*Fracture of Brittle Solids by Applied Loads**

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1. **Introduction.** As the condition for slip or yield, *i.e.*, plastic deformation, one usually assumes an equation involving the components of the stress field and a modulus of the solid which depends on strain hardening in practical applications; as examples we may mention the von Mises and Tresca yield conditions. This point of view regarding the initiation and maintenance of plastic deformation is well established in the literature. Nevertheless it cannot be applied effectively to obtain the solution of even the simplest mechanical problems due to the difficulties caused by strain hardening, *e.g.* the problem of the theoretical derivation of the linear relationships which were obtained by Bridgman in his experiments on the plastic deformation of steel cylinders under simple tension and hydrostatic pressure. Indeed Bridgman, using the von Mises yield condition, was not able to deduce these relationships nor was he able to account for a single one of the excellent experimental results described and documented in his book on *Large Plastic Flow and Fracture* [1]. Such facts constitute a severe indictment of the efficacy of conventional criteria for plastic deformation and flow in continuum mechanics.

I have investigated this matter under the assumption that the yield condition should imply only a *tendency* for the occurrence of slip over a surface element in the medium rather than the actual realization of slip as is commonly asserted. This investigation eventually led to the following two inequalities, namely

$$(1.1) \quad [(1 + k^2)^{1/2} + k]s_1 - [(1 + k^2)^{1/2} - k]s_3 \geq 2K,$$

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$$(1.2) \quad [(1 + q^2)^{1/2} + q]s_1 - [(1 + q^2)^{1/2} - q]s_3 \geq 2Q,$$

as the simplest conditions for the occurrence of slip from the elastic state and fracture from the plastic state, respectively. The quantities k and K in the inequality (1.1) are constants which may be labeled *slip moduli* from the interpretation of this inequality; the corresponding constants q and Q appear as *fracture moduli* for ductile materials admitting significant plastic deformation. Attention will also be called to the fact that the moduli k and q express the contribution of the normal stress on the surface element in the derivation of these inequalities. Finally, the above quantities s_1 and s_3 are the algebraically greatest and least of the principal stresses at points of the stress field under consideration. No difficulties will now arise from strain hardening in the treatment of ductile solids since there is sufficient leeway in the inequality (1.2) to permit this effect without changing the values of the moduli q and Q . The inequalities (1.1) and (1.2) provide the possibility of a simple derivation of the above mentioned relationships obtained by Bridgman in his experiments on steel cylinders. These results have recently been published as well as a number of other theoretical results concerning large plastic deformations in ductile solids [2]–[6].

I would now like to consider the role of the slip inequality (1.1) in the problem of determining the applied load which is just sufficient to cause fracture in a brittle solid; this load will henceforth be referred to as the *critical load* and the term fracture as here used implies the complete rupture or separation of the solid into distinct parts. It is my contention that the occurrence of fracture, regardless of the mechanism by which it is accomplished, is related essentially to the overall stress field in the solid. On the basis of this viewpoint the problem of determining the critical load is reduced to the problem of the determination of a stress field appropriate for fracture and in this connection the following simple assumptions will be made with regard to the fracture characteristics of the brittle solid. These assumptions have been formulated with reference to problems of the type considered in this article and they are not intended to be applicable, as stated, under all conditions.

(1) *The solid will undergo an abrupt change from its elastic state to a state of plastic deformation when the critical load is reached.*

(2) *The strict equality (1.1) will be satisfied in the plastically deformed solid.*

(3) *Fracture will occur immediately after the plastic deformation of the solid.*

When fracture is initiated in the brittle solid it will proceed with great rapidity and this fact has been recognized in the above assumptions. It is clear from assumption (3) that the brittle solid is here regarded as a limiting case of the ductile solid, *i.e.* as a solid of such insignificant ductility that it will fracture without the necessity of increasing the load beyond the point required to produce the initial plastic deformation. Also from assumption (2) we must have

$$(1.3) \quad [(1 + k_2)^{1/2} + k]s_1 - [(1 + k^2)^{1/2} - k]s_3 = 2K$$

where s_1 and s_3 are the algebraically greatest and least of the principal stresses in the plastically deformed solid. The condition imposed by the equation (1.3) can roughly be described by saying that the brittle solid possesses a sharp yield point. It is implicitly assumed in this theory that the brittle solid is well constructed in the sense that it contains only those minute and randomly distributed flaws or imperfections which are unavoidable and which are rightly neglected in continuum mechanics. I believe that the above assumptions will provide us with a macroscopic description of the behavior of a brittle solid during the fracture process which is sufficiently accurate for the determination of the critical load in many problems.

