*An Integral Identity Due to Ramanujan which Occurs in Neutron Transport Theory**

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1. **Introduction.** The large t asymptotic behavior of

$$(1) \quad N(t) = \int_0^\infty \frac{e^{-tx} dx}{x\{\pi^2 + (\log x)^2\}}$$

and a related family of functions is described incorrectly in the well-known Bateman Manuscript Project on higher transcendental functions. $N(t)$ is related to the special function $\nu(t)$ through an identity due to Ramanujan [1],

$$(2) \quad \nu(t) = \int_0^\infty \frac{t^x dx}{\Gamma(x+1)} = e^t - N(t).$$

The results stated in Bateman are incorrectly attributed to W. B. Ford [3]. They were apparently obtained by overlooking a restriction on a related theorem due to Ford. The correct asymptotic behavior is $N(t) = 1/\log t + O(1/(\log t)^2)$. Furthermore, the related functions $\nu(t, \alpha) - e^t$ differ radically from the description given in Bateman. All of these functions diverge when $\alpha > 0$ as $t \rightarrow \infty$ rather than approach zero as stated there.

Direct numerical evaluations of $N(t)$ have been performed which verify the results of our analysis. The incorrect description of $N(t)$ mentioned above gave physically unreasonable results for a problem in neutron transport theory which stimulated our original interest in this function and Ramanujan's identity.

The Laplace transform method is often used when investigating time-dependent problems of neutron transport theory. Typical difficulties which arise are the inversion of the related contour integrals and the determination of the asymptotic behavior of solutions for large times. The investigation of the time-dependence of a fast neutron field (with simultaneous fission regeneration and

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slowing-down) leads to an inverse Laplace transformation which reduces to the real integral given in (1). The asymptotic behavior of $N(t)$ represents the time-dependent amplitude of a decaying fast-neutron field following a burst of neutrons at time zero. It corresponds to a physical situation which arises when the dimensions of the system supporting the neutron field are made sufficiently small so that pure exponential time-decay no longer exists. That is, no isolated pole of the Laplace transform occurs, but rather the pole is coincident with a branch point.

$\nu(t)$, as defined in (2), is a classical higher transcendental function which occurs frequently in operational calculus because of the simplicity of its Laplace transform [2] which is

$$(3) \quad \mathfrak{L}\{\nu(t)\} = \int_0^\infty e^{-st} \nu(t) dt = \frac{1}{s \log s}.$$

Furthermore, the asymptotic behavior of a more general class of functions to which $\nu(t)$ belongs

$$(4) \quad \nu(t, \alpha) = \int_0^\infty \frac{t^{\alpha+x}}{\Gamma(\alpha+x+1)} dx, \quad \alpha > -1$$

is also given incorrectly in [2]. Here, the corresponding error is even more serious. Reference [2] (see p. 221) states that

$$\nu(t) = e^t + O(|t|^{-N}), \quad t \rightarrow \infty, \quad |\arg t| \leq \frac{\pi}{2}$$

and

$$\nu(t, \alpha) = e^t + O(|t|^{\alpha-N}), \quad t \rightarrow \infty, \quad |\arg t| \leq \frac{\pi}{2}$$

for any positive integer N . However, we will show below that

$$\nu(t) = e^t - \frac{1}{\log t} + O\left(\frac{1}{(\log t)^2}\right), \quad t \rightarrow \infty, \quad |\arg t| = 0$$

and

$$\nu(t, \alpha) = e^t - \frac{t^\alpha}{\Gamma(\alpha+1) \log t} + O\left(\frac{\log(\alpha+1)}{\Gamma(\alpha+1)} \frac{t^\alpha}{(\log t)^2}\right), \quad t \rightarrow \infty, \quad |\arg t| = 0.$$

Apparently, the error in reference [2] was made because the limitations in W. B. Ford's theorem (see [3], p. 265, especially eq. (10)) were overlooked.

