\theta - \xi\rho + \xi^3/3, \theta_1 - \xi\rho_1) = i(\xi^2 - \rho)f_m - i\rho_1 f_{m+2}.$$

Integrating (5.2) and assuming that f_{m+3} vanishes at $|\xi| = \infty$ yields

$$(5.3) \quad \int_{-\infty}^{\infty} \xi^2 f_m(\theta - \xi\rho + \xi^3/3, \theta_1 - \xi\rho_1) d\xi = \rho \int_{-\infty}^{\infty} f_m d\xi + \rho_1 \int_{-\infty}^{\infty} f_{m+2} d\xi.$$

We now substitute (5.1) into the wave equation (2.1) and use (4.2), (4.3) and (5.2). Then we find that (2.1) is satisfied if θ , ρ , g_m and h_m satisfy (3.4)–(3.7) with additional terms on the right sides of (3.6) and (3.7) for $m \geq 1$, and if θ_1 and ρ_1 satisfy

$$(5.4) \quad 2[\nabla\theta \cdot \nabla\theta_1 - c^{-2}\theta_{1t}] + \rho_1[(\nabla\rho)^2 - c^{-2}\rho_t^2] \\ + 2\rho[\nabla\rho \cdot \nabla\rho_1 - c^{-2}\rho_t\rho_{1t}] = 0,$$

$$(5.5) \quad \nabla\rho \cdot \nabla\theta_1 - c^{-2}\rho_t\theta_{1t} + \nabla\theta \cdot \nabla\rho_1 - c^{-2}\theta_t\rho_{1t} = 0.$$

We have already shown how to solve (3.4)–(3.7), and the additional terms in (3.6) and (3.7) involve only g_j and h_j with $j < m$. Therefore they may be considered to be known, and the previous solution formulas apply with additional terms in the integrand for $m \geq 1$. To solve (5.4) and (5.5) we introduce χ^+ and χ^- defined by

$$(5.6) \quad \chi^{\pm} = \theta_1 \pm \rho^{1/2}\rho_1.$$

Then (5.4) and (5.5) become

$$(5.7) \quad \nabla\phi^{\pm} \cdot \nabla\chi^{\pm} - c^{-2}\phi_t^{\pm}\chi_t^{\pm} = 0.$$

These equations show that χ^+ and χ^- are constant on bicharacteristics, so we have

$$(5.8) \quad \chi^\pm = \chi^\pm(\gamma).$$

Then from (5.6) and (5.8) we obtain

$$(5.9) \quad \theta_1 = \frac{1}{2}[\chi^+(\gamma) + \chi^-(\gamma)], \quad \rho_1 = \frac{1}{2}\rho^{-1/2}[\chi^+(\gamma) - \chi^-(\gamma)].$$

This completes the determination of all the functions occurring in the representation (5.1). They contain arbitrary quantities which can be adjusted to solve particular problems.

To show the relation between the representation (5.1) and that of Lewis, Bleistein and Ludwig [9], we set

$$(5.10) \quad f_m(\theta - \xi\rho + \xi^3/3, \theta_1 - \xi\rho_1) \\ = e^{-2\pi i/3} \frac{k^{(1-m)/3}}{2\pi} \exp \{ik(\theta - \xi\rho + \xi^3/3) + ik^{1/3}(\theta_1 - \xi\rho_1)\}.$$

This function satisfies (4.2) and (4.3). We also suppose that c is given by (2.15), θ by the right side of (2.16) and we assume that ρ , θ_1 , ρ_1 , g_m and h_m are independent of t . Then (5.1) becomes

$$(5.11) \quad u(t, \mathbf{x}) = \exp \{-i\omega t + ik\hat{\theta}(\mathbf{x}) + ik^{1/3}\theta_1(\mathbf{x})\} \\ \cdot \left\{ Ai[(-k^{2/3}\rho - \rho_1)e^{2\pi i/3}] \sum_{m=0}^{\infty} k^{-m/3} g_m(\mathbf{x}) \right. \\ \left. - e^{i\pi/6} k^{-1/3} Ai'[(-k^{2/3}\rho - \rho_1)e^{2\pi i/3}] \sum_{m=0}^{\infty} k^{-m/3} h_m(\mathbf{x}) \right\}.$$

Here Ai is the Airy function. The representation (5.11) is exactly that of [9], which applies to time harmonic waves.

6. Surface-diffracted progressing waves for first order systems. Progressing wave solutions, such as those of Section 2, have been obtained for general systems of linear first order partial differential equations [3]. Uniform representations of progressing waves valid at caustics, such as those of Section 3, have also been found for first order systems [7]. We shall now show how to obtain non-uniform representations of surface-diffracted progressing waves, such as those of Section 4, for first order systems. Uniform representations like those of Section 5 will be found in Section 7.

