Product Integral Formulae for Nonlinear Expansive Semi-Groups and Non- Expansive Evolution Systems

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1. Introduction. Suppose that T is a non-expansive strongly continuous one-parameter semi-group of linear transformations on the Banach space S and that A is the infinitesimal generator of T (see [6], especially Sec. 10.3, 11.6, 12.2, 12.3). One may prove first that $T(x)p = \lim_{n \rightarrow \infty} [(I - (x/n)A)^{-n}]p$ for all p in S , $x \geq 0$. Having this result one can now consider instead of a non-expansive semi-group, a strongly continuous semi-group L (with infinitesimal generator B) which for some $c > 0$ satisfies $|L(x)| \leq e^{cx}$ for all $x \geq 0$. To prove a somewhat similar exponential formula for L consider the semi-group K defined by $K(x) = e^{-cx}L(x)$ for all $x \geq 0$. K is clearly non-expansive and has infinitesimal generator $-cI + B$. Hence, if $x \geq 0$ and p is in S ,

$$\begin{aligned} e^{-cx}L(x)p &= K(x)p \\ &= \lim_{n \rightarrow \infty} [I - (x/n)(-cI + B)]^{-n}p = \lim_{n \rightarrow \infty} [(1 + (xc/n))I - (x/n)B]^{-n}p \\ &= \lim_{n \rightarrow \infty} (1 + (xc/n))^{-n} [I - ((x/n)/(1 + (xc/n)))B]^{-n}p. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} (1 + (xc/n))^{-n} = e^{-xc}$ it follows that

$$\lim_{n \rightarrow \infty} [I - ((x/n)/(1 + (xc/n)))B]^{-n}p \text{ exists and is equal to } L(x)p.$$

One purpose of the present note is to extend the above developement to a nonlinear setting. There are two problems in attempting such an extension. The first is that there doesn't seem to be known a general exponential formula for non-expansive nonlinear semi-groups. The second is that even in cases where one has such a formula and L is a semi-group which satisfies $|L(x)| \leq e^{cx}$

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(Lipschitz norm) for all $x \geq 0$, it would not follow that the function K specifies a semi-group. A way to at least start dealing with these problems is presented in what follows.

Recently there have been a number of papers ([2], [3], [4], [5], [8], [9], [10], [11], [12], [13], [15]) which, for the most part, concern generation of one-parameter semi-groups on a Banach space. All of these except [3], [9], [10] (which use certain uniform continuity hypotheses not used in the present note) are primarily concerned with semi-groups of non-expansive transformations. In the present note some results are established for semi-groups which do not have the non-expansive property. At first we retain the non-expansive property but dispense with the semigroup property in order to prove a theorem (A) which generalizes [11] in the direction of [9] and [10] and a theorem (B) which gives a generalization of [12] to what are here called evolution systems. From these theorems a product integral formula for an expansive nonlinear semi-group is obtained.

2. Evolution systems. Suppose that S is a Banach space. An evolution system on S is a function M with domain the set of all ordered pairs (x, t) so that $0 \leq t \leq x$ and with range a set of continuous transformations from S to S so that (1) if $0 \leq s \leq t \leq x$, then $M(x, t)M(t, s) = M(x, s)$ and $M(x, x) = I$, and (2) if $t \geq 0$, p is in S and $g(x) = M(x, t)p$ for all $x \geq t$, then g is continuous.

The evolution system M on S is called non-expansive if each member of the range of M has Lipschitz norm not exceeding one, i.e., (3) $\|M(x, t)p - M(x, t)q\| \leq \|p - q\|$ if $0 \leq t \leq x$ and each of p and q is in S .

If p is in S , denote $M(\cdot, 0)p$ by g_p .