2. Asymptotic behavior of $\nu(t)$. Ramanujan's identity (2) enables us to restrict our attention to the integral $N(t)$ which is the physical quantity appearing in the fast neutron problem. It is convenient to introduce two additional equivalent expressions

$$(5a) \quad N(t) = \int_{-\pi/2}^{\pi/2} \exp(-te^{\pi \tan z}) dz$$

and

$$(5b) \quad N(t) = \int_{-\infty}^{+\infty} \frac{e^{-te^y}}{\pi^2 + y^2} dy.$$

The former is suitable for numerical calculations (see Section 4, Table I), and the latter is a convenient starting point for the asymptotic evaluation of $N(t)$.

We begin by dividing $N(t)$ into two parts

$$(6) \quad N(t) = N_A(t) + N_B(t)$$

where

$$(6a) \quad N_A(t) = \int_{-\infty}^0 \frac{e^{-te^y}}{\pi^2 + y^2} dy$$

and

$$(6b) \quad N_B(t) = \int_0^{\infty} \frac{e^{-te^y}}{\pi^2 + y^2} dy.$$

$N_B(t)$ can be easily shown to be at most $O(e^{-t}/t)$ since

$$(7) \quad N_B(t) \leq \frac{1}{\pi^2} \int_0^{\infty} e^{-te^y} dy \leq \frac{1}{\pi^2} \int_0^{\infty} e^{-t(1+y)} dy = \frac{1}{\pi^2} \frac{e^{-t}}{t}.$$

Thus, we may limit our investigation to $N_A(t)$ which we can rewrite as

$$(8) \quad N_A(t) = \int_0^1 \frac{e^{-tx}}{x\{\pi^2 + (\log x)^2\}} dx.$$

Starting from the Laplace transform

$$(9) \quad \frac{1}{\pi^2 + |y|^2} = \int_0^{\infty} e^{-|y|\alpha} \frac{\sin \pi \alpha}{\pi} d\alpha,$$

we will establish, in Section 5, the following useful integral representation for $1/(\pi^2 + (\log x)^2)$,

$$(10) \quad \frac{1}{\pi^2 + (\log x)^2} = \int_0^M \frac{\sin \pi \alpha}{\pi} x^{\alpha} d\alpha + \frac{(-1)^M x^M}{1+x} \int_0^1 \frac{\sin \pi \alpha}{\pi} x^{\alpha} d\alpha,$$

which is valid for $0 \leq x \leq 1$ and for each integer $M \geq 1$. Combining identity (10) with (8) and (6) yields

$$(11) \quad N(t) = \int_0^1 dx e^{-tx} \int_0^M d\alpha \frac{\sin \pi \alpha}{\pi} x^{\alpha-1} + (-1)^M \int_0^1 dx \frac{x^M}{1+x} e^{-tx} \int_0^1 d\alpha \frac{\sin \pi \alpha}{\pi} x^{\alpha-1} + O\left(\frac{e^{-t}}{t}\right),$$

and changing orders of integration leads to

$$(12) \quad N(t) = \int_0^M d\alpha \frac{\sin \pi \alpha}{\pi} \int_0^1 dx e^{-tx} x^{\alpha-1} + (-1)^M \int_0^1 d\alpha \frac{\sin \pi \alpha}{\pi} \int_0^1 dx e^{-tx} \frac{x^{M+\alpha-1}}{1+x} + O\left(\frac{e^{-t}}{t}\right).$$

We first consider the quantity

$$\begin{aligned}
 (13) \quad \varepsilon_M(t) &= \int_0^1 d\alpha \frac{\sin \pi \alpha}{\pi} \int_0^1 dx e^{-tx} \frac{x^{M+\alpha-1}}{1+x} \\
 &\leq \int_0^1 d\alpha \frac{\sin \pi \alpha}{\pi} \int_0^1 dx e^{-tx} x^{M-1} \\
 &= \int_0^1 d\alpha \frac{\sin \pi \alpha}{\pi} \left\{ \frac{(M-1)!}{t^M} - e^{-t} \sum_{r=0}^{M-1} \frac{(M-1)!}{(M-1-r)!} \frac{1}{t^{r+1}} \right\} \\
 &\leq \frac{2}{\pi^2} \frac{(M-1)!}{t^M} + O\left(\frac{e^{-t}}{t}\right).
 \end{aligned}$$

Therefore (12) becomes

$$(14) \quad N(t) = \int_0^M d\alpha \frac{\sin \pi \alpha}{\pi} \int_0^1 dx e^{-tx} x^{\alpha-1} + O\left(\frac{(M-1)!}{t^M}\right).$$