Let the vector $u(x)$ satisfy the following system of linear first order partial differential equations:

$$(6.1) \quad \sum_{\nu=0}^n A_\nu(x) \partial_\nu u + B(x)u = 0.$$

Here $x = (x_0, \dots, x_n)$, A_ν and B are square matrices, and henceforth we shall

sum the repeated index ν from 0 to n . We shall seek a solution of the form

$$(6.2) \quad u(x) = \sum_{m=0}^{\infty} e_m[\varphi(x), \chi(x)]z_m(x).$$

The functions e_m are assumed to satisfy (4.2) and (4.3), while φ , χ and the vectors z_m are to be found so that (6.2) satisfies (6.1).

We now substitute (6.2) into (6.1) and make use of (4.2) and (4.3). Then we find that (6.1) is satisfied if the following equations hold:

$$(6.3) \quad (A, \partial_\nu \varphi)z_m = -(A, \partial_\nu \chi)z_{m-2} + A_\nu \partial_\nu z_{m-3} - iBz_{m-3}, \\ m = 0, 1, \dots$$

For $m < 0$, we define $z_m \equiv 0$. Thus for $m = 0$, the right side of (6.3) is zero. If $z_0 \neq 0$, it follows that

$$(6.4) \quad \det(A, \partial_\nu \varphi) = 0.$$

This first order partial differential equation for $\varphi(x)$ is the characteristic equation of (6.1). Then it follows from (6.3) that z_0 is given by

$$(6.5) \quad z_0(x) = \sigma_0(x)R(x).$$

Here $\sigma_0(x)$ is a scalar, and $R(x)$ is a particular solution of

$$(6.6) \quad (A, \partial_\nu \varphi)R = 0.$$

We assume that the nullity of the matrix $A, \partial_\nu \varphi$ is unity, so that R is unique except for a scalar factor.

For $m = 1$, the right side of (6.3) is also zero, so z_1 is given by

$$(6.7) \quad z_1(x) = \sigma_1(x)R(x)$$

with $\sigma_1(x)$ a scalar. For $m > 1$, (6.3) will have a solution for z_m if and only if the right side is orthogonal to a left null vector of $A, \partial_\nu \varphi$. Let $L(x)$ be a particular solution of

$$(6.8) \quad LA, \partial_\nu \varphi = 0.$$

Then the solvability conditions for (6.3) with m replaced by $m + 2$ is

$$(6.9) \quad -L(A, \partial_\nu \chi)z_m + LA, \partial_\nu z_{m-1} - iLBZ_{m-1} = 0.$$

When (6.9) holds, the solution of (6.3) for z_m can be written as

$$(6.10) \quad z_m = \sigma_m(x)R(x) + y_m(x).$$

Here y_m is a particular solution of (6.3).

By using (6.10) in (6.9), we obtain

$$(6.11) \quad -L(A, \partial_\nu \chi)(R\sigma_m + y_m) + LA, \partial_\nu(\sigma_{m-1}R + y_{m-1}) \\ - iLB(\sigma_{m-1}R + y_{m-1}) = 0.$$

Upon setting $m = 0$ in (6.11), recalling that $z_{-1} = 0$ and $y_0 = 0$, and assuming that $\sigma_0 \neq 0$, we obtain

$$(6.12) \quad L(A, \partial_\nu \chi)R = 0.$$

This is a linear first order partial differential equation for $\chi(x)$. It states that the derivative of χ is zero in the direction of the vector LA, R . Thus χ is constant along any curve which has this direction at each point x . These are just the characteristic curves of (6.4), which are also the bicharacteristics of (6.1). Multiplication of (6.6) on the right by R shows that φ is also constant on bicharacteristics.

In view of (6.12), we can simplify (6.11) to the form

$$(6.13) \quad (LA, R)\partial_\nu \sigma_{m-1} + \sigma_{m-1}(LA, \partial_\nu R - iLBR) \\ = L(A, \partial_\nu \chi)y_m - LA, \partial_\nu y_{m-1} + iLBy_{m-1}.$$

This is a linear first order partial differential equation for σ_{m-1} , which is just an ordinary differential equation along a bicharacteristic of (6.1). By starting with $m = 1$, we can solve successively for the σ_m using (6.13). This is so because y_m does not depend upon σ_m or σ_{m-1} but only upon σ_{m-2} and σ_{m-3} , as we see from the right side of (6.3).