The determination of the critical load will require a knowledge of the stress field in the plastically deformed solid. The usual equations of dynamical equilibrium must of course be satisfied by this stress field, *i.e.* the equations

$$(1.4) \quad g^{ik} \tau_{ij;k} = 0,$$

in which the τ_{ij} denote the plastic stress components, the g^{ik} are the contravariant components of the metric tensor in the coordinate system selected, and the semicolon denotes covariant differentiation; also the indices in the above and following equations may have the values 1, 2, 3 unless otherwise stated and it is to be understood that repeated indices are to be summed over this range in accordance with the summation convention. Since the above equations (1.4) do not in general suffice for the determination of the complete stress field, they must be supplemented by a suitable system of constitutive equations. We shall assume these equations to be given by

$$(1.5) \quad \tau_{ij}^* = \psi e_{ij}; \quad g^{ij} e_{ij} = 0$$

for the purpose of the following discussion. The equations (1.5) are formally equivalent to the Hencky stress-strain equations for plastic equilibrium in the absence of elastic deformations which can justifiably be neglected in the case of the brittle solid. The quantities τ_{ij}^* and the e_{ij} are the components of the deviator of the plastic stress tensor and the components of the plastic strain tensor, respectively; the latter tensor is the same as its deviator due to the last equation (1.5) which expresses the condition of plastic incompressibility. The components of the above tensors are defined by the equations

$$(1.6) \quad \tau_{ij}^* = \tau_{ij} - \frac{1}{3}(g^{km} \tau_{km}) g_{ij}; \quad e_{ij} = \frac{1}{2}(u_{i;j} + u_{j;i})$$

where the g_{ij} are the covariant components of the metric tensor and the u_i are the covariant components of the plastic deformation. The proportionality factor ψ is assumed to be a positive function and is regarded as one of the dependent variables of the problem. The remaining dependent variables are the six components τ_{ij} of the plastic stress and the three components u_i of the plastic deformation. *Hence there are ten dependent variables and there are ten equations (1.3), (1.4) and (1.5) for the determination of these variables.* In certain problems,

FIGURE 1.

however, the equations (1.3) and (1.4) will suffice for the determination of the critical load and in such cases the constitutive equations (1.5) can be disregarded.

In the following sections I have used this theory to deduce formulas for the critical loads in a variety of problems and the results obtained would appear to be of some significance from the standpoint of structural mechanics. Other problems can undoubtedly be devised and solved by the interested reader.

Remark. In using the equations (1.5) it is assumed that the elastic deformations are negligible in the brittle solid. It will also be assumed in the following discussion that the plastic deformation, which occurs immediately prior to the fracture of the brittle solid, is extremely small and that as a consequence the physical dimensions of the solid will effectively be unaltered by such deformations. This is one of the essential differences between brittle and ductile solids since the dimensions of the latter may be changed considerably by plastic deformation preceding fracture.

2. Block under tension or compression. Consider a brittle rectangular block of constant cross section as shown in Fig. 1. Choose a rectangular co-

ordinate system with origin at the center of the block and such that the x^3 axis coincides with the central axis of the block; the x^1 and x^2 axes will be chosen perpendicular to the lateral sides of the block. An elastic stress field created in the block by a uniform tensile load applied over its ends ABCD and PQRS will then have components σ_{ij} such that $\sigma_{33} > 0$ while the remaining components of the field will be equal to zero. Now suppose that this field is given by

$$(2.1) \quad \sigma_{33} = \tau(>0); \quad \sigma_{ij} = 0, \quad \text{otherwise,}$$

when the critical load is reached. The stress τ will be referred to as the *critical stress* and the problem of determining the critical stress τ will obviously be equivalent to the problem of determining the critical load.

The above equations (2.1) suggest that the plastic stress components τ_{ij} can also be taken to be

$$(2.2) \quad \tau_{33} = \tau; \quad \tau_{ij} = 0, \quad \text{otherwise,}$$

and these stress components will of course satisfy the equilibrium conditions (1.4). The components u_i of the plastic deformation may be assumed to be functions of the form

$$(2.3) \quad u_1 = u_1(x^1); \quad u_2 = u_2(x^2); \quad u_3 = u_3(x^3)$$

from the symmetry of the problem. Substituting the values of τ_{ij} and u_i given by (2.2) and (2.3), respectively, into the equations (1.5) subject to the requirement that u_1 and u_2 vanish along the central or x^3 axis and that u_3 vanishes for $x^3 = 0$, *i.e.* over the central cross section of the block, it follows readily that

$$(2.4) \quad u_1 = -\lambda x^1; \quad u_2 = -\lambda x^2; \quad u_3 = 2\lambda x^3$$

in which the quantity λ is a positive constant; also the proportionality factor ψ will have the constant value

$$(2.5) \quad \psi = \frac{\tau}{3\lambda}.$$

Hence the equations (2.2), (2.4) and (2.5) furnish a valid solution of the equations (1.4) and (1.5) governing the plastic deformation and it remains only to take account of the condition (1.3). But the principal values of the plastic stress tensor are given as the solutions s of the determinantal equation