Theorem A. Suppose that in addition to (1)–(3) there holds: (4) for all p in some dense subset of S , g'_p is continuous with domain $[0, \infty)$. Then, if $x \geq 0$, p is in S and $\epsilon > 0$, there is a $\delta > 0$ so that if $0 = t_0 \leq t_1 \leq \dots \leq t_{n+1} = x$ and $\max_{i=0,1,\dots,n} t_{i+1} - t_i < \delta$, then

$$\limsup_{\delta_0, \delta_1, \dots, \delta_n \rightarrow 0} \left\| \left\{ \prod_{i=0}^n [I - ((t_{i+1} - t_i)/\delta_i)(M(t_{i+1} + \delta_i, t_{i+1}) - I)]^{-1} \right\} p - g_p(x) \right\| < \epsilon.$$

Denote by A the function with domain $[0, \infty)$ so that for each $x \geq 0$ $A(x)$ is the transformation so that (i) the domain of $A(x)$ consists of all p in S so that $\lim_{\delta \rightarrow 0} (1/\delta)[M(x + \delta, x) - I]p$ exists and (ii) if p is in the domain of $A(x)$, then $A(x)p$ is this limit. Note that if $x \geq 0$, q is in S and $g'_q(x)$ exists, then $g_q(x)$ is in the domain of $A(x)$ and $g'_q(x) = A(x)g_q(x)$.

Theorem B. In addition to the hypothesis of Theorem A, suppose that (5) if $x \geq 0$ then there is a positive number y so that $R(I - yA(x)) = S$. Then for each $x, y \geq 0$, $(I - yA(x))^{-1}$ exists and has a unique nonexpansive extension $L(y, x)$ to all of S . Moreover, if $x \geq 0$, p is in S and $\epsilon > 0$, there is a $\delta > 0$ so that if $0 = t_0 \leq t_1 \leq \dots \leq t_{n+1} = x$ and $\max_{i=0,1,\dots,n} t_{i+1} - t_i < \delta$, then

$$\left\| \left[\prod_{i=0}^n L(t_{i+1} - t_i, t_i) \right] p - g_p(x) \right\| < \epsilon.$$

In [1], [7], [14] the problem of solving the equation $Y' = AY$ is considered. Solving this equation may be considered equivalent to constructing an evolution system the members of which satisfy in an actual or generalized sense the given equation. In the present work we start with an evolution system and seek to reconstruct it from the function A .

3. Semi-groups. Suppose that T is a strongly continuous semi-group of transformations on S , i.e., that T is a function from $[0, \infty)$ to the set of continuous transformations from S to S so that (a) $T(0) = I$ and $T(x)T(y) = T(x + y)$ if $x, y > 0$, (b) if p is in S and $h_p(x) = T(x)p$ for all $x \geq 0$, then h_p is continuous. Suppose in addition that T satisfies the following conditions: (c) for all p in some dense subset of S h'_p is continuous with domain $[0, \infty)$, (d) c is a positive number so that $\|T(x)p - T(x)q\| \leq e^{cx} \|p - q\|$ for all $x \geq 0$, p, q in S . Denote by B the infinitesimal generator of T , i.e., B is the transformation with domain all p such that $h'_p(0)$ exists and $Bp = h'_p(0)$ for all such p .

If $0 \leq t \leq x$, define

$$M(x, t) = e^{-cx} T(x - t) e^{ct}.$$

It is easy to check that M is an evolution system satisfying (1)–(4) of Section 2.

Theorem C. Suppose that in addition to (a)–(d), T satisfies (e) $\overline{R(I - y(B - cI))} = S$ for some $y > 0$. Then M , as defined above, satisfies (1)–(5) of Section 2. If L is defined from M as in Section 2, p is in S , $x \geq 0$ and $\epsilon > 0$, there is a $\delta > 0$ so that if $0 = t_0 \leq t_1 \leq \dots \leq t_{n+1}$ and $\max_{i=0,1,\dots,n} t_{i+1} - t_i < \delta$, then

$$\left\| e^{cx} \left[\prod_{i=0}^n L(t_{i+1} - t_i, t_i) \right] p - T(x)p \right\| < \epsilon.$$

Corollary. $T(x)p = e^{cx} \lim_{n \rightarrow \infty} [\prod_{i=0}^n L(x/n, ix/n)]p$ for all p in S , $x \geq 0$.