We can rewrite (14) in terms of the incomplete gamma function [4] as

$$(15) \quad N(t) = \int_0^M \frac{\sin \pi \alpha}{\pi} t^{-\alpha} \gamma(\alpha, t) d\alpha + O\left(\frac{(M-1)!}{t^M}\right).$$

Using the asymptotic representation of $\gamma(\alpha, t)$ for large t , and bounded α [4],

$$\begin{aligned}
 (16) \quad \gamma(\alpha, t) &\sim \Gamma(\alpha) - t^{\alpha-1} e^{-t} \left[1 + \frac{\alpha-1}{t} + \frac{(\alpha-1)(\alpha-2)}{t^2} + \dots \right], \\
 &t \rightarrow \infty, \quad |\arg t| < \frac{3\pi}{2},
 \end{aligned}$$

we obtain

$$(17) \quad N(t) = \int_0^M \frac{\sin \pi \alpha}{\pi} t^{-\alpha} \Gamma(\alpha) d\alpha + O\left(\frac{(M-1)!}{t^M}\right)$$

$$(17a) \quad = \int_0^M \frac{t^{-\alpha}}{\Gamma(1-\alpha)} d\alpha + O\left(\frac{(M-1)!}{t^M}\right).$$

In particular for $M = 1$, we have

$$(18) \quad N(t) = \int_0^1 \frac{t^{-\alpha}}{\Gamma(1-\alpha)} d\alpha + O\left(\frac{1}{t}\right).$$

An exact series expansion of the integral in (17a)

$$\begin{aligned}
 (19) \quad \int_0^M \frac{t^{-\alpha}}{\Gamma(1-\alpha)} d\alpha &= \int_0^M \frac{e^{-\alpha \log t}}{\Gamma(1-\alpha)} d\alpha \\
 &= \frac{1}{\log t} \left\{ \sum_{m=0}^{\infty} \frac{1}{(\log t)^m} \frac{d^m}{dz^m} \frac{1}{\Gamma(1-z)} \Big|_{z=0} \right\} \\
 &\quad - \frac{1}{t^M \log t} \left\{ \sum_{m=0}^{\infty} \frac{1}{(\log t)^m} \frac{d^m}{dz^m} \frac{1}{\Gamma(1-z)} \Big|_{z=M} \right\}
 \end{aligned}$$

is obtained by successive integrations by parts. The convergence of this ex-

pansion will be proven in Section 6. Using expansion (19) in (17a) we obtain

$$(20) \quad N(t) = \frac{1}{\log t} \sum_{m=0}^{\infty} \frac{1}{(\log t)^m} \frac{d^m}{dz^m} \frac{1}{\Gamma(1-z)} \Big|_{z=0} + O\left(\frac{(M-1)!}{t^M}\right) \\ = \frac{1}{\log t} \sum_{m=0}^{\infty} \frac{(-1)^m}{(\log t)^m} \frac{d^m}{dz^m} \frac{1}{\Gamma(z)} \Big|_{z=1} + O\left(\frac{(M-1)!}{t^M}\right)$$

which is our main result.

It follows from (20) and (2) that

$$(21) \quad \nu(t) = e^t - \frac{1}{\log t} \left\{ \sum_{m=0}^{\infty} \frac{(-1)^m}{(\log t)^m} \frac{d^m}{dz^m} \frac{1}{\Gamma(z)} \Big|_{z=1} \right\} + O\left(\frac{(M-1)!}{t^M}\right), \\ t \rightarrow \infty, \quad |\arg t| = 0$$

for any integer $M \geq 1$, and in particular

$$(22) \quad \nu(t) = e^t - \frac{1}{\log t} + O\left(\frac{1}{(\log t)^2}\right), \quad t \rightarrow \infty, \quad |\arg t| = 0.$$

Q.E.D.