$$(2.6) \quad |\tau_{ij} - s \delta_{ij}| = \begin{vmatrix} \tau - s & 0 & 0 \\ 0 & -s & 0 \\ 0 & 0 & -s \end{vmatrix} = 0$$

in which the δ_{ij} are the Kronecker deltas. Denoting these solutions s by s_1 , s_2 and s_3 with the ordering

$$(2.7) \quad s_1 \geq s_2 \geq s_3,$$

it follows from (2.6) that

$$(2.8) \quad s_1 = \tau; \quad s_2 = s_3 = 0.$$

If we substitute the values of s_1 and s_3 from (2.8) into the condition (1.3) we now obtain the following result. *When the rectangular block is subjected to a uniform tensile load over its ends, the critical stress τ is given by the equation*

$$(2.9) \quad \tau = 2[(1 + k^2)^{1/2} - k]K, \quad (\text{tension}).$$

The problem of a uniform compressive load over the ends ABCD and PQRS of the rectangular block can be treated in a similar manner. In this case the quantity τ in the equations (2.1) and (2.2) must be replaced by $-\tau$ with $\tau > 0$ if we wish to think of the critical stress as a positive quantity τ in this problem. The principal stresses subject to the ordering (2.7) will now be given by

$$s_1 = s_2 = 0; \quad s_3 = -\tau.$$

When these values of s_1 and s_3 are substituted into the condition (1.3), we find that

$$(2.10) \quad \tau = 2[(1 + k^2)^{1/2} + k]K, \quad (\text{compression}),$$

and hence the following result can be stated. *The critical stress τ is given by the equation (2.10) when the rectangular block is subjected to a uniform compressive load over its ends.*

It is seen from the equations (2.9) and (2.10) that the critical stress in compression is greater than the critical stress in tension and this result can be verified observationally. However the difference in the critical stresses is small in relation to the magnitude of these stresses since the modulus k is extremely small compared to unity.

3. Circular bar under torsion. We shall now consider the problem of a brittle circular bar which is in a state of torsion produced by a continuous distribution of opposing couples over its ends. Cylindrical coordinates r, θ, z will be employed in the treatment of this problem; the origin 0 of the coordinate system will be located at the center of the bar and the z axis will be taken to coincide with the central axis of the bar. If we denote the coordinates r, θ and z by x^1, x^2 and x^3 , respectively, the covariant components g_{ij} and the contravariant components g^{ij} of the metric tensor will be given by

$$(3.1) \quad ||g_{ij}|| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 0 \end{vmatrix}; \quad ||g^{ij}|| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

Also the Christoffel symbols Γ_{jk}^i will have the values

$$(3.2) \quad \Gamma_{22}^1 = -r; \quad \Gamma_{12}^2 = \frac{1}{r}; \quad \Gamma_{jk}^i = 0, \quad \text{otherwise},$$

in the cylindrical system. These formulas will be needed in the following discussion.

If an elastic stress field with components σ_{ij} is created by the action of the above couples, the moment M of either set of opposing couples will be given, quite generally, by the surface integral

$$(3.3) \quad M = \iint \sigma_{ij} \nu^i \xi^j dS$$

over the end of the bar on which the couples act. As so defined the moment M is independent of the coordinate system employed. In the case of the couples over the upper end of the bar, the quantities ν^i are the components of the unit vector ν in the direction of the z axis and the ξ^i are the components of a vector ξ of magnitude r which is perpendicular both to the above vector ν and to the radial direction as shown in Fig. 2. Now the vector ν obviously has components $(0, 0, 1)$ and the vector ξ has components $(0, 1, 0)$ as follows readily from the condition that the magnitude of this vector is equal to the radial coordinate. Making these substitutions in the integral (3.3) the moment M is seen to be given by the formula

$$(3.4) \quad M = \iint \sigma_{23} dS$$

which can likewise be observed to give the moment of the opposing couples over the lower end of the bar. But the elastic stress field in the bar is given by the equations

$$(3.5) \quad \sigma_{23} = \mu\beta r^2; \quad \sigma_{ij} = 0, \quad \text{otherwise,}$$

where μ is the rigidity of the bar and β is the twist, *i.e.* the angular deformation produced per unit length by the applied couples. Substituting the value of σ_{23} from (3.5) into (3.4) and performing the indicated integration, we obtain

$$(3.6) \quad M = \frac{1}{2}\pi\mu\beta a^4$$

where a is the radius of the circular bar [7].

