Inequalities Concerning the Modules of Curve Families

WILLIAM R. DERRICK

Communicated by G. SPRINGER

1. Introduction. Let A be the topological image of an n -cube in n -dimensional Euclidean space and v, w be nonnegative Lebesgue-integrable real-valued functions defined on A . Then

$$(*) \quad \int_{\text{Int } A} vw \, dH^n \geq \left(\inf_{\Gamma} \int_c v \, dH^1 \right) \left(\inf_{\Sigma} \int_Q w \, dH^{n-1} \right),$$

where c is an arc joining, and Q is a set separating, the images of a fixed pair of opposite sides of the n -cube, and H^k is the k -dimensional Hausdorff measure in n -space. (*) was conjectured, for $n = 3$, by R. J. Duffin [3] based on an analogous inequality for networks. Using (*) one obtains, for arbitrary $p > 1$, the module inequality

$$M_p(\Gamma)M_p(\Sigma) \geq 1,$$

generalizing the identity

$$M_n(\Gamma)M_n(\Sigma) = 1,$$

due to Gehring [5] (for $n = 3$) and Ziemer [6].

2. Preliminaries.

2.1. Definitions. Let E^n denote Euclidean n -space, $\text{Bdry } A$, $\text{Int } A$, and $\text{Diam } A$, the *boundary*, *interior*, and *diameter* of a set A in E^n respectively, $S_\epsilon(x) = \{y \text{ in } E^n : |x - y| < \epsilon\}$, $I^n = [0, 1]^n$ the closed unit n -cube, and r_{xy} stands for the straight line segment from x to y .

A function $f : E^n \rightarrow E^m$ is *approximately differentiable* at x if for some linear transformation $D_{ap}f(x) : E^n \rightarrow E^m$ and for every $r > 0$, the point x is a point of density of the set

$$\{x : |f(x + h) - f(x) - D_{ap}f(x)(h)| \leq r |h|\}.$$

The *Jacobian* of f , Jf , is defined on the set of points of approximate differentiability of f by

$$Jf(x) = \Delta(D_{av}f(x)),$$

where, for any $m \times n$ -matrix A , we have

$$\Delta(A) = \left\{ \sum_y (\det A_y)^2 \right\}^{1/2},$$

with y an m -tuple $(y_1, \dots, y_m)$ of integers satisfying $1 \leq y_1 < \dots < y_m \leq n$, A_y the minor of A obtained by taking the m columns whose indices are elements of y , and $\det A_y$ the *determinant* of A_y . If $m = 1$, then the *gradient* of f satisfies

$$\text{grad } f(x) = D_{av}f(x)$$

at all points of approximate differentiability and

$$|\text{grad } f(x)| = Jf(x).$$

Let $H^k(A)$ be the *Hausdorff k -dimensional measure* of A in E^n which equals the limit, as $r \rightarrow 0^+$, of

$$\inf_F \sum_{S \in F} 2^{-k} \alpha(k) (\text{diam } S)^k$$

over all countable coverings F of A such that $\text{diam } S < r$ for S in F , with

$$\alpha(k) = \Gamma(\frac{1}{2})^k / \Gamma(1 + (k/2)), \quad k \geq 0.$$

Then $H^0(A)$, $H^1(A)$, $H^n(A)$ represent the cardinality, length, and Lebesgue n -dimensional measure of A (see [1], 2.8, 2.12, 2.17) respectively.

If A is an H^n -measurable subset of E^n , then $\mathfrak{L}^p(A)$ will denote those functions f for which $|f|^p$ is H^n -integrable over A and $\|f\|_p$ will be its $\mathfrak{L}^p$ norm.

2.2. Theorem. Let $f: E^n \rightarrow E^k$, $n \geq k$, be Lipschitzian and g be an H^n -integrable function on E^n . Then

$$\int_{E^n} g(x) Jf(x) dH^n x = \int_{E^k} \int_{f^{-1}(\{y\})} g(x) dH^{n-k} x dH^k y.$$

Proof. See [4, 3.1]. Clearly the theorem holds on any H^n -measurable subset of E^n .

2.3. Definition. Let A be an arcwise connected subset of E^n and $v: E^n \rightarrow E^1$ be a nonnegative Borel measurable function. Then the v -weighted distance between the points x and y in A is given by

$$\lambda_v(x, y) = \inf \int_c v(x) dH^1 x,$$

where the infimum is taken over all arcs, c joining x to y which lie entirely in A . The v -weighted distance between the subsets A' , A'' of A is given by

$$\lambda_*(A', A'') = \inf \lambda_*(x', x''),$$

for all x' in A' , x'' in A'' .