3. Asymptotic behavior of $\nu(t, \alpha)$. The expression for $\nu(t, \alpha)$ given in (4) above can be rewritten (by simply changing the integration variable) as

$$(23) \quad \nu(t, \alpha) = \int_{\alpha}^{\infty} \frac{t^y dy}{\Gamma(1+y)} = \nu(t) - \int_0^{\alpha} \frac{t^y dy}{\Gamma(1+y)}.$$

Repeated integration by parts yields

$$(24) \quad \nu(t, \alpha) = \nu(t) - \frac{t^{\alpha}}{\log t} \sum_{m=0}^M \frac{(-1)^m}{(\log t)^m} \frac{d^m}{dz^m} \frac{1}{\Gamma(1+z)} \Big|_{z=\alpha} \\ + \frac{1}{\log t} \sum_{m=0}^M \frac{(-1)^m}{(\log t)^m} \frac{d^m}{dz^m} \frac{1}{\Gamma(z)} \Big|_{z=1} - \frac{(-1)^{M+1}}{(\log t)^{M+1}} \int_0^{\alpha} t^y \frac{d^{M+1}}{dy^{M+1}} \frac{1}{\Gamma(1+y)} dy.$$

Since the two sums in (24) represent a convergent expansion for the integral in (23) (The proof follows from the fact that $1/\Gamma(z)$ is an entire function, and is demonstrated in a similar manner as in Section 6) we may write

$$(25) \quad \nu(t, \alpha) = \nu(t) - \frac{t^{\alpha}}{\log t} \sum_{m=0}^{\infty} \frac{(-1)^m}{(\log t)^m} \frac{d^m}{dz^m} \frac{1}{\Gamma(1+z)} \Big|_{z=\alpha} \\ + \frac{1}{\log t} \sum_{m=0}^{\infty} \frac{(-1)^m}{(\log t)^m} \frac{d^m}{dz^m} \frac{1}{\Gamma(z)} \Big|_{z=1} \quad \text{for } t > 1 \text{ and } \alpha > -1.$$

Noting that the last term in (25) is just the asymptotic representation for $N(t)$ (see 20 above) and using the identity

$$\nu(t) = e^t - N(t)$$

due to Ramanujan [1] we have the asymptotic representation

$$(26) \quad \nu(t, \alpha) = e^t - \frac{t^{\alpha}}{\log t} \sum_{m=0}^{\infty} \frac{(-1)^m}{(\log t)^m} \frac{d^m}{dz^m} \frac{1}{\Gamma(1+z)} \Big|_{z=\alpha} + O\left(\frac{(M-1)!}{t^M}\right),$$

for any integer $M \geq 1$, $\alpha > -1$, $t \rightarrow \infty$, $\arg t = 0$, and in particular

$$(27) \quad \nu(t, \alpha) = e^t - \frac{t^\alpha}{\log t} + O\left(\frac{t^\alpha}{(\log t)^2}\right), \quad \text{for } t \rightarrow \infty, \quad \arg t = 0. \quad \text{Q.E.D.}$$

Relations (26) and (27) contrast radically with the incorrect assertion of [2] where it is stated that

$$(28) \quad \nu(t, \alpha) - e^t$$

dies down more rapidly than any power of t . Indeed, as we have shown above, the expression in (28) actually *diverges* as t goes to plus infinity when α is greater than zero!

(26), of course, reduces to (21) when α is taken as zero.

4. Numerical verification. Direct numerical integrations of an expression for $N(t)$, given by (5a), have been performed as verification of the asymptotic expansion for $\nu(t)$ derived in Section 2 above. The results of these computations are compared with successive terms of the asymptotic expansion for $N(t)$, given by (20), and are tabulated in Table I below.

*TABLE I

t	$N(t)$ Eq. (5a)	$\mathfrak{N}(t)$ Eq. (18)	$S_0(t)$	$S_1(t)$	$S_2(t)$	$S_3(t)$	$S_4(t)$
10^1	.276334	.282166	.434295	.325425	.217975	.226941	.228692
10^2	.179176	.179469	.217147	.189930	.176499	.177059	.178989
10^3	.129083	.129099	.144765	.132668	.128689	.128799	.129053
10^4	.100192	.100193	.108574	.101769	.100090	.100125	.100186
10^5	.0816804	.0816805	.0868589	.0825041	.0816445	.0816589	.0816786
10^6	.0688763	.0688763	.0723824	.0693583	.0688608	.0688677	.0688757
10^7	.0595146	.0595146	.0620421	.0598202	.0595070	.0595107	.0595144
10^8	.0523801	.0523801	.0542868	.0525857	.0523759	.0523781	.0523799
10^9	.0467660	.0467660	.0482549	.0469109	.0467635	.0467648	.0467659
10^{10}	.0422348	.0422348	.0434295	.0422408	.0422333	.0422342	.0422348
10^{11}	.0385019	.0385019	.0394813	.0385816	.0385008	.0385015	.0385018
10^{12}	.0353737	.0353737	.0361912	.0354352	.0353730	.0353734	.0353737