The applied load in this problem is the moment M given by the equation (3.6). Now suppose that β is the twist when the moment M has its critical value; the formula (3.6) will then give the critical moment M . Denoting by τ_{ij} the stress components in the plastically deformed state which is initiated in accordance with the assumption (1) in Section 1 when the moment M reaches its critical value, we must expect that

$$(3.7) \quad \tau_{23} = \tau_{23}(r); \quad \tau_{ij} = 0, \quad \text{otherwise,}$$

from the symmetry of the problem and in conformity with the equations (3.5). Computing the components $\tau_{ij;k}$ of the covariant derivative of the stress field (3.7) with the aid of the above equations (3.1) and (3.2), it is seen that the equilibrium conditions (1.4) are satisfied. Disregarding the constitutive equa-

FIGURE 2.

tions (1.5) since they are unnecessary for the solution of this problem, we can proceed immediately to the determination of the principal stresses of the plastic stress field (3.7).

The principal stresses of the stress field (3.7) are the solutions s of the equation

$$|\tau_{ii} - sg_{ii}| = \begin{vmatrix} -s & 0 & 0 \\ 0 & -sr^2 & \tau_{23}(r) \\ 0 & \tau_{23}(r) & -s \end{vmatrix} = 0.$$

Solving this equation for s we see that

$$s_1 = \frac{\tau_{23}(r)}{r}; \quad s_2 = 0; \quad s_3 = -\frac{\tau_{23}(r)}{r}$$

since the quantity $\tau_{23}(r)$ is obviously non-negative. Substituting these values of s_1 and s_3 into the relation (1.3) we find that the component $\tau_{23}(r)$ of the plastic stress field (3.7) is given by the equation

$$(3.8) \quad \tau_{23} = \frac{Kr}{(1 + k^2)^{1/2}}.$$

Since the critical load is assumed to remain unchanged during the transition of the bar from the elastic to the plastic state we can obtain another expression for the critical moment M if we replace the elastic stress component σ_{23} in (3.4) by the plastic stress component τ_{23} given by (3.8). Making this substitution and evaluating the resulting integral, we find that

$$(3.9) \quad M = \frac{2\pi K a^3}{3(1 + k^2)^{1/2}}.$$

Hence fracture will occur when the moment M reaches the value given by (3.9) in which a is the radius of the bar. If we equate the right members of the equations (3.6) and (3.9) we obtain the equation

$$(3.10) \quad \beta = \frac{4K}{3\mu a(1 + k^2)^{1/2}}$$

for the *critical twist* β , i.e. the twist immediately preceding the occurrence of plastic deformation and fracture.

Finally let us observe that the principal stresses in the elastic stress field (3.5) are given by

$$s_1 = \mu\beta r; \quad s_2 = 0; \quad s_3 = -\mu\beta r,$$

and hence the general slip inequality (1.1) reduces to

$$(3.11) \quad \mu\beta r \geq \frac{K}{(1 + k^2)^{1/2}}$$

It is now seen from (3.10) that the strict equality (3.11) holds when $r = 3a/4$ and β has its critical value. But this implies that the inequality (3.11) is valid in the region

$$(3.12) \quad \frac{3a}{4} \leq r \leq a$$

of the bar when the critical load is reached. In other words the tendency for the occurrence of slip which is expressed by the inequality (3.11) will not result in actual slip or plastic deformation and subsequent fracture until the inequality is satisfied in the region (3.12) of the bar.

Remark. One must expect that the fracture process will be arrested if the applied load is decreased during the transition of the bar from the elastic to the plastic state. The explicit assumption made in the present section, namely that the load remains unchanged during this change of state, is therefore a basic assumption of the theory. A similar assumption was implicitly made in Section 1 and the condition of the constancy of the loads, to the extent that this is possible during the fracture process, will be assumed without special mention in the following sections of this article.

4. Tension combined with torsion. Suppose that a tensile load of magnitude T is applied to the ends of a circular cylinder which we shall refer to a system of cylindrical coordinates as in Section 3; it will be assumed that the tensile force creating the load T is in the direction of the central or z axis of the cylinder and that the load is distributed uniformly over the ends. An additional load M , given by the twisting moment M in Section 3, will also be imposed so that the total load L will be the combination of the loads T and M . Our problem is now to find the possible loads T and M which will produce a critical load L , *i.e.* a load L just sufficient to cause the fracture of the cylinder.

The nonvanishing components τ_{ij} of the plastic stress tensor will be τ_{33} and τ_{23} as is clear from the treatment of the related problems in the two preceding sections; these components may be assumed to be given by equations of the form

$$(4.1) \quad \tau_{33} = \tau_{33}(r) \geq 0; \quad \tau_{23} = \tau_{23}(r) \geq 0$$

in view of the symmetry conditions and the nature of the applied loads. Now construct the equation

$$|\tau_{ij} - s g_{ij}| = \begin{vmatrix} -s & 0 & 0 \\ 0 & -s r^2 & \tau_{23} \\ 0 & \tau_{23} & \tau_{33} - s \end{vmatrix} = 0$$

for the determination of the principal stresses s of the plastic stress field (4.1). Solving this equation we find that

$$s_1 = \frac{\tau_{33}}{2} + \frac{1}{2}(\tau_{33}^2 + 4\tau_{23}^2/r^2)^{1/2}; \quad s_2 = 0; \quad s_3 = \frac{\tau_{33}}{2} - \frac{1}{2}(\tau_{33}^2 + 4\tau_{23}^2/r^2)^{1/2}.$$