2.4. Definition. Let Γ be a family of curves in E^n . The p -module of Γ is given by

$$M_p(\Gamma) = \inf_v \frac{\int_{E^n} v^p dH^n}{\left(\inf_{\Gamma} \int_c v dH^1 \right)^p},$$

the infimum being taken over all admissible functions v , that is, all nonnegative Borel-measurable real valued functions for which the quotient has meaning.

Let A_1, A_2 be disjoint subsets of E^n . A set Q separates A_1 from A_2 in E^n provided

$$E^n - Q = B_1 \cup B_2,$$

where B_1, B_2 are disjoint open sets such that $A_i \subset B_i$, $i = 1, 2$. Let Σ be the family of sets separating A_1 from A_2 . Then the p -module of Σ is defined by

$$M_p(\Sigma) = \inf_w \frac{\left(\int_{E^n} w^p dH^n \right)^{p-1}}{\left(\inf_{\Sigma} \int_Q w^{p-1} dH^{n-1} \right)^p},$$

where w is admissible. Modules for families on a subset A of E^n are defined by integrating only over A in the numerator.

3. Duffin's Inequality. Throughout this section A will be a topological cylinder, that is, there exists a homeomorphism g mapping I^n onto A . Let $A' = g(I^{n-1} \times \{0\})$, $A'' = g(I^{n-1} \times \{1\})$. The results obtained are also valid when A is a bounded open connected subset of E^n and A', A'' are disjoint compact subsets of A , the techniques of proof being identical.

3.1. Theorem. Let the nonnegative Borel-measurable function $v : A \rightarrow E^1$ belong to $\mathcal{L}^1(\text{Int } A)$. Then $\lambda_*(A', A'') < \infty$.

Proof. The proof is similar to [2, 4.1].

3.2. Theorem. Let v, w be nonnegative Borel measurable functions belonging to $\mathcal{L}^1(\text{Int } A)$. Then

$$\int_{\text{Int } A} vw dH^n \geq \left(\inf_c \int_c v dH^1 \right) \left(\inf_Q \int_Q w dH^{n-1} \right),$$

where c is an arc joining and Q is a set separating A' and A'' .

Proof. To begin assume v is positive and continuous on A . Define $A'_m = g([1/m, (m-1)/m]^{n-1} \times \{1/m\})$, $A''_m = g([1/m, (m-1)/m]^{n-1} \times \{(m-1)/m\})$, and let $f_m : \text{Int } A \rightarrow E^1$ be given by

$$f_m(x) = \min(\lambda_m, \lambda_m(x)),$$

where $\lambda_m = \lambda_v(A'_m, A''_m)$, $\lambda_m(x) = \lambda_v(A'_m, x)$. Clearly $f_m(\text{Int } A) = [0, \lambda_m]$.

For x in $\text{Int } A$, let $\epsilon(x) = \text{dist}(x, \text{Bdry } A)/2$, and y belong to $S_{\epsilon(x)}(x)$. Then

$$|f_m(x) - f_m(y)| \leq \lambda(x, y) \leq \int_{r_{xy}} v \, dH^1 \leq M |x - y|,$$

where $M = \max_A v(x)$, and, by Lindelöf's Theorem, since

$$\text{Int } A = \bigcup_{x \in \text{Int } A} S_{\epsilon(x)}(x),$$

f_m is Lipschitzian in each set of a countable subcovering $\{S_i\}$ of $\text{Int } A$. Now by the mean-value theorem

$$\begin{aligned} |\text{grad } f_m(x)| &= \limsup_{y \rightarrow x} \frac{|f_m(y) - f_m(x)|}{|y - x|} \leq \limsup_{y \rightarrow x} \frac{\int_{r_{xy}} v \, dH^1}{|y - x|}, \\ &= \limsup_{y \rightarrow x} v(x + t(y - x)), \quad 0 < t = t(y) < 1, \\ &= v(x). \end{aligned}$$

Hence

$$\int_{S_1} v w \, dH^n \geq \int_{S_1} w(x) |\text{grad } f_m(x)| \, dH^n x = \int_0^{\lambda_m} \left(\int_{f_m^{-1}\{v\} \cap S_1} w(x) \, dH^{n-1} x \right) dH^1 y,$$

and letting $T_i = S_i - \bigcup_{j=1}^{i-1} S_j$, for $j > 1$,

$$\int_{T_i} v w \, dH^n \geq \int_0^{\lambda_m} \left(\int_{f_m^{-1}\{v\} \cap T_i} w(x) \, dH^{n-1} x \right) dH^1 y.$$

