Motion of Solids in Viscous and Electrically Conducting Fluids

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Introduction. The general equations of magnetohydrodynamics are non-linear and are more complicated than the Navier-Stokes equations of hydrodynamics. The properties and the solutions of even the latter equations are not completely known and we should therefore expect that we know even much less about the solutions of the system of equations governing hydromagnetic flows. However, for certain physical situations the Navier-Stokes equations can be linearized giving rise to the well known Stokes and Oseen flow equations. Validity of these two sets of equations is limited to small Reynolds numbers only. The same can be done to the general equations of magnethydrodynamics as well. In addition to the classical Reynolds number, various other dimensionless parameters also emerge and two of them play a vital role in linearizing this system. These are the magnetic Reynolds number and the Hartmann number. The linearized systems of equations again fall into two categories—the Stokes approximation and Oseen approximation. Various solutions of these equations have been discussed in the last decade. Our aim is to review and extend these solutions and present a unified account of various boundary value problems for the steady flow of an incompressible, viscous and magnetized fluid.

The canonical problems discussed in this paper are the steady rotation of an insulated solid about its axis of symmetry and uniform motion of the fluid past such a body. Both plane and spatial problems are of interest but, to keep the article to a manageable size, we limit the discussion to three-dimensional problems only. The corresponding analysis for the two-dimensional problems can be given on the same lines and, in fact, is simpler in general. In addition to obtaining various solutions of the flow problems, we display an interesting spectrum of modern approximation techniques of applied mathematics. For example, the method of singular perturbations and matched asymptotic expansions, an integral equation perturbation technique for small as well as large parameters and methods based on Watson transformation are illustrated with appropriate boundary value problems. The analogy between many problems in magnetohydrodynamics and the classical hydrodynamics is established and exploited. Furthermore, an analogy between many of the problems of this

article and certain problems in the theory of diffraction also plays an interesting role.

The plan of the paper is as follows. In Section 2 we analyze the Stokes and Oseen type linearization of the general equations governing the flow of an incompressible, viscous and electrically conducting fluid along with the appropriate boundary conditions in the presence of the obstacles in the flow. It turns out that these systems of equations are valid for small as well as large Hartmann numbers, *i.e.*, for weak as well as strong magnetic fields. We therefore partition the rest of the article into two sections. Section 3 is devoted to the study of flows with a small Hartmann number. We discuss the steady rotation of an axially symmetric body about its axis of symmetry both for Stokes as well as Oseen flows. Thereafter, we proceed to solve the problem of the steady and uniform motion of the fluid past an insulated body and again present solutions for both linearized systems. Formulas for the velocity field as well as the drag and torque are derived for all these problems. The corresponding discussion for large Hartmann numbers is given in Section 4. These problems are solved under the assumption that the magnetic field is parallel to the undisturbed flow.

In the process of the derivation of the above-mentioned formulas there emerge many other interesting features such as the shear layers, upstream and downstream wakes, regions of stagnant flow and cores rotating like rigid bodies. For large Hartmann numbers the viscous effects are found to be important only in thin layers in which the velocity gradients are large. Although the results discussed in this article pertain to unbounded fluids, they can be extended to fluids which are bounded by cylindrical vessels or between two parallel planes.

2. Equations of motion. The equations governing the steady flow of an incompressible, viscous and electrically conducting fluid in *MKS* units are

$$\begin{aligned}
 \mathbf{v} \cdot \nabla \mathbf{v} &= -\frac{1}{\rho} \nabla p + \frac{\mu_e}{\rho} \mathbf{J} \times \mathbf{H} + \nu \nabla^2 \mathbf{v}, \\
 (2.1) \quad \nabla \cdot \mathbf{v} &= 0, \quad \nabla \cdot \mathbf{H} = 0, \quad \nabla \times \mathbf{E} = 0, \\
 \mathbf{J} &= \sigma(\mathbf{E} + \mu_e \mathbf{v} \times \mathbf{H}) = \nabla \times \mathbf{H},
 \end{aligned}$$

where ρ , p , ν , μ_e and σ are the fluid density, pressure, kinematic viscosity, magnetic permeability and electrical conductivity, respectively. Furthermore, $\mathbf{v}$, $\mathbf{J}$, $\mathbf{H}$, and $\mathbf{E}$ are the velocity field, the electric current density, magnetic field strength and electric field strength respectively. To non-dimensionalize the above system of equations we put dashes on the physical variables and set

$$\begin{aligned}
 \mathbf{v} &= \frac{\mathbf{v}'}{U}, \quad \mathbf{r} = \frac{\mathbf{r}'}{a}, \quad p = \frac{p'a}{\mu U}, \\
 \mathbf{H} &= \frac{\mathbf{H}'}{H_0}, \quad \mathbf{E} = \frac{\mathbf{E}'}{\mu_e H_0 U}, \quad \mathbf{J} = \frac{\mathbf{J}'}{\mu_e H_0 \sigma U},
 \end{aligned}$$

where U , a , and H_0 are the typical velocity, characteristic length and the uniform magnetic field. This uniform field is oriented in the $\mathbf{e}_1$ direction, *i.e.*, along the x -axis. The quantity μ is the shear viscosity of the fluid and $\mathbf{r}$ stands for the position vector of a point (x, y, z) .

The non-dimensional form of the system (2.1) therefore is (after dropping the dashes)

$$(2.2) \quad \begin{aligned} R\mathbf{v} \cdot \nabla \mathbf{v} &= -\nabla p + \frac{M^2}{R_m} (\text{curl } \mathbf{H}) \times \mathbf{H}, \\ \nabla \cdot \mathbf{v} &= 0, \quad \nabla \cdot \mathbf{H} = 0, \quad \nabla \times \mathbf{E} = 0, \\ R_m \mathbf{J} &= \nabla \times \mathbf{H} = R_m (\mathbf{E} + \mathbf{v} \times \mathbf{H}), \end{aligned}$$

where the nondimensional parameters R , R_m and M are defined as

$$(2.3) \quad \begin{aligned} R &= aU/\nu = \text{Reynolds number}, \\ R_m &= \sigma\mu_e aU = \text{magnetic Reynolds number}, \\ M &= \mu_e H_0 a (\sigma/\mu)^{1/2} = \text{Hartmann number}. \end{aligned}$$

The equation (2.2)₃ gives

$$(2.4) \quad \text{curl curl } \mathbf{H} = R_m \text{curl } (\mathbf{v} \times \mathbf{H}).$$

2.1 Stokes approximation. The Stokes approximation is obtained [1] by taking $R \ll 1$, $R_m \ll 1$, $M \gg R$ and $M \gg R_m$ although M may itself be small. Furthermore, in view of the symmetry conditions the electric field $\mathbf{E}$ is taken zero. From (2.4) we observe that if we neglect the right hand term, the magnetic field becomes independent of the velocity field. The equations for the velocity field $\mathbf{v}$, then follow from (2.2)_{1,2} by setting $\mathbf{H} = \mathbf{e}_1$, $\nabla \times \mathbf{H} = R_m(\mathbf{v} \times \mathbf{H})$ and putting $R = 0$. We finally have

$$(2.5a) \quad \nabla \cdot \mathbf{v} = 0, \quad -\nabla p + \nabla^2 \mathbf{v} + M^2(\mathbf{v} \times \mathbf{e}_1 \times \mathbf{e}_1) = 0,$$

or

$$(2.5b) \quad \nabla \cdot \mathbf{v} = 0, \quad -\nabla p + \nabla^2 \mathbf{v} - M^2\{\mathbf{v} - (\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1\}.$$

For steady rotation problems of axially symmetric solids the typical linear velocity U is $U = a\Omega$, where Ω is the uniform angular velocity of the solid. For the flow of a fluid past the solid, U is the undisturbed velocity of the stream in the $\mathbf{e}_1$ direction.