4. Proofs of theorems. *Proof of Theorem A.* Suppose that $0 \leq t < x$, $y \geq 0$, $\delta > 0$ and p is in S . The following are true:

- (i) $[I - ((x - t)/\delta)(M(x + \delta, x) - I)]^{-1}$ exists, has domain S and is non-expansive,
- (ii) $\| [I - ((x - t)/\delta)(M(x + \delta, x) - I)]^{-1} p - M(x, t)p \| \leq (x - t) \| (x - t)^{-1} [M(x, t) - I]p - \delta^{-1} [M(x + \delta, t) - M(x, t)]p \|$.

The first two parts of (i) are proved in the same way as Lemma 1 of [11] and the third part of (i) like Lemma 2 of [11]. (ii) follows in the same manner as Lemma 3 of [11].

Similar to Lemma 4 of [11] one has that if each of $B_0, B_1, \dots, B_n, C_0, C_1, \dots, C_n$ is a non-expansive transformation from S to S and p is in S , then,

using from here on the convention $\prod_{i=n+1}^n B_i = I = \prod_{i=0}^{-1} C_i$,

$$\begin{aligned} & \left\| \left[\prod_{i=0}^n B_i \right] p - \left[\prod_{i=0}^n C_i \right] p \right\| \\ & \leq \sum_{i=0}^n \left\| \left[\prod_{j=i}^n B_j \right] \left[\prod_{j=0}^{i-1} C_j \right] p - \left[\prod_{j=i+1}^n B_j \right] \left[\prod_{j=0}^i C_j \right] p \right\| \\ & \leq \sum_{i=0}^n \left\| B_i \left[\prod_{j=0}^{i-1} C_j \right] p - C_i \left[\prod_{j=0}^{i-1} C_j \right] p \right\|. \end{aligned}$$

Now suppose that $x > 0$ and p is in S . Suppose that $\epsilon > 0$ and denote by r a member of S so that $\|p - r\| < \epsilon/4$ and so that g_r is continuous with domain $[0, \infty)$. Denote by δ a number in $(0, 1)$ so that if $0 \leq u, v \leq x + 1$ and $0 < |u - v| < \delta$, then

$$\max_{u \in [u, v]} \|(u - v)^{-1}[g_r(u) - g_r(v)] - g'_r(w)\| < \epsilon/(8x).$$

Denote by $t_0, t_1, \dots, t_{n+1}$ an increasing sequence so that $t_0 = 0, t_{n+1} = x$ and $\max_{i=0,1,\dots,n} t_{i+1} - t_i < \delta$. Denote by each of $\delta_0, \delta_1, \dots, \delta_n$ a positive number $< \delta$. Then since $g_r(x) = [\prod_{i=0}^n M(t_{i+1}, t_i)]r$,

$$\begin{aligned} & \left\| \left[\prod_{i=0}^n [I - ((t_{i+1} - t_i)/\delta_i)(M(t_{i+1} + \delta_i, t_{i+1}) - I)]^{-1} \right] r - g_r(x) \right\| \\ & \leq \sum_{i=0}^n \left\| [I - ((t_{i+1} - t_i)/\delta_i) \right. \\ & \quad \cdot (M(t_{i+1} + \delta_i, t_{i+1}) - I)]^{-1} g_r(t_i) - M(t_{i+1}, t_i) g_r(t_i) \left. \right\| \\ & \leq \sum_{i=0}^n (t_{i+1} - t_i) \left\| (t_{i+1} - t_i)^{-1} [M(t_{i+1}, t_i) - I] g_r(t_i) \right. \\ & \quad \left. - \delta_i^{-1} [M(t_{i+1} + \delta_i, t_i) - M(t_{i+1}, t_i)] g_r(t_i) \right\| \\ & = \sum_{i=0}^n (t_{i+1} - t_i) \left\| (t_{i+1} - t_i)^{-1} [g_r(t_{i+1}) - g_r(t_i)] \right. \\ & \quad \left. - \delta_i^{-1} [g_r(t_{i+1} + \delta_i) - g_r(t_{i+1})] \right\| \\ & \leq \sum_{i=0}^n (t_{i+1} - t_i) \left\| (t_{i+1} - t_i)^{-1} [g_r(t_{i+1}) - g_r(t_i)] - g'_r(t_{i+1}) \right\| \\ & \quad + \sum_{i=0}^n (t_{i+1} - t_i) \left\| g'_r(t_{i+1}) - \delta_i^{-1} [g_r(t_{i+1} + \delta_i) - g_r(t_{i+1})] \right\| \\ & \leq 2 \sum_{i=0}^n (t_{i+1} - t_i) \epsilon/(8x) = \epsilon/4. \end{aligned}$$

