*On Some Extremal Quantities Associated with Unbounded Sets in the Plane**

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Let E be a compact set in the complex plane, and set

$$M_n = \min \max_{z \in E} |P(z)| ,$$

the minimum to be taken over all monic polynomials of degree n . It is well known that

$$(1) \quad \lim_{n \rightarrow \infty} M_n^{1/n} = C(E),$$

the logarithmic capacity of E . This was proved by G. Szegő in 1924 [1]. The inequality

$$(2) \quad \max_{z \in E} |P(z)| \geq C(E)^n$$

holds for all P as a consequence of the maximum principle, and this gives

$$\liminf_{n \rightarrow \infty} M_n^{1/n} \geq C(E).$$

The opposite inequality

$$\limsup_{n \rightarrow \infty} M_n^{1/n} \leq C(E)$$

is more delicate and depends on the construction of polynomials which are not too big on E . Of course these polynomials depend very strongly on the set E itself.

A trivial extension of this is obtained by taking a function $\omega(z)$ which vanishes outside E and satisfies

$$0 < c \leq \omega(z) \leq C < \infty$$

on E , and setting

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$$(3) \quad M_n = \min \sup |P(z)| \omega(z);$$

the supremum is taken over all z and the minimum is as before. Formula (1) still holds.

We shall be concerned with the extension of this to the case where ω has unbounded support. Perhaps surprisingly it will transpire that the asymptotic behavior of M_n (more exactly an asymptotic formula for $\log M_n$) is determined not by a delicate construction but simply by showing that the polynomial z^n cannot be much improved upon in determining the minimum (3). This of course is not always so; we must assume that ω is, roughly speaking, not concentrated on too thin a set. The exact condition is given in Theorem 1 below.

Similarly, let μ be a positive Borel measure in the plane and set

$$(4) \quad m_n = \min \int |P(z)| d\mu(z),$$

the minimum to be taken as before over all monic polynomials of degree n . We shall show that for a large class of measures with unbounded support the quantity $\log m_n$ is obtained asymptotically by taking z^n for $P(z)$ in the right side of (4). Analogous results can also be established for more general integrals of the form

$$\int \Phi(P(z)) d\mu(z).$$

We leave this to the reader.

As usual E' will denote the complement of E . It is also convenient to introduce the notations

$$[E]_r = E \cap \{z : |z| \leq r\}$$

$$[E]_{r_1, r_2} = E \cap \{z : r_1 \leq |z| \leq r_2\}.$$

A closed set E is said to have *capacity density zero* (at infinity) if

$$\lim_{r \rightarrow \infty} r^{-1} C([E]_r) = 0.$$

These are the thin sets referred to above.

The following lemma is useful.

Lemma. *Let $\{\varphi_r\}$ be a family of nonnegative functions of a Borel set with the property that for any closed set E of capacity density zero*

$$\liminf_{r \rightarrow \infty} \varphi_r([E']_{\lambda^{-1}r, \lambda r}) > 0.$$

(Here λ is a fixed number greater than 1.) Then there is a $\delta > 0$ with the following property: For any closed set E satisfying

$$C([E]_r) \leq \delta r$$

for all r we have

$$\varphi_r([E']_{\lambda^{-1}r, \lambda r}) \geq \delta$$

for all $r \geq \delta^{-2}$.

Proof. Suppose not. Then we can find a sequence $\delta_k \searrow 0$ and closed sets E_k such that

$$(5) \quad C([E_k]_r) \leq \delta_k r \quad \text{for all } r,$$

$$(6) \quad \varphi_{r_k}([E_k]_{\lambda^{-1}r_k, \lambda r_k}) < \delta_k \quad \text{for some } r_k \geq \delta_k^{-2}.$$