2.2 Oseen approximation. In the above equations we have neglected the disturbance of the magnetic field and the inertia of the fluid. Oseen approximation takes some of these neglected effects into account. The electric field is, however, again taken to be zero for the same geometrical considerations as before. The velocity field $\mathbf{v}$ and the magnetic field $\mathbf{H}$ are set as

$$\mathbf{v} = \mathbf{e}_1 + \mathbf{v}', \quad \mathbf{H} = \mathbf{e}_1 + \mathbf{h},$$

in the system of equations (2.2). After neglecting the quadratic terms that system reduces to [2, 3, 4]

$$(2.6) \quad \mathbf{v} = \frac{1}{R_{-1} - R_1} [(R_{-1} - R)\mathbf{v}_1 - (R_1 - R)\mathbf{v}_{-1}],$$

$$(2.7) \quad \mathbf{h} = \frac{R_m}{R_{-1} - R_1} (\mathbf{v}_1 - \mathbf{v}_{-1}),$$

$$(2.8) \quad \text{div } v_s = 0, \quad s = \pm 1,$$

$$(2.9) \quad R_s \frac{\partial \mathbf{v}_s}{\partial x} = -\text{grad } P + \nabla^2 \mathbf{v}_s, \quad s = \pm 1,$$

and R_1 and R_{-1} are the roots of

$$R_s^2 - (R + R_m)R_s + RR_m - M^2 = 0,$$

while R , R_m and M are the numbers defined by the equation (2.3). The quantity P is

$$P = p + \frac{M^2}{R_m} \mathbf{h} \cdot \mathbf{e}_1.$$

Waechter [5, 6] has examined this approximation more closely for the discussion of the steady rotation problems. He points out that the appropriate approximation is achieved by requiring that $R_m \ll 1$, and $R/M^2 \ll 1$, which implies that the magnetic field dominates over the inertia forces. Furthermore we need not take $\mathbf{E} = 0$; the equation for the electric field decouples from the rest of the equations. If we take spherical polar coordinates (r, θ, φ) and examine the steady rotation of axially symmetric solids we can set

$$\mathbf{v} = (0, 0, v_\varphi(r, \theta)); \quad \mathbf{H} = (\cos \theta, -\sin \theta, h_\varphi(r, \theta)),$$

where we have neglected the φ -dependence in view of the axial symmetry. The equations for v_φ and h_φ then take the form [6]

$$(2.10) \quad \nabla_1^2 v_\varphi + \frac{M^2}{R_m} \frac{\partial h_\varphi}{\partial x} = 0,$$

$$(2.11) \quad \nabla_1^2 h_\varphi + R_m \frac{\partial v_\varphi}{\partial x} = 0,$$

where

$$(2.12) \quad \nabla_1^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta} - \frac{1}{r^2 \sin^2 \theta}.$$

We can write the equations (2.10) and (2.11) in the form of classical Oseen equations by introducing the functions u_s , $s = \pm 1$;

$$(2.13) \quad u_s = v_\varphi + s \frac{M}{R_m} h_\varphi.$$

The functions u_1 and u_{-1} satisfy the uncoupled equations

$$(2.14) \quad \nabla_1^2 u_s + sM \frac{\partial u_s}{\partial x} = 0, \quad s = \pm 1.$$

The boundary conditions are the same for all these systems of linearized equations. Firstly, we have the no slip condition. Secondly, in view of the fact that the solid is insulated, the induced magnetic field is constant inside as well as on the solid. At infinity we expect the perturbations to vanish. For a detailed discussion of the boundary conditions, reader is referred to Waechter's work [5, 6]. An interesting feature of the linearized systems is that they are valid for small as well as large Hartmann numbers.

3. Analysis for small Hartmann numbers.

3.1 Steady rotation of axially symmetric solids: Stokes approximation. Let the x -axis be the axis of symmetry and let Ω be the uniform angular velocity of the solid S . In the system of equations (2.5b) we set (recall that (Ωa) is used as the characteristic velocity for rotation problems):

$$(3.1) \quad v_x = 0, \quad v_y = z, \quad v_z = -y,$$

where v_x , v_y and v_z are the components of $\mathbf{v}$ in the x , y and z directions. We take cylindrical polar coordinates (ρ, φ, x) and observe that (2.5b) and (3.1) are satisfied by writing

$$(3.2) \quad v_x = 0, \quad v_y = \frac{z}{\rho} V, \quad v_z = -\frac{y}{\rho} V, \quad p = a \text{ const.},$$

while the second equation (2.5b) becomes

$$(3.3) \quad \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial V}{\partial \rho} - \frac{V}{\rho^2} + \frac{\partial^2 V}{\partial x^2} - M^2 V = 0.$$

As we shall see in the sequel this equation can be solved approximately for many configurations. For a sphere an exact solution can be presented. In fact by reverting to spherical polar coordinates, the equation (3.3) takes the form

$$(3.4) \quad (\nabla_1^2 - M^2)V = 0,$$

where the operator ∇_1^2 is defined by the relation (2.12). For a spherical boundary S , the solution is [7]

$$(3.5) \quad V = \frac{M\Omega a^3 e^{Ma}}{1 + Ma} \frac{e^{-Mr}}{r} \left(1 + \frac{1}{Mr}\right) \sin \theta,$$

where a is the radius of the sphere.

The Maxwell stress is zero at S and torque depends on the viscous stresses only. The contribution comes only from the tangential stress component τ

$$(3.6) \quad \tau = \mu \frac{\partial}{\partial n} \left(\frac{V}{\rho} \right),$$

in the direction of φ -increasing; n denotes the normal to the surface. For a sphere this value is $\mu(\partial/\partial r)(V/r)$, therefore the value of the torque G experienced by the sphere is

$$G = 2\pi\mu \int_0^\pi \left[r \frac{\partial}{\partial r} \left(\frac{V}{r} \right) \right]_{r=1} \sin^2 \theta \, d\theta.$$

Evaluating this integral and reverting to physical units we obtain

$$(3.7) \quad G = -\pi\mu a^3 \Omega \frac{3 + 3Ma + M^2 a^2}{1 + Ma}.$$

As $M \rightarrow 0$, $G \rightarrow 8\pi\mu a^3 \Omega$, the well known result for the non-conducting fluids.

3.2 Steady rotations: Oseen approximation. We seek the solution of the system of equations (2.14):

$$(3.8) \quad \left(\nabla_1^2 + sM \frac{\partial}{\partial x} \right) u_s = 0, \quad s = \pm 1.$$

For a sphere an exact solution can be presented in this case also as follows. Let us write [6]

$$(3.9) \quad \begin{aligned} m &= \frac{1}{2}M, \\ u_s &= \exp(-srm \cos \theta) \alpha(r, \theta; m), \end{aligned}$$

and substitute it in (3.8). There results the equation

$$(3.10) \quad (\nabla_1^2 - m^2) \alpha = 0,$$

whose solution can be obtained by separation of variables. Indeed, in terms of spherical polar coordinates the values of u_s are

$$(3.11) \quad \begin{aligned} u_1 &= \exp(-rm \cos \theta) \sum_{n=0}^{\infty} A_n k_n(rm) P_n^1(\cos \theta), \\ u_{-1} &= \exp(rm \cos \theta) \sum_{n=0}^{\infty} B_n k_n(rm) P_n^1(\cos \theta), \end{aligned}$$

where P_n^m are the associated Legendre functions and k_n are the modified spherical Bessel functions of the third kind:

$$(3.12) \quad k_n(rm) = (\pi/2rm)^{1/2} K_{n+(1/2)}(rm).$$

To determine the constants A_n and B_n we appeal first to the no-slip condition:

$$(3.13) \quad v_\varphi(1, \theta; m) = \frac{1}{2}(u_1(1, \theta; m) + u_{-1}(1, \theta; m)) = \sin \theta,$$

on the sphere $r = 1$. Secondly, since the sphere is insulated, the induced magnetic field h_φ satisfies the condition

$$(3.14) \quad h_\varphi(1, \theta; m) = 0.$$

From (2.13), (3.13) and (3.14), it follows that $u_1 = u_{-1} = \sin \theta$ on S . The constants A_n and B_n are determined by substituting these boundary conditions on u_s on one hand and differentiating the series ($s = \pm 1$):

$$(3.15) \quad \exp(sm \cos \theta) = \sum_{n=0}^{\infty} (s)^n (2n+1) i_n(m) P_n(\cos \theta),$$

with respect to θ , on the other, and comparing. The functions i_n are the modified spherical Bessel functions of the first kind:

$$(3.16) \quad i_n(m) = (\pi/2m)^{1/2} I_{n+(1/2)}(m).$$

Thus

$$(3.17) \quad \begin{aligned} u_1 &= -\frac{1}{m} \exp(-rm \cos \theta) \sum_{n=0}^{\infty} \frac{(2n+1) i_n(m) k_n(rm) P_n^1(\cos \theta)}{k_n(m)}, \\ u_{-1} &= \frac{1}{m} \exp(rm \cos \theta) \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1) i_n(m) k_n(rm) P_n^1(\cos \theta)}{k_n(m)}. \end{aligned}$$

We shall need the following symmetry relations which flow out of the equations (3.17):

$$(3.18) \quad \begin{aligned} u_1(r, \theta; m) &= u_{-1}(r, \pi - \theta; m), \\ v_\varphi(r, \theta; m) &= v_\varphi(r, \pi - \theta; m) = \frac{1}{2}(u_1 + u_{-1}), \\ h_\varphi(r, \theta; m) &= -h_\varphi(r, \pi - \theta; m) = \frac{R_m}{2M}(u_1 - u_{-1}). \end{aligned}$$