But $\|g_p(x) - g_r(x)\| \leq \|p - r\| < \epsilon/4$ and $\|Dp - Dr\| \leq \|p - r\| < \epsilon/4$ where

$$D = \prod_{i=0}^n [I - ((t_{i+1} - t_i)/\delta_i)(M(t_{i+1} + \delta_i, t_{i+1}) - I)]^{-1}.$$

Hence

$$\left\| \left[\prod_{i=0}^n [I - ((t_{i+1} - t_i)/\delta_i)(M(t_{i+1} + \delta_i, t_{i+1}) - I)]^{-1} \right] p - g_p(x) \right\| < 3\epsilon/4.$$

This completes a proof of Theorem A.

Proof of Theorem B. Suppose that $x \geq 0$. If $\delta > 0$, denote $\delta^{-1}(M(x+\delta, x) - I)$ by A_δ . Then $A(x)$ is the transformation on a subset of S to S such that p is in the domain of $A(x)$ if and only if $\lim_{\delta \rightarrow 0} A_\delta p$ exists and for all such p , $A(x)p$ is this limit.

Suppose that q is in the domain of $A(x)$ and y is a positive number so that $R(I - yA(x)) = S$. Then,

$$\begin{aligned} \|(I - yA_\delta)^{-1}(I - yA(x))q - q\| &\leq \|(I - yA(x))q - (I - yA_\delta)q\| \\ &= y \|A_\delta q - A(x)q\| \rightarrow 0 \end{aligned}$$

as $\delta \rightarrow 0$. Hence if p is in $R(I - yA(x))$, then $\lim_{\delta \rightarrow 0} (I - yA_\delta)^{-1}p$ exists. But for each $\delta > 0$, $(I - yA_\delta)^{-1}$ is non-expansive and $R(I - yA(x))$ is, by hypothesis, dense in S . Hence $\lim_{\delta \rightarrow 0} (I - yA_\delta)^{-1}p$ exists for all p in S and, if $L(y, x)p = \lim_{\delta \rightarrow 0} (I - yA_\delta)^{-1}p$ for all p in S , then $L(y, x)$ is non-expansive. Moreover, since if p is in $R(I - yA(x))$ and hence $p = (I - yA(x))q$ for some q in S , then $L(y, x)p = \lim_{\delta \rightarrow 0} (I - yA_\delta)^{-1}(I - yA(x))q = q$, it follows that $(I - yA(x))^{-1}$ exists and that $L(y, x)$ is the unique non-expansive extension to all of S .

Now suppose that $z > y/2$. Following [12] one has that

$$I - zA(x) = (z/y)[I - (1 - y/z)L(y, x)](I - yA(x)).$$

Since $|1 - y/z| < 1$, then $[I - (1 - y/z)L(y, x)]$ is a homeomorphism (recall that $L(y, x)$ is non-expansive) and since $R(I - yA(x))$ is dense in S , then so must be $R(I - zA(x))$. To summarize: There is a positive number y such that $\overline{R(I - yA(x))} = S$ and it is true that if $y > 0$ and $\overline{R(I - yA(x))} = S$ then $\overline{R(I - zA(x))} = S$ for all $z > y/2$. From this it follows that $\overline{R(I - zA(x))} = S$ for all $z > 0$.

