On the Right Weak Solution of the Cauchy Problem for a Quasilinear Equation of First Order

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Consider the quasilinear equation of first order

$$(1) \quad u_t + f(u)_x = 0$$

with t, x, u being real scalars and $f(u)$ being a given smooth function on $(-\infty, \infty)$. Wanted is a solution $u = u(t, x)$, $t > 0$, that is prescribed for $t = 0$. The solution can be found by means of the classical method of characteristics. Such a smooth solution exists, in general, only in some neighborhood of the initial line $t = 0$, $u = u_0(x)$. Under wide assumptions about $f(u)$, however, the solution can be continued beyond this neighborhood. The continued solution has, in general, lines of discontinuity in analogy to the shock waves of compressible fluid flow. It is only a weak solution of the Cauchy problem for (1). The property of being a weak solution does not, however, characterize the function $u(t, x)$. The Cauchy problem for (1) has always more than one weak solution if $f(u)$ is not linear in $(-\infty, \infty)$. It must be remembered that the particular weak solution $u(t, x)$ spoken of above is originally thought of as

$$(2) \quad u(t, x) = \lim_{\epsilon \downarrow 0} u(t, x; \epsilon)$$

where $u(t, x; \epsilon)$ is the solution of the parabolic second order equation ($\epsilon > 0$)

$$(3) \quad u_t + f(u)_x = \epsilon u_{xx}$$

with the same initial data $u_0(x)$ at $t = 0$. This is, of course, a complicated description of the particular weak solution $u(x, t)$, and it is an important and apparently not completely solved problem to characterize this solution $u(x, t)$ in substantially simpler ways. The present note contains another approach to this question.

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In order to justify the subsequent procedure we begin by quoting known results about the Cauchy problem for (3) and about the limit (2). The solution $u(t, x; \epsilon)$ of (3) with given initial values $u_0(x)$ at $t = 0$ exists if $f(u)$ is "sufficiently smooth" and if $u_0(x)$ is bounded, $|u_0(x)| \leq M$, and smooth, $u'_0(x)$ bounded; the solution $u(t, x)$ has the same bound M (see [2], end of page 277). If $u_0(x)$ is bounded and monotone the functions $u(t, x; \epsilon)$ converge in the mean, as $\epsilon \rightarrow 0$, to a function $u(t, x)$ in any finite interval of any straight line $t = t_0 \geq 0$ (see [2], theorem 1).

Let $h(u) \in C'(-\infty, \infty)$ and let

$$(4) \quad \begin{aligned} I(u) &= \int h(u) du, \\ F(u) &= \int h(u) df(u) = \int h(u)f'(u) du. \end{aligned}$$

On multiplying the differential equation (3) on both sides by $h(u)$ we may write the resulting equation

$$(5) \quad \begin{aligned} I(u)_t + F(u)_x &= \epsilon(u_x h(u))_x - \epsilon h'(u) u_x^2 \\ &= \epsilon I(u)_{xx} - \epsilon h'(u) u_x^2. \end{aligned}$$

In this equation $u = u(t, x; \epsilon)$ is supposed to be substituted. On multiplying (5) on both sides by a test function $\varphi(t, x)$ in the open half plane $t > 0$ (function with compact support in this half plane and of class C') and on integrating over this half plane we obtain (after some integrations by part)

$$(6) \quad \begin{aligned} \iint_{t>0} [I(u)\varphi_t + F(u)\varphi_x] dt dx \\ = -\epsilon \iint_{t>0} I(u)\varphi_{xx} dt dx + \epsilon \iint_{t>0} h'(u) u_x^2 \varphi dt dx \end{aligned}$$

with $u = u(t, x; \epsilon)$. Now let $\epsilon > 0$ go to zero. From the convergence $u(t, x; \epsilon) \rightarrow u(t, x)$ as explained above and from $|u(t, x; \epsilon)| \leq M$ it is easily inferred that

$$(7) \quad I(u(t, x; \epsilon)) \rightarrow I(u(t, x))$$

holds also in the mean on every finite interval on any line $t = \text{const.}$; the same holds for F . The integral on the left of (6) stays uniformly bounded as $\epsilon \rightarrow 0$. Consequently, the integral on the left, $u = u(t, x; \epsilon)$, tends toward the same integral with $u = u(t, x)$. The first term on the right is easily seen to go to zero. The second term is always ≥ 0 as long as $h'(u)\varphi(t, x) \geq 0$ holds for all t, x, u . Hence we conclude:

The limit function $u(t, x)$ satisfies the inequality

$$(8) \quad \iint_{t>0} [I(u)\varphi_t + F(u)\varphi_x] dt dx \geq 0$$

whenever

$$(8') \quad h'(u)\varphi(t, x) \geq 0$$

holds for all t, x, u .