With the above solution, the boundary condition $v_\varphi \rightarrow 0$ at ∞ is also satisfied. The value of v_φ is thus correctly and exactly determined. For $m \ll 1$, we can expand these functions and find [8]

$$(3.19) \quad \begin{aligned} v_\varphi(r, \theta; m) &= \sin \theta \left[\frac{1}{r^2} + m^2 \left\{ \frac{1}{10r^4} (5 \cos^2 \theta - 1) \right. \right. \\ &\quad \left. \left. + \frac{1}{5r^2} (3 - 5 \cos^2 \theta) - \frac{1}{2} \sin^2 \theta \right\} \right] + O(m^3), \quad r > 1, \quad 0 \leq \theta \leq \pi. \end{aligned}$$

To evaluate the torque G on the sphere we use the formula (3.6) for the tangential stress component τ and obtain [6]

$$(3.20) \quad \begin{aligned} G &= 2\pi\mu \int_0^\pi \left[\frac{\partial v_\varphi}{\partial r} - v_\varphi \right]_{r=1} \sin^2 \theta \, d\theta \\ &= 2\mu \int_0^\pi \left[\left(\frac{\partial u_{-1}}{\partial r} \right)_{r=1} - \sin \theta \right] \sin^2 \theta \, d\theta \\ &= \frac{2\pi}{m} \left(\frac{\pi}{2m} \right)^{1/2} \int_0^\pi \exp(-m \cos \theta) \sin^2 \theta \sum_{n=0}^{\infty} \frac{(2n+1) P_n^1(\cos \theta)}{K_{n+(1/2)}(m)} d\theta, \end{aligned}$$

wherein we have used (3.17) for u_{-1} . For $m \ll 1$, the above formula gives [8, 9, 10]

$$(3.21) \quad G = -8\pi\mu a^3\Omega\{1 + \frac{4}{15}m^2 - \frac{1}{3}m^3\} + O(m^4).$$

For other axially symmetric bodies such as a spheroid, the method of separation of variables becomes rather involved and an approximate technique has to be devised. Fortunately there are various perturbation techniques, regular as well as singular, which are applicable in solving various boundary value problems in magnetohydrodynamics. The present problem of the steady rotation can be solved by the following perturbation method. We shall illustrate it with the case of a thin circular disk which is rotating about its axis [8]. The method is, however, general enough to solve the problem for various interesting configurations. Taking cylindrical polar coordinates, the equations (2.14) become

$$(3.22) \quad \frac{\partial^2 u_s}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u_s}{\partial \rho} - \frac{u_s}{\rho^2} + \frac{\partial^2 u_s}{\partial x^2} + sM \frac{\partial u_s}{\partial x} = 0.$$

The boundary conditions are $u_1 - u_{-1} = 0$, and $(u_1 + u_{-1})/2 = \rho$, or

$$(3.23) \quad \begin{aligned} u_1 = u_{-1} = \rho, \quad \text{on the disk } S: x = 0, \quad \rho \leq 1, \\ u_1, u_{-1} \rightarrow 0, \quad \text{at } \infty. \end{aligned}$$

The equations (3.22) can be transformed to the Helmholtz equation by setting

$$(3.24) \quad w_s = e^{smx} u_s \cos \varphi,$$

where $m = M/2$ as before, and

$$(3.25) \quad (\nabla^2 - m^2)w_s = 0.$$

Let $G(\rho, \varphi, x; \rho', \varphi', x')$ or $G(\mathbf{r}, \mathbf{r}')$ be the Green's function for this problem, *i.e.*,

$$(3.26) \quad (\nabla^2 - m^2)G(\mathbf{r}, \mathbf{r}') = -4\pi \delta(\mathbf{r} - \mathbf{r}'),$$

with

$$(3.27) \quad G_1(\mathbf{r}, \mathbf{r}') = G - \frac{\exp(-m|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|},$$

finite in the limit as $\mathbf{r}' \rightarrow \mathbf{r}$. We can now use the representation formula for the solution $w_s(\mathbf{r})$ of the Helmholtz equation and obtain

$$(3.28) \quad w_s(\mathbf{r}) = \int_S \sigma_s(\rho', x') \cos \varphi' G(\mathbf{r}, \mathbf{r}') dS',$$

where $\mathbf{r}' \in S$, and $\mathbf{r} \in T$; T being the domain outside S . The density σ_s is defined as

$$(3.29) \quad \sigma_s(\rho', x') = -\frac{1}{4\pi} \rho' e^{smx'} \left\{ \frac{\partial}{\partial n} \left(\frac{u_s}{\rho} \right) \right\},$$

evaluated on S . The no-slip boundary condition (3.23) on S leads us to the Fredholm integral equation of the first kind (recall that on $x = 0$, $\rho \leq 1$, the value of w_s is $\rho \cos \varphi$):

$$(3.30) \quad \rho \cos \varphi = \int_S \sigma_s(\rho', x') \cos \varphi' G(\mathbf{r}, \mathbf{r}') dS',$$

with $\mathbf{r}$ and $\mathbf{r}'$ both on S . Let $G_1^{(1)}(\rho, x; \rho', x')$ be the coefficient of $\cos(\varphi - \varphi')$ in the Fourier expansion of $G_1(\mathbf{r}, \mathbf{r}')$ defined above, then an integration over φ' reduces (3.30) to

$$(3.31) \quad \rho \cos \varphi = \int_S \sigma_s(\rho', x') \cos \varphi' \frac{\exp(-m|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} dS' \\ + \pi \cos \varphi \int_C \sigma_s(\rho', x') G_1^{(1)}(\rho, x; \rho', x') \rho' d\ell',$$

where C is the bounding curve of S in the meridian plane and $d\ell'$ denotes the element of arc length of C . Now suppose that σ_s has an expansion of the form

$$\sigma_s = \sum_n m^n \sigma_{sn}.$$

We substitute it in (3.31), equate the coefficients of equal powers of m and omit terms which trivially vanish:

$$(3.32) \quad \rho \cos \varphi = \int_S \sigma_{s0}(\rho', x') |\mathbf{r} - \mathbf{r}'|^{-1} dS',$$

$$(3.33) \quad 0 = \int_S \sigma_{s1}(\rho', x') |\mathbf{r} - \mathbf{r}'|^{-1} \cos \varphi' dS',$$

$$(3.34) \quad 0 = \int_S \sigma_{s2}(\rho', x') |\mathbf{r} - \mathbf{r}'|^{-1} \cos \varphi' dS' + \frac{1}{2} \int_S \sigma_{s0}(\rho', x') |\mathbf{r} - \mathbf{r}'| \cos \varphi' dS'$$

and so on. Thus the quantities σ_{s0} , σ_{s1} , σ_{s2} etc. are determined by solving the electrostatic problem in free space of finding the charge density on S necessary to raise S to a potential of the form $f(\rho, z) \cos \varphi$. The solutions for these integral equations are [8]

$$(3.35) \quad \sigma_{-10} = \sigma_{10} = \frac{2\rho}{\pi^2(1 - \rho^2)^{1/2}}, \quad \sigma_{-11} = \sigma_{11} = 0, \\ \sigma_{-12} = \sigma_{12} = \frac{\rho(2 - \rho^2)}{3\pi^2(1 - \rho^2)^{1/2}},$$

etc. The tangential stress τ is

$$(3.36) \quad \tau = \mu\rho \frac{\partial}{\partial n} \left(\frac{v_\varphi}{\rho} \right) = \frac{1}{2}(\tau_1 + \tau_{-1}), \\ \tau_s = \mu\rho \frac{\partial}{\partial n} \left(\frac{u_s}{\rho} \right), \quad s = \pm 1.$$

Setting $G = (G_1 + G_{-1})/2$, we get, in view of the relation (3.29), the value of G_s as

$$G_s = -8\pi^2\mu \int_C \rho^2 e^{-2mx} \sigma_s(\rho, x) d\ell,$$

and thus

$$(3.37) \quad G = -4\pi^2 \mu \int_c \rho^2 (e^{-mx} \sigma_1 + e^{mx} \sigma_{-1}) d\ell.$$

For the case of the surface of the disk $x = 0$, we use the relations (3.35) to obtain in physical dimensions the value of the torque G :

$$(3.38) \quad G = -\frac{32}{3} \mu a^3 \Omega \left\{ 1 + \frac{1}{5} m^2 - \frac{4}{9\pi} m^3 \right\} + O(m^4).$$

3.3 Motion of the conducting fluid past a solid: Stokes approximation. The system of equations (2.5) can be studied by setting [1]

$$(3.39) \quad \begin{aligned} \mathbf{v} &= \mathbf{e}_1 + e^{Mx} \nabla \varphi_1 + e^{-Mx} \nabla \varphi_{-1}, \\ p &= M e^{Mx} \frac{\partial \varphi_1}{\partial x} - M e^{-Mx} \frac{\partial \varphi_{-1}}{\partial x}. \end{aligned}$$

The functions φ_1 and φ_{-1} satisfy the equations

$$(3.40) \quad \nabla^2 \varphi_s + sM \frac{\partial \varphi_s}{\partial x} = 0, \quad s = \pm 1.$$

The representation formulas (3.39) and (3.40) can be shown to be complete. On writing $m = M/2$ and

$$(3.41) \quad \varphi_s = \exp(smz) \beta(x, y, z),$$

the equation (3.40) reduce to

$$(3.42) \quad (\nabla^2 - m^2) \beta(x, y, z) = 0.$$

For a sphere we can again use the method of separation of variables and obtain solutions similar to the formulas (3.11). The value of the operator ∇^2 in spherical polar coordinates is

$$(3.43) \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.$$

The values of φ_1 and φ_{-1} can be readily calculated to give [1]

$$(3.44) \quad \varphi_s = e^{-smr \cos \theta} \sum_{n=0}^{\infty} (s)^{n+1} A_n k_n(rm) P_n(\cos \theta).$$

One can now evaluate the velocity field and the drag exactly for the sphere. The calculations again become tedious for no-spherical bodies but the problem can be solved approximately by other techniques. We use the method of matched asymptotic expansions to solve this problem [11].