If $\delta > 0$, $0 \leq t \leq s$, denote $[I - ((s - t)/\delta)(Ms + \delta, s) - I]^{-1}$ by $C(\delta; s, t)$. Now suppose that $\epsilon > 0$, $x > 0$ and δ is a positive number so that if (*) $0 = t_0 < t_1 < \dots < t_{n+1} = x$ and $\max_{i=0,1,\dots,n} t_{i+1} - t_i < \delta$, then

$$\limsup_{\delta_0, \delta_1, \dots, \delta_n \rightarrow 0} \left\| \left[\prod_{i=0}^n C(\delta_i; t_{i+1}, t_i) \right] p - g_p(x) \right\| < \epsilon/2.$$

Denote by $t_0, t_1, \dots, t_n$ a sequence so that (*) holds and denote by $\delta_0, \delta_1, \dots, \delta_n$ a positive number sequence so that

$$(I) \quad \|L(t_1 - t_0, t_0)p - C(\delta_0; t_1, t_0)p\| < \epsilon/2(n + 1)$$

and

$$\left\| \left[L(t_{i+1} - t_i, t_i) \left[\prod_{j=0}^{i-1} L(t_{j+1} - t_j, t_j) \right] p - C(\delta_i; t_{i+1}, t_i) \right. \right. \\ \left. \left. \cdot \left[\prod_{j=0}^{i-1} L(t_{j+1} - t_j, t_j) \right] p \right\| < \epsilon/2(n+1), \quad i = 1, 2, \dots, n,$$

and

$$(II) \quad \left\| \left[\prod_{i=0}^n C(\delta_i; t_{i+1}, t_i) \right] p - g_p(x) \right\| < \epsilon/2.$$

Hence

$$\left\| \left[\prod_{i=0}^n L(t_{i+1} - t_i, t_i) \right] p - \left[\prod_{i=0}^n C(\delta_i; t_{i+1}, t_i) \right] p \right\| \\ \leq \sum_{i=0}^n \left\| L(t_{i+1} - t_i, t_i) \left[\prod_{j=0}^{i-1} L(t_{j+1} - t_j, t_j) \right] p \right. \\ \left. - C(\delta_i; t_{i+1}, t_i) \left[\prod_{j=0}^{i-1} L(t_{j+1} - t_j, t_j) \right] p \right\| < \epsilon/2$$

and so $\|[\prod_{i=0}^n L(t_{i+1} - t_i, t_i)]p - g_p(x)\| < \epsilon$. This completes a proof to Theorem B.

Proof of Theorem C. Assume the definitions and conditions (a)–(e) of Section 3. It is easy to check that conditions (1)–(4) of Section 2 are satisfied. It remains to show that (5) holds. By taking a derivative one can see that $A(0) = -cI + B$. Hence condition (e) is equivalent to saying that there is a positive number y so that $\overline{R(I - yA(0))} = S$. Since M , as defined in Section 3, is non-expansive it follows, exactly as in the proof of Theorem B, that one has $\overline{R(I - zA(0))} = S$ for all $z > 0$.

Suppose $x > 0$. Denote by r a point of S so that $(T(\cdot)r)'$ is continuous with domain $[0, \infty)$. Denote $M(x, 0)r$ by p . Then,

$$\begin{aligned} \lim_{\delta \rightarrow 0} (1/\delta)(M(x + \delta, x) - I)p \\ &= \lim_{\delta \rightarrow 0} (1/\delta)[M(x + \delta, 0)r - M(x, 0)r] \\ &= \lim_{\delta \rightarrow 0} (1/\delta)[e^{-(x+\delta)c}T(x + \delta)r - e^{-xc}T(x)r] \\ &= -ce^{-xc}T(x)r + e^{-xc}BT(x)r = e^{-xc}(-cI + B)T(x)r \\ &= e^{-xc}(-cI + B)e^{xc}p. \end{aligned}$$

Hence p is in the domain of $A(x)$ and $A(x)p = e^{-xc}(-cI + B)e^{xc}p$. Hence $A(x)$ is an extension of $e^{-xc}(-cI + B)(e^{xc}I)$ and therefore, if $y > 0$, $I - yA(x)$ is an extension of $I - ye^{-xc}(-cI + B)(e^{xc}I) = e^{-xc}(I - y(-cI + B))(e^{xc}I)$. Since $\overline{R(I - y(-cI + B))} = S$ for all $y > 0$ it follows that $\overline{R(I - yA(x))} = S$ for all $x, y > 0$. Hence (5) of Section 2 holds.

The remaining part of Theorem C is essentially a restatement of the “more-over” part of Theorem B.

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