The method for constructing the solution (6.2) has now been described completely, and we shall summarize it. The e_m are assumed to be given and to satisfy (4.2) and (4.3), or else just e_0 can be given and the other e_m can be found from these equations. Then φ and χ are constant on bicharacteristics, and z_m is given by (6.9). In (6.9), R is a solution of (6.6), y_m is a particular solution of (6.3), and σ_m is a solution of (6.13). The σ_m and y_m can be found successively from these equations.

7. Uniform representation of surface-diffracted progressing waves for first order systems. We shall now obtain a uniform representation, like that of Section 5, of surface-diffracted progressing wave solutions of the first-order system (6.1). We seek a solution of the form (5.1) in which g_m and h_m are vectors, and in which the independent variables are denoted x . We then substitute (5.1) into (6.1), use (4.2) and (4.3) and write $\mathcal{L} = A, \partial_\nu + B$, to obtain

$$(7.1) \quad \sum_{m=0}^{\infty} \left\{ A, \int (g_m - \xi h_m)[(\partial_\nu \theta - \xi \partial_\nu \rho) if_{m-3} + (\partial_\nu \theta_1 - \xi \partial_\nu \rho_1) if_{m-1}] d\xi \right. \\ \left. + \int [\mathcal{L}(g_m - \xi h_m)]f_m d\xi \right\} = 0.$$

We now use (5.3) to eliminate the integrals of $\xi^2 f_m$ from (7.1). Then we equate to zero the coefficients of the integrals of f_{m-3} and ξf_{m-3} in (7.1) to obtain

$$(7.2) \quad (A, \partial_\nu \theta)g_m + \rho(A, \partial_\nu \rho)h_m = i\mathcal{L}g_{m-3} - (A, \partial_\nu \theta_1)g_{m-2} \\ - \rho_1(A, \partial_\nu \rho)h_{m-2} - (A, \partial_\nu \rho_1)(\rho h_{m-2} + \rho_1 h_{m-4}),$$

$$(7.3) \quad (A, \partial_\nu \rho)g_m + (A, \partial_\nu \theta)h_m = i\mathcal{L}h_{m-3} - (A, \partial_\nu \rho_1)g_{m-2} - (A, \partial_\nu \theta_1)h_{m-2}.$$

To solve these equations we multiply (7.3) by $\pm \rho^{1/2}$ and add it to (7.2). The resulting equations can be written simply in terms of $\varphi^\pm$ and $G_m^\pm$ defined by (3.8) and (3.9). Using these functions we have

$$(7.4) \quad (A, \partial_\nu \varphi^\pm)G_m^\pm = i\mathcal{L}g_{m-3} - (A, \partial_\nu \theta_1)g_{m-2} - \rho_1(A, \partial_\nu \rho)h_{m-2} \\ - (A, \partial_\nu \rho_1)(\rho h_{m-2} + \rho h_{m-4}) \pm \rho^{1/2}[i\mathcal{L}h_{m-3} - (A, \partial_\nu \rho_1)g_{m-2} - (A, \partial_\nu \theta_1)h_{m-2}].$$

For $m = 0$ the right side of (7.4) vanishes. Then if $G_0^\pm \neq 0$, as we assume, $\varphi^\pm$ must satisfy the equation

$$(7.5) \quad \det(A, \partial_\nu \varphi^\pm) = 0.$$

Now let $R^\pm$ be a right null vector of $A, \partial_\nu \varphi^\pm$,

$$(7.6) \quad (A, \partial_\nu \varphi^\pm)R^\pm = 0.$$

Then (7.4) with $m = 0$ yields

$$(7.7) \quad G_0^\pm = \sigma_0^\pm R^\pm.$$

The right side of (7.4) also vanishes for $m = 1$ so

$$(7.8) \quad G_1^\pm = \sigma_1^\pm R^\pm.$$

Let $L^\pm$ be a left null vector of $A, \partial_\nu \varphi^\pm$,

$$(7.9) \quad L^\pm A, \partial_\nu \varphi^\pm = 0.$$

Then multiplying (7.4) on the left by $L^\pm$ and replacing m by $m + 2$ yields the solvability condition

$$(7.10) \quad L^\pm \{i\mathcal{L}g_{m-1} - (A, \partial_\nu \theta_1)g_m - \rho_1(A, \partial_\nu \rho)h_m - (A, \partial_\nu \rho_1)(\rho h_m + \rho_1 h_{m-2}) \\ \pm \rho^{1/2}[i\mathcal{L}h_{m-1} - (A, \partial_\nu \rho_1)g_m - (A, \partial_\nu \theta_1)h_m]\} = 0.$$