By passing to a subsequence if necessary we may assume that $\delta_{k+1} < r_k^{-1}$. In particular

$$(7) \quad r_{k+1} > r_k / \delta_{k+1}.$$

We now set

$$E = \bigcup_{k=1}^{\infty} [E_k]_{\lambda^{-1}r_k, \lambda r_k}.$$

If $\delta_1 < \lambda^{-2}$, which may clearly be assumed, then (7) gives $r_{k+1} > \lambda^2 r_k$. Therefore the annuli

$$\lambda^{-1}r_k \leq |z| \leq \lambda r_k, \quad k = 1, 2, \dots,$$

are disjoint and

$$[E']_{\lambda^{-1}r_k, \lambda r_k} = [E'_k]_{\lambda^{-1}r_k, \lambda r_k},$$

and so by (6)

$$\varphi_{r_k}([E']_{\lambda^{-1}r_k, \lambda r_k}) < \delta_k.$$

We need only show that E has capacity density zero to obtain the desired contradiction. It is known [2, p. 63] that for closed sets A and B contained in the disc $|z| \leq \frac{1}{2}$

$$\left[\log \frac{1}{C(A \cup B)} \right]^{-1} \leq \left[\log \frac{1}{C(A)} \right]^{-1} + \left[\log \frac{1}{C(B)} \right]^{-1}.$$

Consequently if A and B are contained in $|z| \leq \rho$

$$(8) \quad \left[\log \frac{2\rho}{C(A \cup B)} \right]^{-1} \leq \left[\log \frac{2\rho}{C(A)} \right]^{-1} + \left[\log \frac{2\rho}{C(B)} \right]^{-1}.$$

Now take an arbitrary r . We distinguish two cases.

Case 1. $\lambda^{-1}r_k \leq r \leq \lambda r_k$. Then

$$[E]_r \subset \{z: |z| \leq \lambda r_{k-1}\} \cup [E_k]_{\lambda r_k}$$

and we shall apply (8) with A and B the two sets on the right and $\rho = \lambda r_k$. Since the capacity of a disc equals its radius,

$$\frac{2\rho}{C(A)} = \frac{2\lambda r_k}{\lambda r_{k-1}} > \frac{2}{\delta_k}$$

by (7). Inequality (5) gives

$$\frac{2\rho}{C(B)} \geq \frac{2\lambda r_k}{\delta_k \lambda r_k} = \frac{2}{\delta_k}.$$

Consequently

$$\log \frac{2\lambda r_k}{C(A \cup B)} \geq \frac{1}{2} \log \frac{2}{\delta_k}$$

or

$$(9) \quad C([E]_r) \leq 2\lambda r_k (\delta_k/2)^{1/2} \leq 2\lambda^2 r (\delta_k/2)^{1/2}.$$

Thus for r in case 1

$$\lim_{r \rightarrow \infty} r^{-1} C([E]_r) = 0.$$

Case 2. $\lambda r_k < r < \lambda^{-1} r_{k+1}$. Then

$$[E]_r = [E]_{\lambda r_k}$$

and λr_k is in case 1. Therefore the first inequality of (9) gives

$$C([E]_r) \leq 2\lambda r_k (\delta_k/2)^{1/2} < 2r (\delta_k/2)^{1/2}.$$

Thus also for r in case 2

$$\lim_{r \rightarrow \infty} r^{-1} C([E]_r) = 0,$$

and the lemma is established.

Suppose now that we are given a nonnegative function $\omega(z)$ defined for all z , and define M_n by (3). We assume that each $M_n < \infty$, which is equivalent to

$$\sup |z|^n \omega(z) < \infty,$$

and that ω does not have bounded support. The supremum in (3) is unchanged if ω is replaced by its upper semicontinuous modification

$$\limsup_{z \rightarrow \zeta} \omega(\zeta),$$

and so we may assume that ω itself is upper semicontinuous. (The “sup” in (3) may then be replaced by “max”.) Set

$$\omega^*(r) = \max_{|z|=r} \omega(z).$$

Theorem 1. *Suppose there is a $\lambda > 1$ such that for any closed set E of capacity density zero,*

$$\liminf_{r \rightarrow \infty} \frac{\sup \{ \omega(z) : z \in [E']_{\lambda^{-1}r, \lambda r} \}}{\omega^*(r)} > 0.$$

(In taking the *lim inf*, r is restricted to those values for which $\omega^*(r) > 0$.) Then

$$\log M_n \sim \log \max_r r^n \omega^*(r), \quad n \rightarrow \infty.$$

Proof. With

$$\varphi_r(A) = \begin{cases} \omega^*(r)^{-1} \sup \{\omega(z) : z \in A\} & \text{if } \omega^*(r) > 0 \\ 1 & \text{if } \omega^*(r) = 0 \end{cases}$$

the hypothesis of the lemma is satisfied. Let δ be as in the conclusion of the lemma.