$$* \text{ Note. } S_N(t) = \frac{1}{\log t} \sum_{m=0}^N \frac{(-1)^m}{(\log t)^m} \frac{d^m}{dz^m} \frac{1}{\Gamma(z)} \Big|_{z=1} \quad (\text{from } (20))$$

$$\left(\text{i.e., } S_0(t) = \frac{1}{\log t}; \quad S_1(t) = S_0(t) - \frac{0.5772156649}{(\log t)^2}; \right.$$

$$S_2(t) = S_1(t) - \frac{1.311756143}{(\log t)^3}; \quad S_3(t) = S_2(t) + \frac{0.252015816}{(\log t)^4};$$

$$S_4(t) = S_3(t) + \frac{3.996926675}{(\log t)^5} \Big).$$

5. A basic identity. We wish to prove the identity

$$(29) \quad \frac{1}{\pi^2 + (\log x)^2} = \int_0^M \frac{\sin \pi \alpha}{\pi} x^\alpha d\alpha + (-1)^M \frac{x^M}{1+x} \int_0^1 \frac{\sin \pi \alpha}{\pi} x^\alpha d\alpha$$

for $0 \leq x \leq 1$ and any integer $M \geq 1$ as stated in (10) of the text. We begin with the Laplace transform of $\sin \pi \beta / \pi$

$$(30) \quad \frac{1}{\pi^2 + |y|^2} = \int_0^\infty e^{-|y|\beta} \frac{\sin \pi \beta}{\pi} d\beta$$

which is valid for $|y| \neq 0$. For $-\infty \leq y < 0$, $|y| = -y$ and (30) becomes

$$(31) \quad \frac{1}{\pi^2 + y^2} = \int_0^\infty e^{y\beta} \frac{\sin \pi \beta}{\pi} d\beta.$$

Changing variables, we have

$$(32) \quad \frac{1}{\pi^2 + (\log x)^2} = \int_0^\infty x^\beta \frac{\sin \pi \beta}{\pi} d\beta, \quad 0 \leq x < 1.$$

This integral can be broken into two parts

$$(33) \quad \frac{1}{\pi^2 + (\log x)^2} = \int_0^M \frac{\sin \pi \beta}{\pi} x^\beta d\beta + \int_M^\infty \frac{\sin \pi \beta}{\pi} x^\beta d\beta, \\ 0 \leq x < 1, \quad 0 \leq M < \infty,$$

the latter of which can be decomposed into a sum of integrals over finite intervals. Thus, we have

$$(34) \quad \frac{1}{\pi^2 + (\log x)^2} = \int_0^M \frac{\sin \pi \beta}{\pi} x^\beta d\beta + \sum_{n=M}^\infty \int_n^{n+1} \frac{\sin \pi \beta}{\pi} x^\beta d\beta \\ = \int_0^M \frac{\sin \pi \beta}{\pi} x^\beta d\beta + \sum_{n=M}^\infty (-1)^n x^n \int_0^1 \frac{\sin \pi \alpha}{\pi} x^\alpha d\alpha \\ = \int_0^M \frac{\sin \pi \beta}{\pi} x^\beta d\beta + (-1)^M x^M \left\{ \sum_{n=0}^\infty (-1)^n x^n \right\} \int_0^1 \frac{\sin \pi \alpha}{\pi} x^\alpha d\alpha; \\ \frac{1}{\pi^2 + (\log x)^2} = \int_0^M \frac{\sin \pi \beta}{\pi} x^\beta d\beta + \frac{(-1)^M x^M}{1+x} \int_0^1 \frac{\sin \pi \alpha}{\pi} x^\alpha d\alpha, \quad 0 \leq x < 1,$$

which is the identity we seek. The case $x = 1$ can be trivially verified separately. Thus, identity (29) is valid for $0 \leq x \leq 1$. The identity is also valid when $M = 0$ in which case it can be rewritten as

$$(35) \quad \frac{1}{\pi^2 + (\log x)^2} = \frac{1}{1+x} \int_0^1 \frac{\sin \pi \alpha}{\pi} x^\alpha d\alpha.$$

