Univalent Functions with Univalent Derivatives III

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1. Introduction. Let D denote the open unit disc. In [8], it was shown that if a function and all of its derivatives are univalent in D , then the function is an entire function and its maximum modulus and coefficients must satisfy certain inequalities. In this paper, we point out that the property of univalence is only one of a class of properties which a function may have for such a conclusion to hold. We further show how this applies to a function with the property that it and all its derivatives are univalent and convex in D .

Next, we discuss what must happen to a function with the property that some, but not all, of its derivatives are univalent in D . In particular, we assume that f is analytic in D and that $\{n_p\}_{p=1}^{\infty}$ is a strictly increasing sequence of non-negative integers such that each $f^{(n_p)}$ is univalent in D . We connect the growth of $\{n_p\}_{p=1}^{\infty}$ with the radius of convergence about the origin of the power series that represents f in D . We show that even if the gaps between successive n_p 's have no upper bound, it may be the case that f must be an entire function.

Finally, we construct an example of a function, f , and a sequence, $\{n_p\}_{p=1}^{\infty}$, such that each $f^{(n_p)}$ is not only univalent, but convex in D , and yet the unit circle is the natural boundary of the domain of f .

2. Properties which force f to be entire. Let (A) be a property which a function f , analytic in D , is able to possess. We say that (A) is an admissible property provided the following hold:

(I) If f has (A) , then $f'(0) \neq 0$.

(II) If f has (A) , and if b and c are complex numbers with $b \neq 0$, then the function, F , defined by $F(z) = bf(z) + c$, also has (A) .

Let T be the family of functions, analytic in D , of the form

$$(2.1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

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Let $T(A)$ be the subfamily of T such that if $f \in T(A)$, then $f^{(n)}$ has property (A) for $n = 0, 1, 2, \dots$. Let

$$(2.2) \quad \alpha_A = \sup \{|a_2| : f \in T(A)\}.$$

(We shall always assume that $T(A)$ is not empty.)

Theorem 1. *Let (A) be an admissible property and let $f \in T(A)$. Suppose that f is of the form (2.1) and that $\alpha_A < \infty$. Then $\alpha_A \neq 0$ and the following hold:*

- (i) $T(A)$ is a normal family in $|z| < R$ for all R .
- (ii) f is an entire transcendental function of exponential type not greater than $2\alpha_A$.
- (iii) $|a_k| \leq (2\alpha_A)^{k-1}/k!$, $k = 1, 2, \dots$.
- (iv) $|f(z)| \leq (\exp \{2\alpha_A |z|\} - 1)/2\alpha_A$, $0 \leq |z| < \infty$.
- (v) $|f^{(n)}(z)| \leq (2\alpha_A)^{n-1} \exp \{2\alpha_A |z|\}$, $0 \leq |z| < \infty$, $n = 1, 2, \dots$.

Proof. Since each $f^{(n)}$ has (A), $f^{(n)}(0) \neq 0$. Therefore, $a_2 \neq 0$ and $\alpha_A \neq 0$. Assume that (iii) and (iv) are true. Using (iv) and Montel's Theorem [6, p. 141], it follows that $T(A)$ is a normal family in $|z| < R$. Using (iii) and [1, p. 8], we have (ii).

The proofs of (iii), (iv), and (v) have essentially been given in [8]. Therefore, we only sketch them here. Let

$$F_n(z) = \frac{f^{(n)}(z) - f^{(n)}(0)}{f^{(n+1)}(0)} = z + \frac{(n+2)a_{n+2}}{2a_{n+1}}z^2 + \dots$$

Since F_n has (A),

$$|a_{n+2}| \leq \frac{2|a_{n+1}|}{n+2} \alpha_A.$$

An induction argument establishes (iii). Using (iii) and letting $r = |z|$, we have

$$|f(z)| = \left| z + \sum_{k=2}^{\infty} a_k z^k \right| \leq \sum_{k=1}^{\infty} \frac{(2\alpha_A)^{k-1}}{k!} r^k = \frac{\exp(2\alpha_A r) - 1}{2\alpha_A}.$$

This proves (iv). (v) is proved similarly.