Take an arbitrary monic polynomial P of degree n and consider the closed set

$$E = \{z : |P(z)| \leq \delta^n |z|^n\}.$$

On $[E]_r$, we have $|P(z)| \leq \delta^n r^n$, so inequality (2) gives

$$C([E]_r) \leq \delta_r$$

for all r . Hence, because of the characteristic property of δ , we can conclude that

$$\varphi_r([E']_{\lambda^{-1}r, \lambda r}) \geq \delta$$

or

$$\sup \{\omega(z) : z \in [E']_{\lambda^{-1}r, \lambda r}\} \geq \delta \omega^*(r)$$

as long as $r \geq \delta^{-2}$. Consequently for such r

$$\begin{aligned} \max |P(z)| \omega(z) &\geq \sup_{E'} |P(z)| \omega(z) \geq \delta^n \sup_{E'} |z|^n \omega(z) \\ &\geq \delta^n \lambda^{-n} r^n \sup \{\omega(z) : z \in [E']_{\lambda^{-1}r, \lambda r}\} \\ &\geq \delta^{n+1} \lambda^{-n} r^n \omega^*(r). \end{aligned}$$

If we take the maximum over all $r \geq \delta^{-2}$, and then the minimum over all P we obtain

$$(10) \quad M_n \geq \delta^{n+1} \lambda^{-n} \max_{r \geq \delta^{-2}} r^n \omega^*(r).$$

We have assumed that ω does not vanish outside a bounded set. Thus we have $\omega(\rho) > 0$ for arbitrarily large ρ . But then

$$\liminf_{n \rightarrow \infty} \{\max_{r \geq \delta^{-2}} r^n \omega^*(r)\}^{1/n} \geq \liminf_{n \rightarrow \infty} \rho \omega^*(\rho)^{1/n} = \rho$$

and so

$$\lim_{n \rightarrow \infty} n^{-1} \log \max_{r \geq \delta^{-2}} r^n \omega^*(r) = \infty.$$

This, together with (10) and the fact that

$$\max_{r \leq \delta^{-2}} r^n \omega^*(r)$$

is at most a constant times δ^{-2n} , gives (for arbitrary $\epsilon > 0$)

$$\log M_n \geq (1 - \epsilon) \log \max_{r \geq 0} r^n \omega^*(r)$$

for large enough n . Since trivially

$$M_n \leq \max |z^n| \omega(z) = \max r^n \omega^*(r)$$

the theorem is established.

The following corollaries give special cases in which the theorem can be applied.

Corollary 1. *Define*

$$A_{r,\delta} = \{\rho \leq r : \omega^*(\rho) \geq \delta \omega^*(r)\}$$

and suppose that for some $\delta > 0$ we have

$$\liminf_{r \rightarrow \infty} r^{-1} |A_{r,\delta}| > 0$$

(where $|A_{r,\delta}|$ denotes the Lebesgue measure of $A_{r,\delta}$). Then the conclusion of the theorem holds.

Proof. For any closed set E in the plane, write

$$E^* = \{r \geq 0 : E \cap \{|z| = r\} \text{ is nonempty}\}.$$

Then we have [2, p. 85]

$$C(E) \geq C(E^*) \geq \frac{1}{4} |E^*|.$$

In particular if E has capacity density zero then

$$\lim_{r \rightarrow \infty} r^{-1} |E^* \cap [0, r]| = 0.$$

Denote by 2η the $\liminf$ in the statement of the corollary. For r large enough we shall have

$$|A_{r,\delta} \cap E^{*'}| > \eta r$$

and so for such r the set

$$A_{r,\delta} \cap E^{*'} \cap [\eta r, r]$$

will be nonempty. It follows that

$$\max \{\omega(z) : z \in [E']_{\eta r, r}\} \geq \delta \omega^*(r)$$

for sufficiently large r , so the hypothesis of the theorem is satisfied.