When (7.10) is satisfied, the solution $G_m^\pm$ of (7.4) is given by

$$(7.11) \quad G_m^\pm = \sigma_m^\pm R^\pm + y_m^\pm.$$

Here $y_m^\pm$ is a particular solution of (7.4) and $\sigma_m^\pm$ is a scalar. Then g_m and h_m are given by (3.11) in which $G_m^\pm$ is given by (7.11). Thus

$$(7.12) \quad g_m = \frac{1}{2}(\sigma_m^+ R_m^+ + \sigma_m^- R_m^- + y_m^+ + y_m^-),$$

$$(7.13) \quad h_m = \frac{1}{2}[\frac{3}{4}(\varphi^+ - \varphi^-)]^{-1/3}(\sigma_m^+ R_m^+ - \sigma_m^- R_m^- + y_m^+ - y_m^-).$$

To satisfy (7.10) let us first set $m = 0$ in it to obtain

$$(7.14) \quad L^\pm A, [(\partial_\nu \theta_1 \pm \rho^{1/2} \partial_\nu \rho_1)g_0 \pm (\partial_\nu \theta_1 \pm \rho^{1/2} \partial_\nu \rho_1 \pm \rho^{-1/2} \rho_1 \partial_\nu \rho) \rho^{1/2} h_0] = 0.$$

In terms of $\chi^\pm$, defined by (5.6), we can write (7.14) in the form

$$(7.15) \quad L^\pm A, [(\partial_\nu \chi^\pm)G_0^\pm \mp \frac{1}{2}\rho^{-1/2} \rho_1(\partial_\nu \rho)G_0^\mp] = 0.$$

Upon using (7.7) for $G_0^\pm$ and (3.10) for ρ we can rewrite (7.15) as

$$(7.16) \quad \sigma_0^\pm L^\pm A, R^\pm \partial, \chi^\pm \mp \frac{\rho_1}{4} [\frac{3}{4}(\varphi^+ - \varphi^-)]^{-2/3} \cdot \sigma_0^\mp [L^\pm A, (\partial, \varphi^\mp) R^\mp - L^\pm A, (\partial, \varphi^\mp) R^\mp] = 0.$$

As a consequence of (7.6) and (7.9), the last two terms in (7.16) vanish. Then if $\sigma_0^\pm \neq 0$, as we assume, (7.16) becomes

$$(7.17) \quad L^\pm A, R^\pm \partial, \chi^\pm = 0.$$

These are two linear first order partial differential equations, one for χ^+ and one for χ^- . They show that $\chi^\pm$ is constant along curves which have the direction $L^\pm A, R^\pm$, i.e. bicharacteristics of (6.1). Multiplication of (7.9) on the right by $R^\pm$ shows that $\varphi^\pm$ is also constant on bicharacteristics.

For any m , (7.10) can be put in the form

$$(7.18) \quad L^\pm \mathcal{L}G_{m-1}^\pm \mp L^\pm A, (\partial, \rho^{1/2}) h_{m-1} = -iL^\pm A, [(\partial, \chi^\pm) G_m^\pm \mp \rho_1 (\partial, \rho^{1/2}) G_m^\pm + \rho_1 (\partial, \rho_1) h_{m-2}].$$

We now use, in (7.18), (7.11) for $G_m^\pm$, (7.13) for h_{m-1} and note that the terms on the right side involving $\sigma_m^\pm R^\pm$ vanish because of (7.16). Then (7.18) becomes

$$(7.19) \quad L^\pm A, R^\pm \partial, \sigma_{m-1}^\pm + L^\pm (BR^\pm + A, \partial, R^\pm) \sigma_{m-1}^\pm + \frac{1}{6(\varphi^+ - \varphi^-)} L^\pm A, (\partial, \varphi^\mp) (\pm R^\pm \sigma_{m-1}^\pm + y_{m-1}^+ - y_{m-1}^-) = -iL^\pm A, [(\partial, \chi^\pm) y_m^\pm \mp \rho_1 (\partial, \rho^{1/2}) y_m^\mp + \rho_1 (\partial, \rho_1) h_{m-2}].$$

This is a pair of uncoupled first order linear partial differential equations for $\sigma_{m-1}^\pm$. From (7.13), h_{m-1} involves $\sigma_{m-1}^\pm$ and h_{m-2} involves $\sigma_{m-2}^\pm$, while from (7.4) y_m involves $\sigma_{m-2}^\pm$, $\sigma_{m-3}^\pm$ and $\sigma_{m-4}^\pm$. Therefore all the $\sigma_m^\pm$ can be found successively from (7.19). In fact (7.19) is an ordinary differential equation along a curve in the direction $L^\pm A, R^\pm$, i.e. along a bicharacteristic of (6.1).

This completes the description of the method for obtaining all the quantities in the progressing wave (5.1).

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