We shall say that f has property (K) provided that f is univalent in D and $f(D)$ is a convex set. We note that (K) is an admissible property. In fact, it implies that if f has form (2.1), then $|a_2| \leq 1$, [6, p. 223]. Hence, $\alpha_K \leq 1$ and the results of the preceding theorem apply to any function in $T(K)$. The only thing to determine in this case is the exact value of α_K . In this direction, we have the following.

Theorem 2. *Let α_K be defined by (2.2). Then $1/2 \leq \alpha_K < .68379$.*

Proof. Let $\phi(z) = e^z - 1$. Then $\phi \in T(K)$. Since the second coefficient of ϕ is $1/2$, it follows that $1/2 \leq \alpha_K$.

Let $f \in T(K)$ and suppose f has form (2.1). Let

$$g(z) = zf'(z) = z + \sum_{k=2}^{\infty} k a_k z^k.$$

Then g is starlike and univalent in D , [6, pp. 221 and 224]. From (v), if $z \in D$, then $|g(z)| < e^{2\alpha_K}$. From [4],

$$|(2a_2)^2 - 3a_3| \leq 1 - \frac{1}{(\exp \{2\alpha_K\})^3}.$$

Using (iii) and the triangle inequality,

$$4|a_2|^2 \leq 1 - e^{-4\alpha_K} + 2\alpha_K^2.$$

Hence,

$$2\alpha_K^2 \leq 1 - e^{-4\alpha_K}.$$

This implies that $\alpha_K < .68379$.

3. Not all derivatives are univalent in D . In [9], we proved the following: Let f be defined in D by

$$(3.1) \quad f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Let R be the radius of convergence of the power series in (3.1) about the origin. Let ρ_n be the largest number such that $f^{(n)}$ is univalent in $|z| < \rho_n$. (If there is no such number, let $\rho_n = 0$.) Then

$$\liminf_{n \rightarrow \infty} n\rho_n \leq 4R.$$

This result is of no interest if an infinite number of ρ_n are zero. The theorems presented below partially remedy this situation. In the proofs, we shall write $a(n)$ for a_n whenever necessary.

Theorem 3. *Let $\{n_p\}_{p=1}^{\infty}$ be a strictly increasing sequence of positive integers. Let f be defined in D by (3.1) and suppose each $f^{(n_p)}$ is univalent in D . Let R be the radius of convergence of (3.1). Then*

$$\liminf_{p \rightarrow \infty} (n_1 n_2 \cdots n_p)^{1/n_p} \leq R \limsup_{p \rightarrow \infty} 4^{p/n_p} \leq 4R.$$

Proof. The univalence of $f^{(n_p)}$ implies [6, p. 218] that for $p = 1, 2, \dots$ and $k = 2, 3, \dots$,

$$|a(n_p + 2)| < \frac{4|a(n_p + 1)|}{n_p}$$

and

$$(3.2) \quad |a(n_p + k)| < \frac{(ek)k!(n_p + 1)!|a(n_p + 1)|}{(n_p + k)!}.$$

There is a positive integer, Q , such that if $p \geq Q$ and $k \geq 3$, then

$$\frac{(ek)k! (n_p + 1)!}{(n_p + k)!} < \frac{4}{n_p}.$$

Hence, if $p \geq Q$ and $k \geq 2$,

$$(3.3) \quad |a(n_p + k)| < \frac{4 |a(n_p + 1)|}{n_p}.$$

Using (3.3) an inductive argument establishes that for $p \geq Q$ and $n_p + 1 < n_p + k \leq n_{p+1} + 1$,

$$|a(n_p + k)| < \frac{4^{p+1-Q} |a(n_Q + 1)|}{n_Q n_{Q+1} \cdots n_p} = \frac{A 4^p}{n_1 n_2 \cdots n_p},$$

where A is a constant. From this, it follows that

$$\limsup_{k \rightarrow \infty} |a_k|^{1/k} \leq \frac{\limsup_{p \rightarrow \infty} 4^{p/n_p}}{\liminf_{p \rightarrow \infty} (n_1 n_2 \cdots n_p)^{1/n_p}}.$$

Since $1/R = \limsup_{k \rightarrow \infty} |a_k|^{1/k}$, the theorem is proved.

Corollary If $n_{p+1} - n_p = o(\log n_{p+1})$, then f is entire.