Corollary 2. *Suppose there is a $\delta > 0$ such that (for large enough r) $r' > r$ implies $\omega^*(r) \geq \delta \omega^*(r')$. Then the conclusion of the theorem holds.*

Proof. With the notation of Corollary 1 we have

$$\lim_{r \rightarrow \infty} r^{-1} |A_{r,\delta}| = 1.$$

One may well wonder whether the hypothesis of the theorem is unnecessarily stringent. Although it may be that an entirely different sort of hypothesis comes closer to being best possible, we shall now show that for hypotheses of the type stated we cannot do much better. In particular the annuli

$$\lambda^{-1}r \leq |z| \leq \lambda r$$

cannot be replaced by the larger annuli

$$r^{1-\epsilon} \leq |z| \leq r^{1+\epsilon}$$

no matter how small the positive number ϵ is taken to be.

Take any sequences $r_k \nearrow \infty$, $\alpha_k \searrow 0$ ($k = 1, 2, \dots$) and define

$$\omega(z) = \begin{cases} \alpha_k & \text{if } |z| = r_k \text{ with } k \text{ odd,} \\ \alpha_k & \text{if } |z| = r_k \text{ with } k \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus ω is supported on the circles $|z| = r_1, r_3, \dots$ and the points $z = r_2, r_4, \dots$. A set of capacity density zero, while perhaps containing all the points $r_2, r_4, \dots$, can only contain finitely many of the circles $|z| = r_1, r_3, \dots$. This is so because the capacity of a circle equals its radius. It follows easily that if the ratios r_{k+1}/r_k are bounded then the hypothesis of the theorem holds. In fact with λ an upper bound for these ratios the $\liminf$ in question is at least 1.

Consider now the case

$$\begin{aligned} r_k &= \exp \{a^k\} \quad (a > 1), \\ \alpha_k &= \exp \{-\tfrac{1}{2}(\log r_k)^2\}. \end{aligned}$$

The hypothesis of the theorem is no longer satisfied (take $E = \{r_2, r_4, \dots\}$ and let $r \rightarrow \infty$ through the values $r_2, r_4, \dots$) although it would be if we were allowed to use the annuli

$$r^{1/a} \leq |z| \leq r^a.$$

We shall show that the conclusion of the theorem fails to hold.

The function

$$r^n \exp \{-\tfrac{1}{2}(\log r)^2\}$$

attains its maximum when $r = e^n$. In fact

$$(11) \quad r^n \exp \{-\tfrac{1}{2}(\log r)^2\} = \exp \{\tfrac{1}{2}n^2 - \tfrac{1}{2}(\rho - 1)^2 n^2\} \text{ if } r = e^{\rho}.$$

If we write

$$r_k = e^{n^{\rho_k}}$$

then

$$(12) \quad \rho_k = \frac{1}{n} a^k = \exp \{k \log a - \log n\}.$$

Now we can find a sequence of integers such that for n belonging to this sequence the following hold:

(a) The distance from $\log n / \log a$ to its nearest integer $k(n)$ tends to zero as $n \rightarrow \infty$.

(b) This nearest integer $k(n)$ is even.

It follows from property (a) and (12) above that $\rho_{k(n)} \rightarrow 1$, and so from (11) that (for any fixed $\epsilon > 0$)

$$\begin{aligned} \max_r r^n \omega^*(r) &= \max_k r_k^n \alpha_k \geq r_{k(n)}^n \exp \left\{ -\frac{1}{2} (\log r_{k(n)})^2 \right\} \\ &\geq \exp \left\{ \left(\frac{1}{2} - \epsilon \right) n^2 \right\} \end{aligned}$$

for large enough n from our sequence.