We can view the Hartmann number as the ratio of two lengths, the typical length of the solid and the length $\{(\rho\nu/\sigma)^{1/2}(1/\mu_e H)\}$. For weak magnetic fields the latter length is large compared with a . Therefore the system of equations (2.4) can be solved by the method of matched asymptotic expansions. For an

appropriate solution $\mathbf{v}$ and p of this system, we assume the expansions

$$(3.45) \quad \begin{aligned} \mathbf{v} &= \mathbf{v}^{(0)}(x, y, z) + M\mathbf{v}^{(1)}(x, y, z) + M^2\mathbf{v}^{(2)}(x, y, z) + \cdots; \\ p &= p^{(0)}(x, y, z) + Mp^{(1)}(x, y, z) + M^2p^{(2)}(x, y, z) + \cdots. \end{aligned}$$

These expansions are the inner-expansions and represent the exact solution $\mathbf{v}$ for small values of M at a fixed point in space. They are, however, not uniform approximations of the exact solution in the sense that the difference between these expansions and the exact solution for a given value of M becomes arbitrary large as $r \rightarrow \infty$. Under the circumstances these expansions are required to satisfy the boundary condition at the surface of the body only and hence are valid only in the vicinity of the body.

By insertion of the above expansions into equation (2.5b), the following equations result

$$(3.46) \quad \begin{aligned} O(1) : \operatorname{div} \mathbf{v}^{(0)} \\ -\nabla p^{(0)} + \nabla^2 \mathbf{v}^{(0)} = 0; \end{aligned}$$

$$(3.47) \quad \begin{aligned} O(M) : \operatorname{div} \mathbf{v}^{(1)} = 0, \\ -\nabla p^{(1)} + \nabla^2 \mathbf{v}^{(1)} = 0; \end{aligned}$$

$$(3.48) \quad \begin{aligned} O(M^2) : \operatorname{div} \mathbf{v}^{(2)} = 0, \\ -\nabla p^{(2)} + \nabla^2 \mathbf{v}^{(2)} = \mathbf{v}^{(0)} - (\mathbf{v}^{(0)} \cdot \mathbf{e}_1)\mathbf{e}_1; \end{aligned}$$

etc.

The equations (3.46) and (3.47) are identical to the classical Stokes equations. Thus in the neighborhood of the body, *i.e.*, for $r = O(1)$, in the limit $M \rightarrow 0$, the flow field is essentially governed by Stokes equations.

Although the solid is arbitrarily small compared with the second length defined above, boundary conditions must be imposed at distances that do not have this feature. It is this singular perturbation nature of the problem which necessitates a second set of expansions of $\mathbf{v}$ and p in the neighborhood of infinity—the so called outer expansions. The first step is to define the new independent variables

$$\tilde{x} = Mx, \quad \tilde{y} = My, \quad \tilde{z} = Mz.$$

The outer expansions are then written as

$$(3.49) \quad \begin{aligned} \mathbf{v} &= \mathbf{e}_1 + M\mathbf{V}^{(1)}(\tilde{x}, \tilde{y}, \tilde{z}) + M^2\mathbf{V}^{(2)}(\tilde{x}, \tilde{y}, \tilde{z}) + \cdots, \\ p &= M^2\tilde{p}^{(1)}(\tilde{x}, \tilde{y}, \tilde{z}) + M^3\tilde{p}^{(2)}(\tilde{x}, \tilde{y}, \tilde{z}) + \cdots. \end{aligned}$$

These expansions may be obtained from the exact solutions $\mathbf{v}$ and p by the limit process $M \rightarrow 0$, with fixed $\tilde{x}$, $\tilde{y}$, $\tilde{z}$, and are expected to be uniformly valid for all $r = O(M^{-1})$.

The second step is to transform the system (2.5b) into this new coordinate system $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ and insert the expansions (3.47) in them. There result the relations

$$(3.50) \quad -\nabla p^{(j)} + \nabla^2 \mathbf{V}^{(j)} - \{\mathbf{V}^{(j)} - \mathbf{V}^{(j)} \cdot \mathbf{e}_1\} = 0, \quad j = 1, 2, \cdots.$$

These equations are identical to the original system (2.4b); however, it transpires that the boundary conditions for these equations are simpler than before. Indeed, since the inner expansions are not valid at infinity, the condition $\mathbf{v} \rightarrow \mathbf{e}_1$, as $r \rightarrow \infty$, does not hold, in general, for the inner expansions. By the same token, the no-slip condition $\mathbf{v} = 0$, at the boundary does not, in general, hold for the outer expansions. These boundary conditions are replaced by the matching conditions which satisfy the requirement that the inner and the outer expansions should agree, term by term, in the overlapping region, *i.e.*, for some intermediate order of r ; $r = O(M^{-\alpha})$, $0 < \alpha < 1$, for which the inner and the outer expansions are both valid.

For the inner solutions, the no-slip condition is valid and therefore $\mathbf{v}^{(i)} = 0$, at the body for all j ; in particular $\mathbf{v}^{(0)} = 0$, on S . On the other hand, by matching conditions, $\mathbf{v}^{(0)}$ must agree with the leading term of the outer expansion for $r \gg 1$:

$$\mathbf{v}^{(0)} = \mathbf{e}_1, \quad \text{as } r \rightarrow \infty.$$

From equations (3.44) and the above-mentioned boundary conditions on $\mathbf{v}^{(0)}$, we find that the solutions for $\mathbf{v}^{(0)}$ and $p^{(0)}$ are simply the solutions of the non-magnetic Stokes equations. For a sphere such solutions are

$$(3.51) \quad \begin{aligned} \mathbf{v}^{(0)} &= \mathbf{e}_1 - \frac{3}{2} \left(\frac{\mathbf{e}_1}{r} - \nabla \frac{x}{2r} \right) + \frac{1}{4} \left(\nabla \frac{\partial}{\partial x} \frac{1}{r} \right), \\ p^{(0)} &= -\frac{3x}{2r^3}. \end{aligned}$$

The drag on the sphere for physical units is $6\pi\mu aU$.

For large values of r the asymptotic expansions of the Stokes solutions for axially symmetric bodies of any shape are

$$(3.52) \quad \begin{aligned} \mathbf{h}^{(0)} &= \mathbf{e}_1 - \frac{F_0}{4\pi} \left(\frac{\mathbf{e}_1}{r} - \nabla \frac{x}{2r} \right) + O\left(\frac{1}{r^2}\right), \\ p^{(0)} &= -\frac{F_0}{4\pi r^3} + O\left(\frac{1}{r^3}\right), \end{aligned}$$

where F_0 is the Stokes drag for the corresponding solid in classical fluids. The first order solutions $\mathbf{V}^{(1)}$ and $\hat{p}^{(1)}$ must, therefore, agree with the expansions (3.52). Thus the solutions for $\mathbf{V}^{(1)}$ and $\hat{p}^{(1)}$ are to be given by solving the system (3.50) for $j = 1$ with the boundary condition $\mathbf{V}^{(1)} \rightarrow 0$ as $r \rightarrow \infty$ and the matching condition mentioned above. These solutions are [11]

$$(3.53) \quad \begin{aligned} \mathbf{V}^{(1)} &= -\frac{F_0}{4\pi} \left[\frac{\mathbf{e}_1}{\tilde{r}} e^{-1/2\tilde{r}} \cosh \frac{\tilde{x}}{2} - \tilde{\nabla} \frac{1}{\tilde{r}} e^{-1/2\tilde{r}} \sinh \frac{\tilde{x}}{2} \right], \\ \hat{p}^{(1)} &= \frac{F_0}{4\pi} \left[\frac{\partial}{\partial \tilde{x}} \frac{1}{\tilde{r}} e^{-1/2\tilde{r}} \cosh \frac{\tilde{x}}{2} - \frac{1}{\tilde{r}} e^{-1/2\tilde{r}} \sinh \frac{\tilde{x}}{2} \right]. \end{aligned}$$

Writing (3.51) in the inner variables, we find that the inner component $\mathbf{v}^{(1)}$ has

to behave like $F_0 \mathbf{e}_1 / 16\pi$ as $r \rightarrow \infty$, while satisfying the no-slip condition on S . From the boundary conditions and the system (3.48) it follows that

$$(3.54) \quad \mathbf{v}^{(1)} = \frac{F_0}{16\pi} \mathbf{v}^{(0)}, \quad p^{(1)} = \frac{F_0}{16\pi} p^{(0)}.$$

Thus, to $O(M)$, the inner solution of the velocity field is

$$(3.55) \quad \mathbf{v} = \mathbf{v}^{(0)} + M \mathbf{v}^{(1)} = \left(1 + \frac{F_0}{16\pi} M\right) \mathbf{v}^{(0)} + O(M^2),$$

which shows that the effect of the magnetic field on the inner Stokes flow is an apparent increase of the free-stream velocity in the ratio $1 : (1 + (F_0/16\pi)M)$. There is a corresponding amount of increase in the drag on the body and we have the drag formula (in physical units):

$$(3.56) \quad F = F_0 \left(1 + \frac{F_0}{16\pi\mu a U} M\right) + O(M^2).$$

The value of F can now be calculated for all bodies for which the Stokes drag F_0 is known. For example, for a sphere and a disk (both of radius a) the values of F are $6\pi\mu a U(1 + M/8)$ and $16\mu a U(1 + M/\pi)$.