Proof. This follows from

$$\liminf_{p \rightarrow \infty} \frac{\log n_{p+1}}{n_{p+1} - n_p} \leq \liminf_{p \rightarrow \infty} \frac{\sum_{i=1}^p \log n_i}{n_p}.$$

Note that the hypothesis of this corollary is equivalent to $n_{p+1} - n_p = o(\log p)$

From what we have done so far, it follows that if $\limsup_{p \rightarrow \infty} (n_{p+1} - n_p) < \infty$, then f is an entire function. Now we obtain results in another direction.

Theorem 4. Assume all of the hypotheses of Theorem 3. Also assume there is an integer, $M > 1$, such that $\liminf_{p \rightarrow \infty} (n_{p+1} - n_p) \geq M$. Then

$$\liminf_{p \rightarrow \infty} \{(n_1 n_2 \cdots n_p)^{1/n_p}\}^{M-1} \leq R.$$

Proof. There is an integer, Q_1 , such that if $p \geq Q_1$ and $k \geq M + 1$, then

$$\frac{(ek)k! (n_p + M)!}{(n_p + k)!} < 1.$$

For these p and k , (3.2) implies that

$$(3.4) \quad |a(n_p + k)| < \frac{|a(n_p + 1)|}{(n_p + 2) \cdots (n_p + M)} < \frac{|a(n_p + 1)|}{n_p^{M-1}}.$$

Also, there is an integer, Q_2 , such that if $p \geq Q_2$ and $k \geq 2$, then

$$\frac{(ek)k! (n_p + 1)!}{(n_p + k)!} < 1.$$

Using (3.2) again, it follows that for $p \geq Q_2$ and $k \geq 2$,

$$(3.5) \quad |a(n_p + k)| < |a(n_p + 1)|.$$

Further, there is an integer, Q_3 , such that if $p > Q_3$, then

$$n_{p+1} - n_p \geq M.$$

Let $Q = \max \{Q_1, Q_2, Q_3\}$ and suppose that $p \geq Q$. Then $n_{p+1} - n_p + 1 \geq M + 1$, and (3.4) implies that

$$|a(n_{p+1} + 1)| < \frac{|a(n_p + 1)|}{n_p^{M-1}}.$$

An induction argument then shows that if $p > Q$,

$$|a(n_p + 1)| < \frac{|a(n_{Q+1})|}{(n_Q n_{Q+1} \cdots n_{p-1})^{M-1}} = \frac{C}{(n_1 n_2 \cdots n_{p-1})^{M-1}},$$

where C is a constant. If $n_p + 1 \leq n_p + k < n_{p+1} + 1$, then (3.5) implies that for $p > Q$,

$$|a(n_p + k)| < \frac{C}{(n_1 n_2 \cdots n_{p-1})^{M-1}}.$$

Hence,

$$\limsup_{k \rightarrow \infty} |a_k|^{1/k} \leq \frac{1}{\liminf_{p \rightarrow \infty} \{(n_1 n_2 \cdots n_p)^{1/n_p}\}^{M-1}}.$$

This completes the proof.

Corollary. If $\lim_{p \rightarrow \infty} (n_{p+1} - n_p) = \infty$ and if $\liminf_{p \rightarrow \infty} (n_1 \cdots n_p)^{1/n_p} > 1$, then f is entire.

This corollary is not contained in the corollary to the previous theorem. For example, let $n_p = 1 + [p \log p]$. Then for large p , $n_{p+1} - n_p > (\log p)/2$ and $\liminf_{p \rightarrow \infty} (n_1 n_2 \cdots n_p)^{1/n_p} \geq e$. But $n_{p+1} - n_p \neq o(\log n_{p+1})$.

In the next theorem, we shall be directly concerned with restraints on the growth of $\{n_p - n_{p-1}\}_{p=2}^{\infty}$ which will insure that f is entire. In particular, we wish to use functions which are slowly oscillating ([3], [5]), hereafter abbreviated to s.o. functions.

Theorem 5. Assume all the hypotheses of Theorem 3. Suppose that θ and ϕ are s.o. functions such that

$$(3.6) \quad 1 \leq \phi(p) \leq n_p - n_{p-1} \leq \theta(p)$$

for $p = 2, 3, \dots$. If

$$(3.7) \quad \limsup_{p \rightarrow \infty} \frac{\theta(p) \log \theta(p)}{\phi(p) \log p} = \alpha < 1,$$

then f is an entire function of order $\Lambda \leq 1/(1 - \alpha)$.