Now consider the polynomial

$$P(z) = z^{n-1} (z - r_{k(n)}).$$

Since this polynomial vanishes at $r_{k(n)}$ and $k(n)$ is even it is easy to see that $\max |P|$ is at most the larger of

$$\max_{k > k(n)} 2r_k^n \exp \left\{ -\frac{1}{2} (\log r_k)^2 \right\}$$

and

$$(13) \quad \max_{k < k(n)} 2r_{k(n)} r_k^{n-1} \exp \left\{ -\frac{1}{2} (\log r_k)^2 \right\} \leq \max_{k < k(n)} 2r_{k(n)} r_k^n \exp \left\{ -\frac{1}{2} (\log r_k)^2 \right\}.$$

In the first case we have

$$k \geq k(n) + 1 \geq \frac{\log n}{\log a} + \frac{1}{2}$$

and so

$$r_k = \exp \{a^k\} \geq \exp \{a^{1/2} n\}$$

and (11) gives

$$r_k^n \exp \left\{ -\frac{1}{2} (\log r_k)^2 \right\} \leq \exp \{\alpha n^2\}$$

with

$$\alpha = \frac{1}{2} - \frac{1}{2} (a^{1/2} - 1)^2 < \frac{1}{2}.$$

In the second case we have similarly

$$k \leq k(n) - 1 \leq \frac{\log n}{\log a} - \frac{1}{2}$$

and

$$r_k^n \exp \left\{ -\frac{1}{2} (\log r_k)^2 \right\} \leq \exp \{\beta n^2\}$$

with

$$\beta = \frac{1}{2} - \frac{1}{2}(1 - a^{-1/2})^2 < \frac{1}{2}.$$

(The factor $r_{k(n)}$ in (13) is $\exp \{O(n)\}$ and so is irrelevant.) We have therefore shown that for a certain sequence $n \rightarrow \infty$

$$\limsup_{n \rightarrow \infty} n^{-2} \log M_n < \frac{1}{2}$$

while

$$\liminf_{n \rightarrow \infty} n^{-2} \log \max_r r^n \omega^*(r) \geq \frac{1}{2}.$$

We turn now to the problem of determining the asymptotic behavior of the quantity m_n defined by (4). It is assumed that each $m_n < \infty$ and that μ is not supported on a finite set. We use the notation

$$K_{r_1, r_2} = \{z : r_1 \leq |z| \leq r_2\}$$

and define

$$\mu^*(r) = \mu(|z| \leq r).$$

Theorem 2. Suppose there are numbers $\lambda > 1$ and $\alpha < 1$ such that for any closed set E of capacity density zero

$$\liminf_{r \rightarrow \infty} \frac{\mu([E']_{\lambda^{-1}r, \lambda r})}{\mu(K_{\alpha r, r})} > 0.$$

(In taking the $\liminf$ r is restricted to those values for which the denominator is positive.) Then

$$\log m_n \sim \log \int_0^\infty r^n d\mu^*(r), \quad n \rightarrow \infty.$$

Proof. With

$$\varphi_r(A) = \begin{cases} \mu(K_{\alpha r, r})^{-1} \mu(A) & \text{if } \mu(K_{\alpha r, r}) > 0 \\ 1 & \text{if } \mu(K_{\alpha r, r}) = 0 \end{cases}$$

the hypothesis of the lemma is satisfied. Let δ be as in the conclusion of the lemma.

As in the proof of Theorem 1, if P is monic of degree n and

$$E = \{z : |P(z)| \leq \delta^n |z|^n\}$$

then

$$C([E]_r) \leq \delta r$$

for all r . Therefore

$$\mu([E']_{\lambda^{-1}r, \lambda r}) \geq \delta \mu(K_{\alpha r, r})$$

for $r \geq \delta^{-2}$. Hence for such r

$$\begin{aligned} \int_{K\lambda^{-1}r, \lambda r} |P| d\mu &\geq \int_{[E']_{\lambda^{-1}r, \lambda r}} |P| d\mu \geq \delta^n \lambda^{-n} r^n \mu([E']_{\lambda^{-1}r, \lambda r}) \\ &\geq \delta^{n+1} \lambda^{-n} r^n \mu(K_{\alpha r, r}) \geq \delta^{n+1} \lambda^{-n} \int_{\alpha r}^r \rho^n d\mu^*(\rho). \end{aligned}$$