Hence the equation (1.3) becomes

$$k\tau_{33} + (1 + k^2)^{1/2}(\tau_{33}^2 + 4\tau_{23}^2/r^2)^{1/2} = 2K$$

and when this equation is solved for the component τ_{23} we have

$$(4.2) \quad \tau_{23} = \frac{r(4K^2 - 4kK\tau_{33} - \tau_{33}^2)^{1/2}}{2(1 + k^2)^{1/2}}.$$

To obtain additional information concerning the plastic stress field we must

turn to the equilibrium equations (1.4) and the constitutive equations (1.5). But the equilibrium equations (1.4) are found to be satisfied identically and hence we must rely entirely on the equations (1.5) for the needed information. Now the components u_i of the plastic deformation may be assumed to be functions of the type

$$(4.3) \quad u_1 = u_1(r); \quad u_2 = u_2(r, z); \quad u_3 = u_3(z).$$

Calculating the components $u_{i,j}$ of the covariant derivative of the plastic deformation (4.3) with the aid of the above relations (3.1) and (3.2), we find that

$$\begin{aligned} u_{1;1} &= \frac{du_1}{dr}; & u_{1;2} &= -\frac{u_2}{r}; & u_{1;3} &= 0, \\ u_{2;1} &= \frac{\partial u_2}{\partial r} - \frac{u_2}{r}; & u_{2;2} &= u_1 r; & u_{2;3} &= \frac{\partial u_2}{\partial z}, \\ u_{3;1} &= 0; & u_{3;2} &= 0; & u_{3;3} &= \frac{du_3}{dz}. \end{aligned}$$

Hence the plastic strain components $e_{i,j}$ are given by

$$(4.4) \quad \begin{cases} e_{11} = \frac{du_1}{dr}; & e_{22} = u_1 r; & e_{33} = \frac{du_3}{dz} \\ e_{12} = \frac{1}{2} \frac{\partial u_2}{\partial r} - \frac{u_2}{r}; & e_{13} = 0; & e_{23} = \frac{1}{2} \frac{\partial u_2}{\partial z} \end{cases}$$

from the equations (1.6) In addition to the equations (4.4) we shall need the equations

$$(4.5) \quad \begin{cases} \tau_{11}^* = -\frac{\tau_{33}}{r}; & \tau_{22}^* = -\frac{r^2 \tau_{33}}{3}; & \tau_{33}^* = \frac{2\tau_{33}}{3} \\ \tau_{12}^* = 0; & \tau_{13}^* = 0; & \tau_{23}^* = \tau_{23} \end{cases}$$

which give the components $\tau_{i,j}^*$ of the deviator of the plastic stress field; the above equations (4.5) are obtained directly from the equations (1.6) in which this deviator is defined. The expanded form of the first set of equations (1.5) can now be obtained immediately from the relations (4.4) and (4.5); thus we have

$$(4.6) \quad \psi \frac{du_1}{dr} = -\frac{\tau_{33}}{3},$$

$$(4.7) \quad \psi r u_1 = -\frac{r^2 \tau_{33}}{3},$$

$$(4.8) \quad \psi \frac{du_3}{dz} = \frac{2\tau_{33}}{3},$$

$$(4.9) \quad \frac{\partial u_2}{\partial r} = 2 \frac{u_2}{r},$$

$$(4.10) \quad \psi \frac{\partial u_2}{\partial z} = 2\tau_{23}.$$

The last equation (1.5), which expresses the condition of plastic incompressibility, does not need to be considered explicitly since it is an algebraic consequence of the first set of these equations.

Now combine equations (4.6) and (4.7) to obtain

$$\frac{du_1}{dr} = \frac{u_1}{r}.$$

This equation admits a solution of the form

$$(4.11) \quad u_1 = -br, \quad b > 0,$$

in which the minus sign has been introduced in the right member to conform with the fact that there will be a displacement toward the center of the cylindrical bar due to the tensile load T . Substituting u_1 from (4.11) into (4.7) we find that

$$(4.12) \quad \psi = \frac{\tau_{33}}{3b}.$$

The equation (4.6) must of course be satisfied as a consequence of (4.11) and (4.12). Next eliminate ψ from (4.8) by means of (4.12) and solve the resulting equation for $u_3(z)$ subject to the condition that $u_3(0)$ must vanish from symmetry; but this gives

$$(4.13) \quad u_3 = 2bz.$$

Continuing we readily find from the equation (4.9) and the equation (4.10) in which ψ has been eliminated by means of (4.12), that