In Section 4.3 we construct an analogy between the theory of diffraction and magnetohydrodynamics Stokes flow. Then one can appeal to the theory of diffraction to sharpen the results above and vice versa.

3.4 Motion of the fluid past a solid: Oseen approximation. This motion is analyzed with the help of the system (2.6) to (2.9). It is possible to solve these equations also by the method of matched asymptotic expansions [12]. We shall, however, present the solution by an integral equation perturbation technique [13]. In this connection we need the Green's tensors $\{\mathbf{T}, \mathbf{p}\}$ which satisfy the system:

$$(3.57) \quad R \frac{\partial \mathbf{T}}{\partial x} = -\text{grad } \mathbf{p} + \nabla^2 \mathbf{T} - \mathbf{I} \delta(\mathbf{r} - \mathbf{r}'),$$

$$\text{div } \mathbf{T} = 0,$$

where $\mathbf{I}$ is the unit tensor and $\mathbf{r}$ and $\mathbf{r}'$ denote the position vectors of two arbitrary points in space. The condition on $\mathbf{T}$ is that it tends to zero at ∞ . The required functions are given by the formulas, as can be verified by direct substitution,

$$(3.58) \quad \mathbf{T} = \mathbf{I} \nabla^2 \varphi - \text{grad grad } \varphi,$$

$$\mathbf{p} = -\text{grad} \left(\nabla^2 \varphi - R \frac{\partial \varphi}{\partial x} \right),$$

with

$$(3.59) \quad \varphi = \frac{1}{|R|} \int_0^{1/2|R|t} \frac{1 - e^{-x}}{x} dx,$$

where $2t = |\mathbf{r} - \mathbf{r}'| + (R/|R|)(x - x')$, and x and x' are the x -coordinates of $\mathbf{r}$ and $\mathbf{r}'$ respectively. Using the expansion $(1 - e^{-x})/x = (1 - x/2! + x^2/3! + \dots)$

in the equation (3.59), integrating term by term and substituting the value of φ so obtained in (3.58) we have

$$(3.60) \quad \mathbf{T} = \mathbf{T}' - \frac{1}{8}R\mathbf{T}^0 - \frac{|R|}{8}(2\mathbf{e}_1\mathbf{e}_1 + 3\mathbf{e}_2\mathbf{e}_2 + 3\mathbf{e}_3\mathbf{e}_3) + \cdots,$$

where the tensors $\mathbf{T}'$ and $\mathbf{T}^0$ are defined by the first of the equations (3.58) with φ replaced by $|\mathbf{r} - \mathbf{r}'|/2$ and $(x - x')|\mathbf{r} - \mathbf{r}'|/2$ respectively. Having found the Green's tensor, one follows the standard techniques of potential theory and Green's integral identities to obtain the integral representation formula for the velocity vector $\mathbf{v}$:

$$(3.61) \quad \mathbf{v} = -\frac{1}{4\pi} \int_S \left[\mathbf{T} \cdot \left(\frac{\partial \mathbf{v}}{\partial n} - p\mathbf{n} \right) - \left(\frac{\partial T}{\partial n} - p\mathbf{n} \right) + R(\mathbf{T} \cdot \mathbf{v})\mathbf{n} \right] dS,$$

where $\mathbf{n}$ is the unit vector along the outward normal to S and n_1 is its x -component.

We can apply the formula (3.61) to (2.9) and get

$$(3.62) \quad v_s = -\frac{1}{4\pi} \int_S \left[\mathbf{T}_s \cdot \left(\frac{\partial \mathbf{v}_s}{\partial n} - P\mathbf{n} \right) - \left(\frac{\partial \mathbf{T}_s}{\partial n} - \mathbf{P}_s\mathbf{n} \right) \cdot \mathbf{v} + R_s(\mathbf{T}_s \cdot \mathbf{v}_s)n_1 \right] dS, \quad s = \pm 1.$$

From the equation (2.6), (2.7), and (3.62), an integral representation can be obtained for $\mathbf{v}$ in terms of the values of $\mathbf{v}$, $\mathbf{h}$, $\partial \mathbf{v}/\partial n$, $\partial \mathbf{h}/\partial n$ and P on S . Since the velocity vector $\mathbf{v}$ is required to be constant on S , it can be shown by using Green's theorem and the above representation formula that the surface integrals involving the values of $\mathbf{v}$ on S vanish. The induced field $\mathbf{h}$ itself is of the first order of magnitude in the small parameters. On neglecting terms which are obviously of second order, we have

$$(3.63) \quad \mathbf{v} = -\frac{M^2}{4\pi R_m(R_{-1} - R_1)} \int_S \left\{ (\mathbf{T}_1 - \mathbf{T}_{-1}) \cdot \frac{\partial \mathbf{h}}{\partial n} + \left[\frac{\partial}{\partial n} (\mathbf{T}_1 - \mathbf{T}_{-1}) + (\mathbf{P}_{-1} - \mathbf{P}_1)\mathbf{n} \right] \cdot \mathbf{h} \right\} dS - \frac{1}{4\pi(R_{-1} - R_1)} \int [(R_{-1} - R)\mathbf{T}_1 - (R_1 - R)\mathbf{T}_{-1}] \cdot \mathbf{f} dS,$$

where $\mathbf{f} = \partial \mathbf{v}/\partial n - p\mathbf{n}$. The quantities $\mathbf{T}_{-1} - \mathbf{T}_1$, $\mathbf{P}_{-1} - \mathbf{P}_1$ are themselves of the first order and hence the terms involving $\mathbf{h}$ and $\partial \mathbf{h}/\partial n$ in the above equations may be neglected. Thus only the second integral is to be taken into account. Now using the boundary condition $\mathbf{q} = \mathbf{e}_1$ on S , we finally have the Fredholm integral equation of the first kind

$$(3.64) \quad \mathbf{e}_1 = -\frac{1}{4\pi(R_{-1} - R_1)} \int_S [(R_{-1} - R)\mathbf{T}_1(\mathbf{r}, \mathbf{r}') - (R_1 - R)\mathbf{T}_2(\mathbf{r}, \mathbf{r}')] \cdot \mathbf{f}(\mathbf{r}') dS,$$

with both $\mathbf{r}$ and $\mathbf{r}'$ on S .

The values of $\mathbf{T}_s$ are given by the relation (3.60) which when substituted in (3.64) give, correct to the first order of small quantities,

$$(3.65) \quad \mathbf{e}_1 - \frac{1}{32\pi(R_{-1} - R_1)} \{[(R_{-1} - R) |R_1| - (R_1 - R) |R_{-1}|] \\ \cdot [2\mathbf{e}_1\mathbf{e}_1 + 3\mathbf{e}_2\mathbf{e}_2 + 3\mathbf{e}_3\mathbf{e}_3]\} = -\frac{1}{4\pi} \int_S \mathbf{T}' \cdot \mathbf{f} dS - \frac{R}{32\pi} \int_S \mathbf{T}^0 \cdot \mathbf{f} dS.$$

The contribution arising from the Maxwell stresses to the drag S is of the second order. Hence we can use the force formula

$$(3.66) \quad \mathbf{F} = \int_S \left(\frac{\partial \mathbf{v}}{\partial n} - p\mathbf{n} \right) dS = \int_S \mathbf{f} dS,$$

valid for a non-conducting fluid to evaluate the drag. Therefore, the surface integrals on the right side of the equation (3.64) may, to the present order of approximation, be replaced by $\mathbf{F}$. The first term on the right side of the equation (3.65) can be evaluated by appealing to the classical Stokes flow. Indeed, on setting $R = 0$, in the equations (3.57) we have the Stokes equations for the Green's tensor $\mathbf{T}'$. By the same token setting $R_s = R = 0$ in the relation (3.65) above, we have the integral equation for the Stokes flow [14]. The solution of this latter problem is known for various shapes of S . We denote this known value of $\mathbf{f}$ by $\mathbf{f}_v^\infty$ and the resulting force on S by $\Phi \cdot \mathbf{v}$, where Φ is called the Stokes resistance tensor.