We now set consecutively $r = \delta^{-2}\alpha^{-1}, \delta^{-2}\alpha^{-2}, \delta^{-2}\alpha^{-3}, \dots$ and add. The right side of the inequality becomes

$$\delta^{n+1} \lambda^{-n} \int_{\delta^{-2}}^{\infty} \rho^n d\mu^*(\rho).$$

As for the left side, note that although the annuli

$$K_{\lambda^{-1}\delta^{-2}\alpha^{-k}, \lambda\delta^{-2}\alpha^{-k}}, \quad k = 1, 2, \dots,$$

need not be disjoint, any point of the plane belongs to at most

$$c = 2 + \frac{\log \lambda^2}{\log \alpha^{-1}}$$

of them. Therefore the left side of the new inequality is at most $c \int |P| d\mu$. We conclude that

$$\int |P| d\mu \geq c^{-1} \delta^{n+1} \lambda^{-n} \int_{\delta^{-2}}^{\infty} \rho^n d\mu^*(\rho).$$

As in the proof of Theorem 1 it is easy to see that because μ has unbounded support

$$\lim_{n \rightarrow \infty} n^{-1} \log \int_{\delta^{-2}}^{\infty} \rho^n d\mu^*(\rho) = \infty.$$

Therefore for each $\epsilon > 0$

$$\log m_n \geq (1 - \epsilon) \log \int_0^{\infty} \rho^n d\mu^*(\rho)$$

for large enough n . Since trivially

$$m_n \leq \int |z^n| d\mu(z) = \int_0^{\infty} \rho^n d\mu^*(\rho)$$

the conclusion of the theorem follows.

The following corollaries are analogous to those following Theorem 1. We denote the derivative of μ^* (which exists almost everywhere) by ν .

Corollary 1. *Define*

$$B_{r, \delta} = \{\rho \leq r : \nu(\rho) \geq \delta [\mu^*(r) - \mu^*(r-1)]\}$$

and suppose that for some $\delta > 0$

$$\liminf_{r \rightarrow \infty} r^{-1} |B_{r, \delta}| > 0.$$

Then the conclusion of the theorem holds.

Proof. Denote the $\liminf$ by 3η . For each r there is a number s satisfying

$$\eta r \leq s \leq r,$$

$$\mu^*(r) - \mu^*(\eta r) \leq r[\mu^*(s) - \mu^*(s-1)].$$

This is so simply because the interval $[\eta r, r]$ can be covered by at most r non-overlapping intervals of the form $[s-1, s]$.

Let E have capacity density zero. Then as in the proof of Corollary 1 to Theorem 1

$$\lim_{r \rightarrow \infty} r^{-1} |E^* \cap [0, r]| = 0.$$

Consequently if r is sufficiently large

$$|\{\rho \leq s : \rho \notin E^*, v(\rho) \geq \delta[\mu^*(s) - \mu^*(s-1)]\}| \geq |B_{s,s}| - |E^* \cap [0, s]| > 2\eta s$$

and so also

$$|\{\rho \in [\eta s, s] : \rho \notin E^*, v(\rho) \geq \delta[\mu^*(s) - \mu^*(s-1)]\}| > \eta s.$$

Call S the set in curly brackets above. Then

$$S \subset [\eta^2 r, r] \cap E^{*'}.$$

so

$$\begin{aligned} \mu([E']_{\eta^2 r, r}) &\geq \int_S d\mu^* \geq \int_S v(\rho) d\rho \geq |S| \delta[\mu^*(s) - \mu^*(s-1)] \\ &\geq \eta s \delta r^{-1} [\mu^*(r) - \mu^*(\eta r)] \geq \eta^2 \delta \mu(K_{2\eta r, r}). \end{aligned}$$

Thus the hypothesis of the theorem is satisfied.

Corollary 2. Suppose μ^* is absolutely continuous and that there is a $\delta > 0$ such that (for sufficiently large r) $r' > r$ implies $v(r) \geq \delta v(r')$. Then the conclusion of the theorem holds.

Proof. Since

$$\mu^*(r) - \mu^*(r-1) = \int_{r-1}^r v(\rho) d\rho$$

the set $B_{r,s}$ contains $[r_0, r-1]$ for an appropriate fixed r_0 . Therefore

$$\lim_{r \rightarrow \infty} r^{-1} |B_{r,s}| = 1.$$

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