Thus, since the integrands are nonnegative measurable functions,

$$\begin{aligned} \int_{\text{Int } A} v w \, dH^n &\geq \int_0^{\lambda_m} \left(\int_{f_m^{-1}\{v\}} w(x) \, dH^{n-1} x \right) dH^1 y, \\ &\geq \left(\inf_Q \int_Q w \, dH^{n-1} \right) \cdot \lambda_m, \end{aligned}$$

since $f_m^{-1}\{y\}$ is a separating set. Hence

$$\int_{\text{Int } A} v w \, dH^n \geq \left(\inf_Q \int_Q w \, dH^{n-1} \right) \cdot \liminf_m \lambda_m,$$

but, as was shown in [2, Theorem 3.6, part 4]

$$\liminf_m \lambda_m \geq \lambda,$$

hence the theorem holds for v positive and continuous. Moreover, [2, Theorem 3.6, part 5], an arc c joining A' to A'' exists satisfying

$$\lambda = \int_c v \, dH^1.$$

For v positive and lower semi-continuous, we can find a nondecreasing sequence $\{v_k\}$ of positive continuous functions such that

$$\begin{aligned} \int_{\text{Int } A} v w \, dH^n &= \lim_k \int_{\text{Int } A} v_k w \, dH^n \geq \left(\inf_Q \int_Q w \, dH^{n-1} \right) \lim_k \lambda_k, \\ \lim_k \lambda_k &= \lambda, \end{aligned}$$

by [2, 4.2], where λ_k is the v_k -weighted distance from A' to A'' , and λ is attained.

A limit argument from above extends the result to nonnegative lower semi-continuous functions, and finally, using the Vitali-Caratheodory Theorem we have our result for v nonnegative and in $\mathcal{L}^1(\text{Int } A)$.

3.3. Corollary. *Let Γ, Σ be the families of arcs joining, and sets separating A' from A'' in A . Then, if $p > 1$,*

$$M_p(\Gamma)M_p(\Sigma) \geq 1.$$

Proof. By (3.2) and Hölder's Inequality we have

$$\begin{aligned} &\left(\inf_{\Gamma} \int_c v \, dH^1 \right)^p \left(\inf_{\Sigma} \int_Q w^{p-1} \, dH^{n-1} \right)^p \\ &\leq \left(\int_{\text{Int } A} v w^{p-1} \, dH^n \right)^p \leq \left(\int_A v w^{p-1} \, dH^n \right)^p \leq \left(\int_A v^p \, dH^n \right) \cdot \left(\int_A w^p \, dH^n \right)^{p-1}, \end{aligned}$$

which holds for all admissible v and w in $\mathcal{L}^p(A)$. But, these are the only functions one need consider.

3.4. Remarks. When $p = n$, the dimension of the space, Gehring [5, Theorem 1] (for $n = 3$) and Ziemer [6] have shown that

$$M_n(\Gamma)M_n(\Sigma) = 1.$$

It would be of interest to discover under what circumstances equality holds for the product of the p -modules.

In [2, 4.5 and 5.3], it was shown that if $\Gamma_i, i = 1, \dots, n$, is the family of arcs joining the i^{th} pair of opposite faces of A , then

$$\int_{\text{Int } A} \left(\prod_{i=1}^n w_i(x) \right) dH^n x \geq \prod_{i=1}^n \left(\inf_{\Gamma_i} \int_c w_i \, dH^1 \right), \quad w_i \in \mathcal{L}^n(\text{Int } A),$$

and consequently

$$\prod_{i=1}^n M_n(\Gamma_i) \geq 1.$$

Clearly, for $\Gamma = \Gamma_1$, we have

$$\prod_{i=2}^n M_n(\Gamma_i) \geq M_n(\Sigma).$$

Does this inequality hold for p -modules?

Observe that, for $p \geq n$, Hölder's Inequality yields

$$\int_{\text{Int } A} \left(\prod_{i=1}^n w_i \right) dH^n \leq \prod_{i=1}^n \left(\int_{\text{Int } A} w_i^p dH^n \right)^{1/p} [H^n(\text{Int } A)]^{(p-n)/p},$$

implying that

$$\prod_{i=1}^n M_p(\Gamma_i) \geq [H^n(\text{Int } A)]^{n-p}, \quad p \geq n.$$

When do we have equality? It is well known that for $p = n = 2$

$$M_2(\Gamma_1)M_2(\Gamma_2) = 1.$$

The following example shows equality need not hold in n -space, $n > 2$.