Condition (8') is satisfied if $h = 1$ or $h = -1$ and if φ is an arbitrary test function in the open half plane $t > 0$. In this case there holds

$$(9) \quad I(u) = \pm u, \quad F(u) = \pm f(u),$$

respectively. The result is that the integral (8) with $I = u$, $F = f$ is both ≥ 0 and ≤ 0 . Therefore the statement made above includes the statement that $u(t, x)$ is a weak solution of (1):

$$(10) \quad \iint_{t>0} [u\varphi_t + f(u)\varphi_x] dt dx = 0$$

holds for every test function $\varphi(t, x)$ in the half plane $t > 0$.

In order to see more clearly what the property (8), (8') means we consider the case where the solution $u(t, x)$ is of class C' in the t - x -plane, except on a finite number of smooth lines of discontinuity along each of which it has limits u^- , u^+ on the two sides. We say in this case that u is of class PC' . Let the test function φ be of the form

$$\varphi(t, x) = \psi(t)\chi(\xi), \quad \xi = x - x(t),$$

where $x = x(t)$ is the equation of the discontinuity line and where χ , with $\epsilon > 0$, is given by

$$(11) \quad \chi(\xi) = \begin{cases} 1 - |\xi|/\epsilon, & |\xi| < \epsilon \\ 0, & |\xi| \geq \epsilon \end{cases}.$$

$\psi(t)$ is a nonnegative test function which is first kept fixed and later made to approach a δ -function around a given value of t . The integral (8) now becomes the sum of the two integrals

$$(12) \quad \iint [-I(u)\dot{x} + F(u)]\chi'(\xi)\psi(t) dt d\xi$$

and

$$(13) \quad \iint I(u)\chi(\xi)\psi(t) dt d\xi.$$

By virtue of (8) the sum is ≥ 0 . From (11) it follows easily that (13) $\rightarrow 0$ as $\epsilon \rightarrow 0$. (12) equals

$$(14) \quad \int_{(t)} \left\{ \frac{1}{\epsilon} \int_{-\epsilon}^0 (-I(u)\dot{x} + F(u)) d\xi - \frac{1}{\epsilon} \int_0^{\epsilon} (-I(u)\dot{x} + F(u)) d\xi \right\} \psi(t) dt,$$

and it is clear that (14) converges to

$$(15) \quad \int_{(t)} \{ (I(u^+) - I(u^-))\dot{x} - (F(u^+) - F(u^-)) \} \psi(t) dt \geq 0.$$

(15) holds (see (8')) if

$$(15') \quad h'(u) \geq 0 \text{ and } \psi(t) \geq 0.$$

If now $\psi(t)$ is made to concentrate upon a single value t (15) becomes simply

$$(16) \quad (I(u^+) - I(u^-))\dot{x} - (F(u^+) - F(u^-)) \geq 0,$$

and this holds whenever

$$(16') \quad h(u) \text{ non-decreasing.}$$

(It is easy to see that condition (15') may be relaxed to (16').)

If the whole argument is applied to (10) in place of (8) there results the well-known formula

$$(17) \quad (u^+ - u^-)\dot{x} - (f(u^+) - f(u^-)) = 0.$$

Altogether we have the following result:

The inequality

$$(18) \quad (I(u^+) - I(u^-)) \frac{f(u^+) - f(u^-)}{u^+ - u^-} - (F(u^+) - F(u^-)) \geq 0$$

holds whenever (16') holds.

The meaning of this inequality becomes clear when the special increasing functions

$$(19) \quad h(u) = \begin{cases} 0, & u < u_0 \\ 1, & u \geq u_0 \end{cases}$$

are used. In this case there follows from (4)

$$I(u) = \begin{cases} 0, & u < u_0 \\ u - u_0, & u \geq u_0 \end{cases}, \quad F(u) = \begin{cases} 0, & u < u_0 \\ f(u) - f(u_0), & u \geq u_0 \end{cases}.$$

We suppose that u_0 lies in the interval u^-, u^+ and we distinguish between the two cases

$$\alpha) \ u^- \leq u_0 \leq u^+ \quad \beta) \ u^+ \leq u_0 \leq u^-.$$

In either case the simple calculations furnish the result that—we write u in place of u_0 —

$$(20) \quad \frac{f(u^+) - f(u)}{u^+ - u} \leq \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

holds. Geometrically this means in the case α) that the curve $v = f(u)$ lies above the chord (the chord subtended between $(u^-, f(u^-))$ and $(u^+, f(u^+))$) in the interval (u^-, u^+) and in the case β) that it lies below the chord in the interval (u^+, u^-) .

This is precisely the condition E introduced by O. A. Oleinik [3]. It would be of considerable interest to prove the uniqueness of the weak solution of the Cauchy problem for (1) under the additional conditions (8), (8').

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