6. Convergence of expansion. We wish to prove the convergence of the expansion

$$(36) \quad \int_0^M \frac{t^{-\alpha}}{\Gamma(1-\alpha)} d\alpha = \frac{1}{\log t} \left\{ \sum_{m=0}^{\infty} \frac{1}{(\log t)^m} \frac{d^m}{dz^m} \frac{1}{\Gamma(1-z)} \Big|_{z=0} \right\} \\ - \frac{1}{t^M \log t} \left\{ \sum_{m=0}^{\infty} \frac{1}{(\log t)^m} \frac{d^m}{dz^m} \frac{1}{\Gamma(1-z)} \Big|_{z=M} \right\}$$

(given by (19) of the text) for $t > 1$. We rewrite the integral

$$(37) \quad \int_0^M \frac{t^{-\alpha}}{\Gamma(1-\alpha)} d\alpha = \int_0^M \frac{e^{-\alpha \log t}}{\Gamma(1-\alpha)} d\alpha$$

and note that $1/\Gamma(z)$ is an entire function with a Taylor series expansion absolutely convergent in the whole complex plane

$$(38) \quad \frac{1}{\Gamma(z)} = \sum_{k=0}^{\infty} c_k z^k.$$

Thus (37) becomes

$$(39) \quad \int_0^M \frac{e^{-\alpha \log t}}{\Gamma(1-\alpha)} d\alpha = \int_0^M e^{-\alpha \log t} \left\{ \sum_{k=0}^{\infty} c_k (1-\alpha)^k \right\} d\alpha.$$

Since $0 \leq e^{-\alpha \log t} \leq 1$, $t \geq 1$, $0 \leq \alpha \leq M$, and since the expansion (38) is absolutely convergent, we can invert the order of integration and summation

$$(40) \quad \int_0^M \frac{e^{-\alpha \log t}}{\Gamma(1-\alpha)} d\alpha = \sum_{k=0}^{\infty} c_k \int_0^M e^{-\alpha \log t} (1-\alpha)^k d\alpha.$$

Performing successive integrations by parts in each integral we obtain

$$(41) \quad \int_0^M \frac{t^{-\alpha}}{\Gamma(1-\alpha)} d\alpha \\ = \sum_{k=0}^{\infty} c_k \left\{ \sum_{n=0}^k \frac{1}{(\log t)^{n+1}} \left[\frac{d^n}{dz^n} (1-z)^k \Big|_{z=0} - \frac{1}{t^M} \frac{d^n}{dz^n} (1-z)^k \Big|_{z=M} \right] \right\}.$$

Because of the absolute convergence of (40) which follows from that of (38), expansion (41) is absolutely convergent and can be rewritten as

$$(42) \quad \int_0^M \frac{t^{-\alpha}}{\Gamma(1-\alpha)} d\alpha = \frac{1}{\log t} \sum_{k=0}^{\infty} c_k \left\{ \sum_{n=0}^{\infty} \frac{1}{(\log t)^n} \frac{d^n}{dz^n} (1-z)^k \Big|_{z=0} \right\} \\ - \frac{1}{t^M \log t} \sum_{k=0}^{\infty} c_k \left\{ \sum_{n=0}^{\infty} \frac{1}{(\log t)^n} \frac{d^n}{dz^n} (1-z)^k \Big|_{z=M} \right\}.$$

(For $n > k$ all terms in the inner summations are identically zero.)

$$(43) \quad \int_0^M \frac{t^{-\alpha}}{\Gamma(1-\alpha)} d\alpha = \frac{1}{\log t} \sum_{n=0}^{\infty} \frac{1}{(\log t)^n} \left\{ \sum_{k=0}^{\infty} c_k \frac{d^n}{dz^n} (1-z)^k \Big|_{z=0} \right\} \\ - \frac{1}{t^M \log t} \sum_{n=0}^{\infty} \frac{1}{(\log t)^n} \left\{ \sum_{k=0}^{\infty} c_k \frac{d^n}{dz^n} (1-z)^k \Big|_{z=M} \right\}.$$