The basic idea of this perturbation technique is to assume the expression on the left side of the equation (3.65) to be the boundary condition on S for the problem in Stokes flow and set $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}^*$, where

$$(3.67) \quad \mathbf{F}_1 = \Phi \cdot \left\{ \mathbf{e}_1 - \frac{1}{32\pi(R_{-1} - R_1)} [(R_{-1} - R) |R_1| \right. \\ \left. - (R_1 - R) |R_{-1}| (2\mathbf{e}_1\mathbf{e}_1 + 3\mathbf{e}_2\mathbf{e}_2 + 3\mathbf{e}_3\mathbf{e}_3)] \cdot (\Phi \cdot \mathbf{e}_1) \right\},$$

and

$$\mathbf{F}^* = \int_S \mathbf{f}^* dS,$$

such that

$$(3.68) \quad -\frac{R}{8} \int_S \mathbf{T}^0 \cdot \mathbf{f}_v^\infty dS = \int_S \mathbf{T}' \cdot \mathbf{f}^* dS.$$

From the equations (3.61) and (3.68) it follows that reversing the direction of motion of S does not change the sign of $\mathbf{f}^*$ and thus $\mathbf{F}^*$ does not change sign when the motion of S is reversed. Hence, if S is restricted to be such that the reversal of its direction of motion in the fluid changes the sign of $\mathbf{F}$, the total drag is given by the formula (3.67) only [12]. Limits of this formula for special cases such as the formula (3.56) can be readily obtained [12, 15].

It is well known for the case of non-conducting viscous fluids that Oseen's approximation predicts a wake downstream. The equations (2.9) are two Oseen type equations and solutions of these equations predict wakes in the vicinity of the solid. One of these wakes is always downstream. However, when the magnetic pressure μH_0^2 is less than the dynamic pressure ρU^2 , the second wake is also downstream, otherwise the second wake is upstream [3, 15, 16].

4. Analysis for large Hartmann numbers.

4.1 *The steady rotations.* The formulas (3.5) and (3.7) for the velocity field and torque on a sphere as derived by using the system of equations (2.4) are exact solutions of the system and are valid for all values of M . However, the corresponding formulas (3.18) and (3.20) based on Oseen system of equations (2.13) are valid for $M \ll 1$. The difficulty lies in the series (3.17) which converge too slowly to be useful for large M . For the case $R_m \ll 1$, $R/M^2 \ll 1$ (that is when the system (2.13) is valid) one can establish an analogue between the magnetohydrodynamics problem and the theory of diffraction [5, 6]. Let us consider the scattering by a soft sphere of a plane incident pulse $\phi_{inc} = \delta(x + t)$. We take spherical polar coordinates and denote the total velocity potential by $\phi(r, \theta, t) = \phi_{inc} + \phi_{sc}$. Then ϕ_{sc} satisfies the boundary value problem [17]

$$(4.1) \quad -\frac{\partial^2 \phi_s}{\partial t^2} + \nabla^2 \phi_s = 0, \quad r \geq 1,$$

$$\phi_s(1, \theta, t) = -\phi_{inc} \quad \text{on } S,$$

while at ∞ , it satisfies the appropriate radiation condition. The operator ∇^2 is defined by the relation (2.43). By taking Laplace transforms of these quantities we get

$$(4.2) \quad \bar{\phi}_{sc} = \int_{-\infty}^{\infty} e^{-mt} \phi_{sc}(r, \theta, t) dt,$$

$$(4.3) \quad \begin{aligned} \bar{\phi}_{inc} &= \overline{\delta(x + t)} = \exp(rm \cos \theta) \\ &= \left(\frac{\pi}{2rm}\right)^{1/2} \sum_{n=0}^{\infty} (2n+1) I_{n+(1/2)}(rm) P_n(\cos \theta). \end{aligned}$$

We now define a new function $\alpha^*(r, \theta; m)$ such that

$$(4.4) \quad \alpha^*(r, \theta; m) = -\bar{\phi}_{sc}(r, \theta; m) = \bar{\phi}_{inc} - \bar{\phi}(r, \theta; m).$$

From (4.1), it follows that $\alpha^*(r, \theta; m)$ satisfies the simple boundary value problem

$$(4.5) \quad \begin{aligned} (\nabla^2 - m^2) \alpha^*(r, \theta; m) &= 0, \\ \alpha^*(r, \theta; m) &= \exp(rm \cos \theta) \quad \text{on } S, \\ \alpha^*(r, \theta; m) &\rightarrow \infty \quad \text{as } r \rightarrow \infty. \end{aligned}$$

By exploiting the equivalence relation (4.4) we can solve our magnetohydro-

dynamic problem. Indeed the solution of the system (4.5) for a sphere is

$$(4.6) \quad \alpha^*(r, \theta; m) = \sum_{n=0}^{\infty} \frac{(2n+1)i_n(m)k_n(rm)P_n(\cos \theta)}{k_n(m)}.$$

By comparing (4.5) and (3.17) we obtain the relation between the functions u_1 and α^* as

$$(4.7) \quad u_1(r, \theta; m) = -\frac{1}{m} \exp(-rm \cos \theta) \frac{\partial}{\partial \theta} [\alpha^*(r, \theta; m)].$$

On the right hand side of (4.7), m denotes the Laplace transform variable, while on the left hand side $m = M/2$. If we can find asymptotic expansions for the function $\alpha^*(r, \theta; m)$, we can then use this relation to obtain the solution to our problem.

The slowly converging series (4.6) is converted into a rapidly converging series, suitable for asymptotic expansions, with the help of the Watson transformation. The series (4.6) can be expressed as a contour integral in the complex ν -plane:

$$(4.8) \quad \alpha^*(r, \theta; m) = -\bar{\Phi}_{sc}(r, \theta; m) = -i \left(\frac{\pi}{2rm} \right)^{1/2} \cdot \int_C \frac{\nu I_\nu(m) K_\nu(rm) P_{\nu-(1/2)}(-\cos \theta) d\nu}{K_\nu(m) \cos \pi \nu},$$

where C encloses the poles of the integrand located at the positive half-integers but excludes the zeroes of $K_\nu(m)$ which are located on the imaginary axis. The proof follows by the application of the residue theorem. The next step is to derive an alternative expression for the function $P_{\nu-(1/2)}(-\cos \theta)/\cos \pi \nu$, in terms of certain more suitable functions. Thereby the geometrical optic field $\alpha_{g.o.}^*$ and the geometrical diffraction field $\alpha_{g.d.}^*$ can be derived [5, 6]. The value of $\alpha_{g.o.}^*$ is

$$(4.9) \quad \alpha_{g.o.}^* = \left[\frac{\cos \alpha \sin \alpha}{r \sin \theta (2r \cos \beta - \cos \alpha)} \right]^{1/2} \exp \{ -m(r \cos \beta - 2 \cos \alpha) \},$$

and α and β are given by the expressions

$$(4.10) \quad \theta + \beta = 2\alpha, \quad \sin \alpha = r \sin \beta.$$

We can now appeal to the equivalence relation (4.7) and obtain

$$(4.11) \quad u_{1.g.o.}(r, \theta; m) = \sin \alpha \left[\frac{\cos \alpha \sin \alpha}{r \sin \theta (2r \cos \beta - \cos \alpha)} \right]^{1/2} \cdot \exp \{ -m(r \cos \beta + r \cos \theta - 2 \cos \alpha) \}.$$