$$(4.14) \quad u_2 = czr^2, \quad c > 0,$$

$$(4.15) \quad 2\tau_{23} = \frac{mr^2\tau_{33}}{a}, \quad m = \frac{ac}{3b},$$

where c is a constant and a is the radius of the bar. The constant c must be positive, as indicated in the equation (4.14), since u_2 must obviously be positive in the upper half and negative in the lower half of the bar due to the action of the couples considered in Section 3; also the radius a of the bar and the constant m , defined by the second equation (4.15), have been introduced to meet the requirements of the following discussion. The equations (4.11), (4.13) and (4.14) determine the plastic deformation and the equation (4.15) provides an additional relation between the components of the plastic stress field. Using (4.15) to eliminate τ_{23} from (4.2) and solving the resulting equation for τ_{33} we have

$$(4.16) \quad \tau_{33} = \frac{2K[(1+k^2)^{1/2}(1+m^2x^2)^{1/2} - k]}{1 + (1+k^2)m^2x^2}; \quad x = \frac{r}{a},$$

in which the variable x is defined by the second of these equations.

The tensile load T and the twisting moment M are given, respectively, by the equations

$$(4.17) \quad T = \iint \tau_{33}(r) dS = 2\pi \int_0^a \tau_{33}(r)r dr,$$

$$(4.18) \quad M = \iint \tau_{23}(r) dS = 2\pi \int_0^a \tau_{23}(r)r dr,$$

where the surface integrals can be taken over any cross section of the bar. Now effect a change of variables from the variable r in (4.17) and (4.18) to the variable x which appears in the equation (4.16). The equations (4.17) and (4.18) can then be written in the form

$$(4.19) \quad T = \pi a^2 K A(m, k),$$

$$(4.20) \quad M = \pi a^3 K m B(m, k),$$

when use is made of the equations (4.15) and (4.16); the functions $A(m, k)$ and $B(m, k)$ will be given by the equations

$$(4.21) \quad A(m, k) = 4 \int_0^1 \frac{[(1+k^2)^{1/2}(1+m^2x^2)^{1/2} - k]x dx}{1 + (1+k^2)m^2x^2},$$

$$(4.22) \quad B(m, k) = 2 \int_0^1 \frac{[(1+k^2)^{1/2}(1+m^2x^2)^{1/2} - k]x^3 dx}{1 + (1+k^2)m^2x^2}.$$

The following result can now be stated. *A critical load L will be determined by the tensile load T and the twisting moment M given by (4.19) and (4.20), respectively.*

It is clear from the above discussion that m can be any positive number, *i.e.*

$$(4.23) \quad 0 < m < \infty$$

and when m satisfies this inequality a critical load L will be produced by *actual* loads T and M given by (4.19) and (4.20), respectively, *i.e.* neither of these latter loads will vanish; conversely if T and M are nonvanishing loads which produce a critical load L , then T and M must be given by the equations (4.19) and (4.20) for some number m satisfying the inequality (4.23). Now it is easily seen from the equations (4.21) and (4.22) that

$$\lim_{m \rightarrow 0} A(m, k) = 2[(1+k^2)^{1/2} - k]; \quad \lim_{m \rightarrow 0} mB(m, k) = 0,$$

$$\lim_{m \rightarrow \infty} A(m, k) = 0; \quad \lim_{m \rightarrow \infty} mB(m, k) = \frac{2}{3(1+k^2)^{1/2}}.$$

Hence we have

$$\lim_{m \rightarrow 0} T = 2\pi a^2 K [(1+k^2)^{1/2} - k]; \quad \lim_{m \rightarrow 0} M = 0,$$

$$\lim_{m \rightarrow \infty} T = 0; \quad \lim_{m \rightarrow \infty} M = \frac{2}{3} \frac{\pi K a^3}{(1+k^2)^{1/2}},$$

from equations (4.19) and (4.20). In the first of these limiting cases the moment M vanishes and fracture is produced entirely by the tensile load T ; the condition for such fracture is equivalent to the previous condition (2.9) for the critical stress. The tensile load T vanishes in the second limiting case and we obtain the condition (3.9) previously found for the critical moment M due to opposing couples on the ends of the bar.

Remark. The reader, if interested, can easily supply the necessary modifications of the formulas in this section when the tensile load T is replaced by a compressive load C in the determination of the critical load L .