Before proving this theorem, we establish the following.

Lemma. *Suppose all the hypotheses of Theorem 3 are true. Suppose that θ is a real-valued function defined on the integers such that*

$$(3.8) \quad n_p - n_{p-1} \leq \theta(p)$$

for $p = 2, 3, \dots$. If $p \geq 2$ and $n_p + 1 \leq n_p + k \leq n_{p+1}$, then

$$\frac{\log |a(n_p + k)|}{n_p + k} < (\log n_{p+1}) \left\{ o(1) - 1 + \frac{\theta(p+1) \log \theta(p+1)}{n_{p+1} \log n_{p+1}} + \frac{1}{n_p \log n_p} \sum_{j=2}^p [\theta(j) + 4] \log \theta(j) \right\}.$$

Proof. From (3.2) with $k = n_{p+1} - n_p + 1$,

$$|a(n_{p+1} + 1)| < \frac{e(n_{p+1} - n_p + 2)! (n_p + 1)! a(n_p + 1)}{(n_{p+1} + 1)!}.$$

An induction argument on p shows that if $p \geq 2$, then

$$|a(n_p + 1)| < \frac{e^{p-1} (n_1 + 1)! |a(n_1 + 1)|}{(n_p + 1)!} \prod_{i=2}^p (n_i - n_{i-1} + 2)!.$$

Using this and (3.2), we have that if $p \geq 2$ and $n_p + 1 \leq n_p + k \leq n_{p+1}$, then

$$|a(n_p + k)| < \frac{e^p (k + 1)! (n_1 + 1)! |a(n_1 + 1)|}{(n_p + k)!} \prod_{i=2}^p (n_i - n_{i-1} + 2)!.$$

We have from Stirling's Formula that, if N is a positive integer, then $N \log N - N < \log N! < \log 3 + (N + \frac{1}{2}) \log N - N$. Using this, it follows that

$$(3.9) \quad \begin{aligned} \log |a(n_p + k)| &< O(p) + \frac{1}{2} \log(k + 1) + (k + 1) \log(k + 1) \\ &\quad - (n_p + k) \log(n_p + k) + n_p \\ &\quad + \sum_{i=2}^p [(n_i - n_{i-1} + \frac{5}{2}) \log(n_i - n_{i-1} + 2) - (n_i - n_{i-1} + 2)]. \end{aligned}$$

It is desirable to replace certain expressions in (3.9) by simpler expressions. To do that, we use the following inequalities:

$$\sum_{i=2}^p (n_i - n_{i-1} + 2) = O(p) + n_p.$$

$$\sum_{i=2}^p \log(n_i - n_{i-1} + 2) = O(p) + \sum_{i=2}^p \log(n_i - n_{i-1}).$$

$$\sum_{i=2}^p (n_i - n_{i-1}) \log(n_i - n_{i-1} + 2) \leq \sum_{i=2}^p (n_i - n_{i-1} + \log 3) \log(n_i - n_{i-1}).$$

Using these, (3.9) becomes

$$\begin{aligned} \log |a(n_p + k)| &< O(p) + 3k + k \log k - (n_p + k) \log (n_p + k) \\ &+ \sum_{i=2}^p (n_i - n_{i-1} + 4) \log (n_i - n_{i-1}). \end{aligned}$$

Or,

$$\begin{aligned} (3.10) \quad \frac{\log |a(n_p + k)|}{n_p + k} &< O(1) + \frac{k \log k}{n_p + k} - \log (n_p + k) \\ &+ \frac{1}{n_p + k} \sum_{i=2}^p (n_i - n_{i-1} + 4) \log (n_i - n_{i-1}). \end{aligned}$$

Now

$$\frac{k \log k}{n_p + k} - \log (n_p + k) < \frac{n_{p+1} - n_p}{n_{p+1}} \log (n_{p+1} - n_p) - \log n_{p+1}.$$

Using this and (3.10), we finally have

$$\begin{aligned} \frac{\log |a(n_p + k)|}{n_p + k} &< (\log n_{p+1}) \\ &\cdot \left\{ o(1) - 1 + \frac{\theta(p+1) \log \theta(p+1)}{n_{p+1} \log n_{p+1}} + \frac{1}{n_p \log n_p} \sum_{i=2}^p [\theta(j) + 4] \log \theta(j) \right\}. \end{aligned}$$