3.5. Example. Let A be the subset of 3-space given, in cylindrical coordinates, by the equations

$$a \leq r \leq b, \quad 0 \leq \theta \leq k\pi, \quad 0 \leq z \leq h,$$

where $0 < a < b$, $0 < k < 2$, $0 < h$. Denote by Γ_1 the family of arcs joining the vertical plane sides, Γ_2 the arcs joining the circular vertical sides, and Γ_3 the arcs joining the horizontal plane sides of A . A function ρ "wedges" with Γ_i , $\rho \wedge \Gamma_i$, if

$$\int_c \rho dH^1 \geq 1,$$

for every arc c in Γ_i . We compute $M_p(\Gamma_i)$.

Let $\rho \wedge \Gamma_1$, $\gamma_{r,z} = \{(r, \theta, z) : r, z \text{ fixed}, 0 \leq \theta \leq k\pi\}$. By Hölder's Inequality

$$1 \leq \left(\int_{\gamma_{r,z}} \rho dH^1 \right)^p \leq (rk\pi)^{p-1} \int_0^{k\pi} \rho^p r d\theta,$$

hence, integrating with respect to r and z , we have

$$\int_A \rho^p dH^3 \geq K_p, \quad \text{implying } M_p(\Gamma_1) \geq K_p,$$

where

$$K_p = \begin{cases} \frac{h[a^{2-p} - b^{2-p}]}{(p-2)(k\pi)^{p-1}}, & p \neq 2, \\ (h/k\pi) \log(b/a), & p = 2. \end{cases}$$

But $\rho = (k\pi r)^{-1} \wedge \Gamma_1$, hence

$$M_p(\Gamma_1) \leq \int_A \rho^p dH^3 = K_p .$$

If $\rho \wedge \Gamma_2, \gamma_{\theta, z} = \{(r, \theta, z) : \theta, z \text{ fixed}, a \leq r \leq b\}$, then by Hölder's Inequality

$$1 \leq \left(\int_a^b r^{-1/(p-1)} dr \right)^{p-1} \int_a^b \rho^p r dr$$

implying, after integrating with respect to θ and z , that

$$M_p(\Gamma_2) \geq N_p ,$$

where

$$N_p = \begin{cases} hk\pi \left[\frac{p-1}{p-2} (b^{(p-2)/(p-1)} - a^{(p-2)/(p-1)}) \right]^{1-p}, & p \neq 2, \\ hk\pi [\log(b/a)]^{-1}, & p = 2. \end{cases}$$

Now

$$\rho = \left[\frac{p-1}{p-2} (b^{(p-2)/(p-1)} - a^{(p-2)/(p-1)}) r^{1/(p-1)} \right]^{-1} \wedge \Gamma_2 \quad (p \neq 2)$$

yielding $M_p(\Gamma_2) \leq N_p$. For $p = 2$ use $[r \log b/a]^{-1}$. Finally if $\rho \wedge \Gamma_3, \gamma_{r, \theta} = \{(r, \theta, z) : r, \theta \text{ fixed}, 0 \leq z \leq h\}$

$$1 \leq \frac{h^{p-1}}{r} \int_0^h \rho^p r dz,$$

$$M_p(\Gamma_3) \geq \frac{k\pi(b^2 - a^2)}{2h^{p-1}},$$

equality resulting when we let $\rho = h^{-1}$. Hence

$$\prod_{i=1}^3 M_p(\Gamma_i) = R_p ,$$

where

$$R_p = \begin{cases} \frac{(hk\pi)^{3-p}(p-2)^{p-2}(a^{2-p} - b^{2-p})(b^2 - a^2)}{2(p-1)^{p-1}(b^{(p-2)/(p-1)} - a^{(p-2)/(p-1)})^{p-1}}, & p \neq 2, \\ \frac{hk\pi(b^2 - a^2)}{2} = H^3(A), & p = 2. \end{cases}$$

In particular, if $p = 3$, we have

$$\prod_{i=1}^3 M_3(\Gamma_i) = \frac{((b)^{1/2} + (a)^{1/2})^2(b + a)}{8ab},$$

and, letting $a = 1, b = 4$, we have

$$\prod_{i=1}^3 M_3(\Gamma_i) = \frac{45}{32} > 1.$$

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University of Utah

Date communicated: MARCH 26, 1969