Finally, since (38) is the Taylor expansion of an entire function, (43) becomes

$$(44) \quad \int_0^M \frac{t^{-\alpha}}{\Gamma(1-\alpha)} d\alpha = \frac{1}{\log t} \sum_{n=0}^{\infty} \frac{1}{(\log t)^n} \frac{d^n}{dz^n} \frac{1}{\Gamma(1-z)} \Big|_{z=0} - \frac{1}{t^M \log t} \sum_{n=0}^{\infty} \frac{1}{(\log t)^n} \frac{d^n}{dz^n} \frac{1}{\Gamma(1-z)} \Big|_{z=-M}. \quad \text{Q.E.D.}$$

7. Alternate proof. (See Acknowledgment.) The asymptotic behavior of $N_A(t)$, derived in Section 2 above, can also be obtained using the representation

$$(45) \quad \frac{1}{s^2 + \pi^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\pi|x|} e^{ixs} dx$$

from which it follows that

$$(46) \quad \frac{d}{dt} N_A(t) = -\frac{1}{2\pi t} \int_{-\infty}^{\infty} e^{-ix \log t} e^{-\pi|x|} f(x, t) dx$$

where

$$(47) \quad f(x, t) = \int_0^t e^{-y} e^{ix \log y} dy.$$

Integrating by parts in (46) twice, rearranging and integrating by parts a third time yields

$$(48) \quad \left[\frac{d}{dt} N_A(t) \right] \left[1 + \frac{\pi^2}{(\log t)^2} \right] = -\frac{1}{t(\log t)^2} \left[f(0, t) + O\left(\frac{1}{\log t}\right) \right].$$

From (48) it follows (because $f(0, t) = 1 - e^{-t}$) that

$$(49) \quad N_A(t) \sim \frac{1}{\log t} + O\left(\frac{1}{(\log t)^2}\right).$$

However to obtain higher order terms in the asymptotic series for $N_A(t)$ it is necessary to integrate by parts several times in (46) to obtain

$$(50) \quad \left[\frac{d}{dt} N_A(t) \right] \left[1 - \frac{\pi^4}{(\log t)^4} \right] = -\frac{f(0, t)}{t(\log t)^2} - \frac{2i \frac{\partial f(0, t)}{\partial x}}{t(\log t)^3} + \frac{\pi^2 f(0, t) + 3 \frac{\partial^2 f(0, t)}{\partial x^2}}{t(\log t)^4} + O\left(\frac{1}{t(\log t)^5}\right),$$

from which it follows that

$$(51) \quad N_A(t) \sim \frac{f(0, t)}{\log t} + \frac{i \frac{\partial f(0, t)}{\partial x}}{(\log t)^2} - \frac{\pi^2 f(0, t) + 3 \frac{\partial^2 f(0, t)}{\partial x^2}}{(\log t)^3} + O\left(\frac{1}{(\log t)^4}\right).$$

The above method can be applied to obtain the general n term expansion; unfortunately, however, the evaluation of the leading terms (constants) of the

$\partial^n f(0, t)/\partial x^n$ when $n > 0$ is extremely cumbersome; in addition the error estimates in the expansion of the $\partial^n f(0, t)/\partial x^n$ are also laborious to obtain. Further, the general term in the expansion of $N_A(t)$ is not simple to obtain. However, this approach is a useful and extremely straightforward method of obtaining the leading term in the asymptotic behavior of $N_A(t)$.

8. Conclusion. The actual large t asymptotic behavior of $\nu(t, \alpha) - e^t$ and $\nu(t) - e^t$ contrasts radically with the incorrect description given in reference [2]. There the asymptotic behaviors of the above functions are given incorrectly, and are erroneously attributed to W. B. Ford [3]. Apparently, an essential restriction stated on page 265 (see [3]) of Ford's manuscript was overlooked by these authors. The results presented here are consistent with the above mentioned restriction given by Ford. In addition independent machine calculations give data which support our conclusions.

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