5. Torsion combined with hydrostatic pressure. If a hydrostatic pressure p is added to the elastic stress field produced in a circular bar of radius a by the action of opposing couples over its ends (see Section 3), the resulting stress field will have components

$$(5.1) \quad \sigma_{23} = \mu\beta r^2; \quad \sigma_{ij} = -pg_{ij}, \quad \text{otherwise,}$$

and the principal stresses s will be given as solutions of the equation

$$|\sigma_{ij} - sg_{ij}| = \begin{vmatrix} -(p+s) & 0 & 0 \\ 0 & -(p+s)r^2 & \sigma_{23} \\ 0 & \sigma_{23} & -(p+s) \end{vmatrix} = 0.$$

We thus find that

$$(5.2) \quad s_1 = \frac{\sigma_{23}}{r} - p; \quad s_2 = -p; \quad s_3 = -\frac{\sigma_{23}}{r} - p.$$

Substituting these values of s_1 and s_3 into the slip inequality (1.1) we have

$$(5.3) \quad (1 + k^2)^{1/2} \frac{\sigma_{23}}{r} - kp \geq K,$$

or

$$(5.4) \quad (1 + k^2)^{1/2} \mu\beta r - kp \geq K,$$

when we replace σ_{23} by its value in (5.1).

The equations (5.1) indicate that the components τ_{ij} of the plastic stress field, which is initiated immediately prior to fracture, may be assumed to be given by equations of the form

$$(5.5) \quad \tau_{23} = \tau_{23}(r) \geq 0; \quad \tau_{ij} = -pg_{ij}, \quad \text{otherwise,}$$

as the result of the addition of the hydrostatic pressure p . The equilibrium equations (1.4) are satisfied identically by the stress field (5.5) and it is readily found that the constitutive conditions (1.5) admit a simple solution u_i which satisfies the obvious symmetry requirements of the problem, namely

$$u_1 = 0; \quad u_2 = czr^2; \quad u_3 = 0$$

where c is a positive constant. Replacing σ_{23} in (5.2) by $\tau_{23}(r)$ we obtain the principal stresses

$$s_1 = \frac{\tau_{23}}{r} - p; \quad s_2 = -p; \quad s_3 = -\frac{\tau_{23}}{r} - p$$

in the plastic stress field (5.5) and when these values of s_1 and s_3 are substituted into the equation (1.3) we find that

$$(5.6) \quad \tau_{23} = \frac{(K + kp)r}{(1 + k^2)^{1/2}}.$$

To determine the critical moment M due to the opposing couples we have merely to substitute the value of τ_{23} from (5.6) into the equation

$$M = \iint \tau_{23} dS = 2\pi \int_0^a \tau_{23} r dr$$

as is clear from the discussion in Section 3; this gives

$$(5.7) \quad M = \frac{2}{3} \frac{\pi(K + kp)a^3}{(1 + k^2)^{1/2}}$$

and the following result has now been obtained. *Fracture will occur in a circular bar subjected to a hydrostatic pressure p and twisted by opposing couples on its ends when the moment M produced by the couples reaches the value given by the equation (5.7) in which a is the radius of the bar.* A comparison of the equations (3.9) and (5.7) shows that the effect of pressure is to delay the occurrence of plastic deformation and fracture.

The critical moment M will also be given by the equation (3.6) in which β is the critical twist. Combining the equations (3.6) and (5.7) we thus obtain

$$(5.8) \quad \beta = \frac{4}{3} \frac{(K + kp)}{\mu a(1 + k^2)^{1/2}}$$

as the equation for the critical twist β . Finally the equation (5.8) implies that plastic deformation and fracture will not occur until the slip inequality (5.4) is satisfied in the region (3.12) corresponding to the result in Section 3.

6. Circular pipe under internal pressure. Cylindrical coordinates r, θ, z will be used in the treatment of this problem and the z axis will, of course, be taken to lie along the central axis of the pipe. Let a be the inner and b the outer radius of the pipe in its unstrained state as indicated by the cross section in Fig. 3. It is assumed that negligibly small changes in the radii are produced by the elastic deformations and also that the radii will remain effectively unchanged during the brief period in which the pipe is in the plastic state prior to fracture (see Section 1). Hence if τ_{ij} are the components of the plastic stress tensor, we must have

FIGURE 3.

$$(6.1) \quad \tau_{11} = -p, \quad \text{for } r = a; \quad \tau_{11} = 0, \quad \text{for } r = b,$$

where p is the internal pressure. Assuming the pipe to be indefinitely long so that end effects need not be considered, the components u_i of the plastic deformation will be given by equations of the form

$$(6.2) \quad u_1 = u_1(r); \quad u_2 = 0; \quad u_3 = 0$$

in view of the symmetry conditions of the problem; also from symmetry we have

$$(6.3) \quad \tau_{ij} = \tau_{ij}(r),$$

i.e. the components τ_{ij} depend only on the radial coordinate.