Proof of Theorem 5. Using (3.6), we have that $n_p \geq \sum_{i=2}^p \phi(j) + n_1$. Since ϕ is an s.o. function, $\sum_{i=2}^p \phi(j) \sim p\phi(p)$ and

$$(3.11) \quad n_p \log n_p \geq [1 + o(1)]p\phi(p) \log p.$$

Further, (3.7) implies that

$$\frac{\theta(p) \log \theta(p)}{p\phi(p) \log (p)} = o(1)$$

and

$$\frac{\log \theta(p)}{\phi(p) \log p} = o(1).$$

Using these and (3.11), it follows that

$$(3.12) \quad \frac{\theta(p+1) \log \theta(p+1)}{n_{p+1} \log n_{p+1}} = o(1)$$

and

$$(3.13) \quad \frac{p \log \theta(p)}{n_p \log n_p} = o(1).$$

Since $[\theta(j) + 4] \log \theta(j)$ is also an s.o. function,

$$\sum_{i=2}^p [\theta(j) + 4] \log \theta(j) = [1 + o(1)]p[\theta(p) + 4] \log \theta(p).$$

Using this, (3.11), (3.12), and (3.13), we have

$$\begin{aligned} \frac{1}{n_p \log n_p} \sum_{j=2}^p [\theta(j) + 4] \log \theta(j) + \frac{\theta(p+1) \log \theta(p+1)}{n_{p+1} \log n_{p+1}} \\ \leq \frac{[1 + o(1)][\theta(p) + 4] \log \theta(p)}{\phi(p) \log p} + o(1) = \frac{\theta(p) \log \theta(p)}{\phi(p) \log p} + o(1). \end{aligned}$$

The lemma now shows that if $p \geq 2$ and $n_p + 1 \leq n_p + k \leq n_{p+1}$, then

$$(3.14) \quad \frac{\log |a(n_p + k)|}{n_p + k} < \left[o(1) - 1 + \frac{\theta(p) \log \theta(p)}{\phi(p) \log p} \right] \log n_{p+1}.$$

Since $\alpha < 1$, (3.7) shows that

$$\lim_{k \rightarrow \infty} \frac{\log |a_k|}{k} = -\infty,$$

and f is an entire function. Further, if Λ is the order of f , then by (3.14)

$$\Lambda = \limsup_{k \rightarrow \infty} \frac{k \log k}{\log (1/|a_k|)} \leq \frac{1}{1 - \alpha}.$$

4. Non-continuable functions with univalent derivatives. Suppose that f is analytic in D and defined there by (3.1). It is known [2] that if $a_1 \neq 0$ and if

$$\sum_{k=2}^{\infty} k^2 |a_k| \leq |a_1|,$$

then f is univalent in D and maps D onto a convex region. (Further, f is continuous on the closure of D and maps the boundary onto a simple, closed, rectifiable, convex curve.) We shall use this to prove the following:

Theorem 6. *There is a function, f , analytic in D , and an increasing sequence of positive integers, $\{n_p\}_{p=1}^{\infty}$, such that f and each $f^{(n_p)}$ map D univalently onto convex domains and yet the unit circle is the natural boundary of f .*

Proof. We shall construct a family of functions, each member of which will satisfy the conclusions of the theorem.

Let η be a function defined on $x \geq 1$ such that $\eta(x) \geq 1$ and

$$(4.1) \quad \lim_{x \rightarrow \infty} \eta(x) = \infty.$$

Define ν by

$$\nu(x) = \frac{x}{\eta(x) \log x}.$$

Assume that η has the additional properties that ν is strictly increasing on $x \geq a > 1$ and

$$(4.1) \quad \lim_{x \rightarrow \infty} \nu(x) = \infty.$$

(For example, let $\eta(x) = \log \log (x + 10)$.)