This equation shows that u_1 is exponentially small outside a boundary layer on the sphere $r = 1$. In fact, if we set $\delta = r - 1$, $\alpha = \theta - \epsilon$, where δ and ϵ are small quantities and use (4.10), the relation (4.11) gives

$$(4.12) \quad u_{1 \text{ g.o.}} \sim \frac{\sin \theta}{[r(2r - 1)]^{1/2}} e^{-\delta M \cos \theta},$$

where θ lies outside the region $((\pi/2) - \theta) = M^{-(1/3)}$. Hence $u_1(r, \theta; m)$ has a boundary layer of thickness δ

$$(4.13) \quad \delta = O[M \cos \theta]^{-1},$$

provided $((\pi/2) - \theta) > O(M^{-(1/3)})$. For the values of θ in the neighborhood of $\pi/2$ (i.e. for $((\pi/2) - \theta) = o(M^{-(1/3)})$), $u_{1 \text{ g.o.}}$ again decays exponentially outside the tangential layer of thickness δ

$$(4.14) \quad \delta = O(M^{-(2/3)}).$$

The value of the geometrical diffraction field $\alpha_{\text{g.d.}}^*$ is

$$(4.15) \quad \alpha_{\text{g.d.}}^* = e^{mr \cos \theta} - \frac{9e^{-\sigma_0 m}}{(2\pi)^{1/2} m^{1/6} (r^2 - 1)^{1/4} (r \sin \theta)^{1/2}} \frac{e^{-\alpha_1 m^{1/3} \delta_0}}{[f'(\alpha_1)]^2},$$

where

$$\delta_0 = \theta - \frac{\pi}{2} - \cos^{-1} \frac{1}{r}, \quad \sigma_0 = \delta_0 + (r^2 - 1)^{1/2},$$

and α_1 is the lowest zero of $f(\alpha_1)$ while

$$(4.16) \quad \begin{aligned} f(\alpha) &= \alpha^{1/2} \left[J_{1/3} \left(\frac{2^{3/2} \alpha^{3/2}}{3} \right) + J_{-(1/3)} \left(\frac{2^{3/2} \alpha^{3/2}}{3} \right) \right] \\ &= \frac{3}{2^{1/6}} A_i(-2^{1/3} \alpha), \end{aligned}$$

where A_i is the Airy function. From the formulas (4.15) and (4.7) we have

$$(4.17) \quad u_{1 \text{ g.d.}} = r \sin \theta - \frac{9e^{-m(\sigma_0 + r \cos \theta)}}{(2\pi)^{1/2} m^{1/6} (r^2 - 1)^{1/4} (r \sin \theta)^{1/2}} \frac{e^{-\alpha_1 m^{1/3} \delta_0}}{[f'(\alpha_1)]^2}.$$

An examination of the exponential decay factors in the formulas (4.11) and (4.17) reveals that there exists a transition layer with thickness of $O[|x|/M^{1/2}]$.

Observing the relations (3.17) and (3.18) and using the asymptotic formula ($m \gg 1$):

$$(4.18) \quad k_n(rm) \sim \frac{\pi}{2rm} e^{-rm},$$

it follows that

	$0 \leq \theta \leq \pi/2$	$\pi/2 < \theta \leq \pi$
	$rm \cos \theta > 0$	$rm \cos \theta < 0$
	u_1 is negligible	u_{-1} is negligible
	$u_{-1} \sim r \sin \theta$	$u_1 \sim r \sin \theta$
	$u_\varphi \sim \frac{1}{2} r \sin \theta$	$u_\varphi \sim \frac{1}{2} r \sin \theta$
(4.19)	$h_\varphi \sim -\frac{1}{2} \left(\frac{R_m}{M}\right) r \sin \theta$	$h_\varphi \sim \frac{1}{2} \left(\frac{R_m}{M}\right) r \sin \theta$

From (4.19), it follows that there exists a core—a region endowed with rotary motion—and since $u_\varphi \sim (r \sin \theta)/2$, it implies that the angular velocity of the fluid in the core is $\Omega/2$. Furthermore, using the formula (2.2)₃,

$$\begin{aligned}
 \mathbf{J} &= \frac{1}{R_m} \operatorname{curl} (0, 0, h_\varphi(r, \theta)) \\
 &= \mp \frac{1}{M} [\cos \theta, -\sin \theta, 0].
 \end{aligned}
 \tag{4.20}$$

Thus there is a constant current in the x -direction, and the magnitude of the current density in the core is (in physical units) $\mp \Omega(\sigma\mu)^{1/2}$. It follows that the Hartmann layer on the sphere drives a uniform current in the core. The current is directed parallel to the applied magnetic field and the axis of rotation of the sphere. Furthermore, it flows toward the sphere. On entering the Hartmann layer the current flows towards the tangential region ($\theta \sim \pi/2$). It then flows back along the transition layers, and the current paths are closed where the transition layers meet the axis of rotation and merge. On the plane $\theta = \pi$, $h_\varphi = 0$, so no current lines can enclose the sphere, and the physical domain exterior to the sphere is divided in two independent parts, each of which behaves like a current generator.

The far field behavior of the velocity and magnetic fields can also be obtained from the far field behavior of the scattered field. Using the large argument approximation ($r \gg 1$) for $\alpha_k^*(r, \theta; m)$ we obtain the Rayleigh approximation

$$\alpha_k^*(r, \theta; m) = \frac{\pi}{2rm} e^{-mr} \sum_{n=0}^{\infty} (2n+1) \frac{I_{n+(1/2)}(m)}{K_{n+(1/2)}(m)} P_n(\cos \theta).
 \tag{4.21}$$

Applying a Watson transformation, we can again convert this series into a contour integral and thereby obtain an asymptotic formula for α_k^* . Thereupon the equivalence formula (4.7) gives

$$u_{1R}(r, \theta; m) \sim \frac{m^2 \sin \theta}{8r} e^{-rm(1+\cos \theta)},
 \tag{4.22}$$

provided $-\pi - \theta < O(m^{-1})$ and $r \gg O(m)$. From the exponential decay factor

of this formula we observe that the velocity and current distribution in the far field are confined to the paraboloidal regions

$$(4.23) \quad r(1 \pm \cos \theta) = \frac{c}{m}, \quad c = a \text{ constant},$$

and the fluid velocity is $O(r^{-1})$ in these regions. The physical features of the flow are shown [6] in the following figure:

FIG. 1 Flow picture (not to scale) at large m for rotating sphere. Current circuits shown dotted

The torque G on the sphere is given by the formula (3.20). To evaluate this formula for large Hartmann numbers we take the following steps. First we appeal to the orthogonality relation of the Legendre functions to obtain

$$(4.24) \quad G = \frac{4\pi^2}{m^3} \sum_{n=0}^{\infty} (-1)^n (n + \frac{1}{2})^3 \frac{I_{n+(1/2)}(m)}{K_{n+(1/2)}(m)} - \frac{\pi^2}{m^3} \sum_{n=0}^{\infty} (-1)^n \frac{(n + \frac{1}{2}) I_{n+(1/2)}(m)}{K_{n+(1/2)}(m)}.$$

This form is suitable for the application of a Watson transformation. The resulting contour integral can be evaluated to give (in physical dimensions)

$$(4.25) \quad G \sim -\pi m \mu \Omega a^3.$$

The above analysis for the sphere can be extended to the case of other axially symmetric bodies by exploiting the equivalence relation with the corresponding diffraction problem. For instance the formula for the torque on a thin circular disk rotating about its axis of symmetry can be calculated [18] by using Jones technique [19, 20]:

$$(4.26) \quad G = -2\pi m \mu \Omega a^3 \left\{ \left[\frac{1}{2} + \frac{1}{m} + \frac{3}{4m^2} - \frac{3}{8m^3} + O\left(\frac{1}{m^4}\right) \right] - i \frac{e^{-2m}}{\pi^{1/2}} \left[\frac{1}{4m^{5/2}} + \frac{253}{256m^{7/2}} + O\left(\frac{1}{m^{9/2}}\right) \right] \right\}.$$

4.2 The steady flow past solids. It is possible to give the analysis for the system of equations (2.6–2.9) because they lead to Oseen type equations resulting in the series solution (3.44). A Watson transformation is applicable to these series directly in view of the presence of the polynomial $P_n(\cos \theta)$ in them. However, it is sufficient to solve the Stokes system (2.5) to give an approximate

view of the flow and field pattern for the following reason. The factor $R(\mathbf{v} \cdot \nabla)\mathbf{v}$ in the equation (2.2), modifies the Stokes solution by the factor $\{1 + O(R)\}$, while the magnetic term modifies that solution by the factor $\{1 + O(M)\}$. In Section 3, although $M \gg R$ still $M \ll 1$. In the present situation $M \gg 1$. In this section we therefore present the solution of the system (2.5) only.

The Stokes equations also lead to Oseen type operators (3.40). The method of Watson transformation of last section can be used for their analysis as well. In the next section we shall present the analogy between this system and the theory of diffraction. For the present we display an integral equation technique to get the leading terms in the required solutions [21–23]. For that purpose we appeal to the representation formulas (3.39). The function p and each component of $(\mathbf{v} - \mathbf{e}_1)$ can be written as the sum of two terms ψ_1 and ψ_{-1} which satisfy the Oseen type differential operators:

$$(4.27) \quad \nabla^2 \psi_s - sM \frac{\partial \psi_s}{\partial x} = 0, \quad s = \pm 1.$$

Let the equation of the solid be $x = f(y, z)$, and let α represent the distance from the body along the outward normal which has direction cosines (ℓ, ℓ', ℓ'') relative to an orthogonal frame of reference. Thus we can write

$$(4.28) \quad \psi_s = A_s(y', z'), \quad \frac{\partial \psi_s}{\partial \alpha} = MB_s(y', z'),$$

at the point (x', y', z') of the body. In the relations (4.28) we have tacitly assumed that the normal derivatives can be $O(M)$ on the body, an assumption which will be justified in the sequel.