Let $n_0 = 0$ and $n_1 = [a] + 3$. Let $p \geq 1$ and assume that n_p is known. Let n_{p+1} be the smallest integer such that

$$(4.2) \quad n_p + 5 \leq \nu(n_{p+1}).$$

(Note that $n_{p+1} > n_p$ for $p = 0, 1, \dots$.) If $k = n_p + 1$ for some p , let $a_k = \exp \{-n_p/\eta(n_p)\}$. Otherwise, let $a_k = 0$. Define f by

$$f(z) = \sum_{k=1}^{\infty} a_k z^k.$$

Since

$$|a(n_p + 1)|^{1/n_{p+1}} = \exp \{-n_p/(n_p + 1)\eta(n_p)\},$$

(4.1) implies that f is analytic in D . Using the Hadamard Gap Theorem [7, p. 315] and (4.2), we conclude that the unit circle is the natural boundary of the domain of f .

For $p = 0, 1, \dots$,

$$f^{(n_p)}(z) = \sum_{k=0}^{\infty} \frac{(n_{p+k} + 1)!}{(n_{p+k} - n_p + 1)!} a(n_{p+k} + 1) z^{n_{p+k} - n_p + 1}.$$

To conclude that $f^{(n_p)}$ has a convex and bounded univalent image, it is enough to show that

$$(4.3) \quad \sum_{k=1}^{\infty} \frac{(n_{p+k} - n_p + 1)(n_{p+k} + 1)!}{(n_{p+k} - n_p)!} |a(n_{p+k} + 1)| \leq (n_p + 1)! |a(n_p + 1)|.$$

Since, for $p = 0, 1, \dots$

$$\sum_{k=1}^{\infty} \frac{1}{n_{p+k}^2} < 1 \leq (n_p + 1)! |a(n_p + 1)|,$$

to prove (4.3) it is sufficient to show that for $p = 0, 1, \dots$ and $k = 1, 2, \dots$,

$$\frac{(n_{p+k} - n_p + 1)(n_{p+k} + 1)! |a(n_{p+k} + 1)|}{(n_{p+k} - n_p)!} \leq \frac{1}{n_{p+k}^2}.$$

Now

$$\begin{aligned} & \frac{n_{p+k}^2 (n_{p+k} - n_p + 1)(n_{p+k} + 1)! |a(n_{p+k} + 1)|}{(n_{p+k} - n_p)!} \\ &= n_{p+k}^2 (n_{p+k} - n_p + 1)(n_{p+k} + 1)(n_{p+k}) \cdots (n_{p+k} - n_p + 1) |a(n_{p+k} + 1)| \\ &< n_{p+k}^2 (n_{p+k} + 1)^{n_p+2} \exp \{-n_{p+k}/\eta(n_{p+k})\} \\ &< (n_{p+k})^{n_p+5} \exp \{-n_{p+k}/\eta(n_{p+k})\}. \end{aligned}$$

Hence, the theorem will be proved if

$$(4.4) \quad (n_{p+k})^{n_p+5} \exp \{-n_{p+k}/\eta(n_{p+k})\} \leq 1.$$

Taking logarithms and rearranging, we discover that (4.4) is equivalent to $n_p + 5 \leq \nu(n_{p+k})$. However, (4.2) and the fact that ν is increasing show the latter to be true.

REFERENCES

- [1] R. P. BOAS, *Entire Functions*, Academic Press, New York, 1954.
- [2] A. W. GOODMAN, Univalent functions and nonanalytic curves, *Proc. Amer. Math. Soc.*, **8** (1957) 598–601.
- [3] G. H. HARDY & W. W. ROGOSINSKI, Note on Fourier series (III): Asymptotic formulae for the sums of certain trigonometrical series, *Quart. J. Math. Oxford Ser.*, **16**(1954) 49–58.
- [4] Z. J. JAKUBOWSKI, Sur les coefficients des fonctions univalentes dans le cercle unité, *Ann. Polon. Math.*, **19**(1967) 207–233.
- [5] J. KARAMATA, Sur un mode de croissance reguliere des fonctions, *Math. (Cluj)*, **4**(1930) 38–53.
- [6] Z. NEHARI, *Conformal Mapping*, McGraw-Hill, New York, 1952.
- [7] W. RUDIN, *Real and Complex Analysis*, McGraw-Hill, New York, 1966.
- [8] S. M. SHAH & S. Y. TRIMBLE, Univalent functions with univalent derivatives, *Bull. Amer. Math. Soc.*, **75**(1969) 153–157.
- [9] ———, Univalent functions with univalent derivatives (II), *Trans. Amer. Math. Soc.*, **145**(1969) (to appear).

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