The next step is to obtain the integral representation formula for the equation (4.27). The fundamental solution of this equation is

$$\frac{1}{R} e^{(1/2) s M (x - x' - sR)}, \quad R = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2};$$

we then get the required integral formula by the standard techniques of potential theory:

$$(4.29) \quad \psi_s(x, y, z) = \frac{1}{4\pi} \iint_s \left\{ M(B_s - s l A_s) \frac{e^{(1/2) s M (x - x' - sR)}}{R} - A_s \frac{\partial}{\partial \alpha} \left(\frac{e^{(1/2) s M (x - x' - sR)}}{R} \right) \right\} ds.$$

We shall give the asymptotic formula for ψ_1 , the corresponding formula for ψ_{-1} can be obtained in a similar fashion. For finding the asymptotic behavior of ψ_1 let us first examine the term

$$(4.30) \quad \iint_s M(B_1 - l A_1) \frac{e^{(1/2) M (x - x' - R)}}{R} dS,$$

for $M \gg 1$. Since $R > |x - x'|$, the integral is exponentially small unless $(y - y')^2 + (z - z')^2$ is small. Thus the integral itself is small unless a line through (x, y, z) parallel to the x -axis cuts the body. If it does cut it then let one of the points of intersection be (ξ, y, z) . We evaluate the contribution of the above integral in the neighborhood of that point. This contribution is

$$(4.31) \quad M\{B_1(y, z) - lA_1(y, z)\} \iint_S \frac{e^{(1/2)M(x-x'-R)}}{R} dS,$$

where ℓ , the x component of the normal, is also a function y and z , and the integration can be taken over the infinite plane $\ell x' \pm my' \pm nz' = \ell\xi \pm my \pm nz$.

The integral (4.31) can be evaluated by using the transformation

$$(4.32) \quad \begin{aligned} x' - \xi &= -(1 - l^2)^{1/2} r l' \sin \theta + (1 - l^2)(x - \xi), \\ (y' - y)(1 - l^2)^{1/2} &= l'' r \cos \theta + l' l r \sin \theta - l l''(1 - l^2)^{1/2}(x - \xi), \\ (z' - z)(1 - l^2)^{1/2} &= -l' r \cos \theta + l'' l r \sin \theta - l l''(1 - l^2)^{1/2}(x - \xi). \end{aligned}$$

This gives the parametric representation of the point (ξ, y, z) on the infinite plane mentioned above. From this formula it follows that $R^2 = r^2 + \ell^2(x - \xi)^2$. With these substitutions in (4.31), there results the integral

$$(4.33) \quad M(B_1 - lA_1) \int_0^\infty \frac{r dr}{\{r^2 + \ell^2(x - \xi)^2\}^{1/2}} \cdot \int_0^{2\pi} \exp \frac{1}{2} M[l^2(x - \xi) + (1 - l^2)^{1/2} r \sin \theta - \{r^2 + \ell^2(x - \xi)^2\}] d\theta.$$

This integral is evaluated as follows. First step is to use the relation

$$(4.34) \quad \int_0^{2\pi} e^{t \sin \theta} d\theta = 2\pi I_0(t).$$

Second step is to use the Laplace transform formula for I_0 function. There results the value of the integral (4.31):

$$(4.35) \quad \begin{cases} \frac{4\pi(B_1 - lA_1)}{|l|} & \text{for } x > \xi, \\ \frac{4\pi(B_1 - lA_1)}{|l|} e^{M\ell^2(x-\xi)}, & \text{for } x < \xi. \end{cases}$$

Proceeding in the same manner we get the asymptotic contribution from

$$(4.36) \quad \begin{aligned} \iint_S A_1 \frac{\partial}{\partial \alpha} \left(\frac{e^{(1/2)M(x-x'-R)}}{R} \right) dS \\ = -A_1 \left\{ l \frac{\partial}{\partial x} + l' \frac{\partial}{\partial y} + l'' \frac{\partial}{\partial z} \right\} \iint_S \frac{e^{(1/2)M(x-x'-R)}}{R} dS \\ = \begin{cases} 0, & \text{for } x > \xi, \\ -4\pi A_1 l^2 \operatorname{sgn} \ell e^{M\ell^2(x-\xi)}, & \text{for } x < \xi. \end{cases} \end{aligned}$$

Similar contributions come from other points of intersection. The final result is that ψ_1 tends exponentially to zero outside a cylinder circumscribing the body with its axis along the x -axis. Inside the cylinder the asymptotic form of ψ_1 is independent of x on the downstream side ($x > 0$) so that $\psi_1 = \psi_1(y, z)$, while on the upstream side ($x < 0$) $\psi_1 = \beta(y, z) \exp \{M\ell^2(x - \xi)\}$, where ξ is the value of x at the point on the body nearest the point (x, y, z) for which $y = y', z = z'$.

Similarly $\psi_{-1} = \psi_{-1}(y, z)$ on the upstream side of the circumscribing cylinder and $\psi_{-1} = \beta'(y, z) \exp \{-M\ell^2(x - \xi)\}$ on the downstream side. For the perturbed field all the components are asymptotically zero outside the circumscribing cylinder. Thus inside this cylinder the following picture emerges:

Upstream ($x < 0$)	Downstream ($x > 0$)
$\psi_1 = \beta(y, z)e^{M\ell^2(x-\xi)}$ $\psi_{-1} = \psi_{-1}(y, z)$	$\psi_1 = \psi_1(y, z)$ $\psi_{-1} = \beta'(y, z)e^{-M\ell^2(x-\xi)}$

From (4.37) we obtain

$$(4.38) \quad \begin{aligned} M^{-1}p &= v_0 - v_1 e^{-M\ell^2(x-\xi)}, \\ \mathbf{v} &= \mathbf{e}_1 + \mathbf{v}_0 + \mathbf{v}_1 e^{-M\ell^2(x-\xi)}, \end{aligned}$$

where v_0 and v_1 are the x -components $\mathbf{v}_0$ and $\mathbf{v}_1$ respectively. Now appealing to the equation of continuity, regularity, the boundary conditions and the requirements of symmetry we find [22] that $\mathbf{v}_1 \equiv 0$, and $v_0 = -1$.

The flow pattern on the upstream side can be discussed by similar arguments. Boundary conditions being unchanged, the velocity fields on the upstream and downstream sides of the body are identical. In view of the interchange in the roles of ψ_1 and ψ_{-1} , the pressures on the two sides of the solid, however, are equal in magnitude but opposite in sign. Thus outside the circumscribing cylinder the uniform flow field parallel to the axis of the body continues unchecked while inside the cylinder the fluid is stagnant. The pressure p is equal to M on the upstream side and $-M$ on the downstream side. Thus the leading term for the drag emerges from the pressure drag and in physical units it is

$$(4.39) \quad \frac{F}{\mu a U} \sim 2M \left(\frac{A}{a^2} \right),$$

where A is the cross-sectional area of the cylinder.

For the case of a circular disk further details can be furnished. It turns out that there is a region of transition when one crosses from out of the circumscribing cylinder to the inside part. Its thickness is $O(|x|^{1/2}/M^{1/2})$. The leading terms in the expression for the drag on the disk are [15, 23]

$$(4.40) \quad \frac{F}{\mu a U} \sim 2M\pi \left(1 + \frac{2}{M} + \frac{1}{M^2} \right).$$

The flow for large values of the Hartmann numbers turns out to be insensitive to the detailed shape of the body. Thus the above-mentioned results hold for a large class of solids.

4.3 *Analogy between the theory of diffraction and magnetohydrodynamics Stokes flow.* It is a matter of simple verification that the following representation formulas satisfy the system of equations (2.5):

$$(4.41) \quad \mathbf{v} = \frac{1}{m} [(m\Phi \cosh mx)\mathbf{e}_1 - \sinh mx \operatorname{grad} \Phi],$$

$$(4.42) \quad p = -2 \left[\cosh mx \frac{\partial \Phi}{\partial x} - m\Phi \sinh mx \right],$$

where $m = M/2$ and Φ satisfies the Helmholtz equation

$$(4.43) \quad (\nabla^2 - m^2)\Phi = 0.$$

The boundary condition $\mathbf{v} = \mathbf{e}_1$ on S is satisfied by setting

$$(4.44) \quad \Phi = e^{mx} \text{ on } S.$$

If we write $m = ik$, where k is the wave number (dimension-less) in the theory of diffraction for the soft body S , the equations (4.43) and (4.44) solve the problem of diffraction of a soft body by a plane incident wave. The radiation condition on Φ leads to the correct boundary conditions at infinity for $\mathbf{v}$. For the case of a sphere an exact solution is known for the diffraction problem and therefore the corresponding results of the previous section can be sharpened. Furthermore, Jones has presented solutions for the case of diffraction by a soft disk for small as well as large values of k [19, 24]. Those results can be used to extend the corresponding results in magnetohydrodynamics, especially the formulas for the drag.

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