It follows from (6.2) that

$$u_{1;1} = \frac{du_1}{dr}; \quad u_{1;2} = 0; \quad u_{1;3} = 0,$$

$$u_{2;1} = 0; \quad u_{2;2} = u_1 r; \quad u_{2;3} = 0,$$

$$u_{3;1} = 0; \quad u_{3;2} = 0; \quad u_{3;3} = 0,$$

and hence the plastic strain components e_{ij} are given by

$$(6.4) \quad \begin{cases} e_{11} = \frac{du_1}{dr}; & e_{22} = u_1 r; & e_{33} = 0 \\ e_{12} = 0; & e_{13} = 0; & e_{23} = 0 \end{cases}$$

from the second set of equations (1.6). Also we have

$$(6.5) \quad \tau_{11}^* = \frac{2}{3} \tau_{11} - \frac{1}{3} \left(\frac{\tau_{22}}{r^2} + \tau_{33} \right),$$

$$(6.6) \quad \tau_{22}^* = \frac{2}{3} \tau_{22} - \frac{1}{3} \left(\tau_{11} + \tau_{33} \right) r^2,$$

$$(6.7) \quad \tau_{33}^* = \frac{2}{3} \tau_{33} - \frac{1}{3} \left(\tau_{11} + \frac{\tau_{22}}{r^2} \right),$$

$$(6.8) \quad \tau_{12}^* = \tau_{12}; \quad \tau_{13}^* = \tau_{13}; \quad \tau_{23}^* = \tau_{23}$$

from the first set of equations (1.6) for the components τ_{ij}^* of the plastic stress deviator. Using (6.4) and (6.5), $\dots$, (6.8) we now see that the constitutive equations (1.5) provide the following relations

$$(6.9) \quad 2\tau_{11} - \left(\frac{\tau_{22}}{r^2} + \tau_{33} \right) = 3\psi \frac{du_1}{dr},$$

$$(6.10) \quad 2 \frac{\tau_{22}}{r^2} - (\tau_{11} + \tau_{33}) = 3\psi \frac{u_1}{r},$$

$$(6.11) \quad \tau_{33} = \frac{1}{2} \left(\tau_{11} + \frac{\tau_{22}}{r^2} \right),$$

$$(6.12) \quad \tau_{12} = \tau_{13} = \tau_{23} = 0.$$

It is also seen from (6.12) that τ_{11} , τ_{22} and τ_{33} are the only nonvanishing components of the stress tensor and these components depend on the radial coordinate alone in accordance with (6.3). Hence it follows that the equilibrium conditions (1.4) reduce to the single equation

$$(6.13) \quad r \frac{d\tau_{11}}{dr} + \tau_{11} - \frac{\tau_{22}}{r^2} = 0.$$

Let us now observe that

$$(6.14) \quad \frac{du_1}{dr} + \frac{u_1}{r} = 0$$

as an algebraic consequence of the equations (6.9), (6.10) and (6.11). Integrating (6.14) we obtain

$$(6.15) \quad u_1 = \frac{\epsilon}{r}$$

where ϵ is a positive constant which can be regarded as a modulus of the solid since this equation gives the radial deformation immediately prior to fracture. Hence (6.10) becomes

$$2 \frac{\tau_{22}}{r^2} - (\tau_{11} + \tau_{33}) = \frac{3\psi\epsilon}{r^2}.$$

Eliminating τ_{33} from the latter equation by means of (6.11), we obtain

$$(6.16) \quad \frac{\tau_{22}}{r^2} - \tau_{11} = \frac{2\psi\epsilon}{r^2} > 0$$

since ψ is positive by assumption (see Section 1). But the value of τ_{33} is midway between the values of τ_{11} and τ_{22}/r^2 from (6.11) and when this fact is used in conjunction with the inequality (6.16) we see that

$$(6.17) \quad \frac{\tau_{22}}{r^2} > \tau_{33} > \tau_{11} .$$

The principal plastic stresses are the solutions s of the equation

$$|\tau_{ij} - sg_{ij}| = \begin{vmatrix} \tau_{11} - s & 0 & 0 \\ 0 & \tau_{22} - sr^2 & 0 \\ 0 & 0 & \tau_{33} - s \end{vmatrix} = 0.$$

Solving this equation for s and taking account of the inequalities (6.17) we now see that

$$(6.18) \quad s_1 = \frac{\tau_{22}}{r^2} ; \quad s_2 = \tau_{33} ; \quad s_3 = \tau_{11} .$$

Substituting s_1 and s_3 from (6.18) into the equation (1.3) and solving for the quantity τ_{22}/r^2 we obtain

$$(6.19) \quad \frac{\tau_{22}}{r^2} = \alpha + \beta\tau_{11}$$

where

$$(6.20) \quad \alpha = 2[(1 + k^2)^{1/2} - k]K; \quad \beta = [(1 + k^2)^{1/2} - k]^2.$$

Next use (6.19) to eliminate τ_{22}/r^2 from the equation (6.13); this gives

$$(6.21) \quad r \frac{d\tau_{11}}{dr} + \gamma\tau_{11} - \alpha = 0$$

where we have put

$$(6.22) \quad \gamma = 1 - \beta = 2k[(1 + k^2)^{1/2} - k].$$

The solution of (6.21) is given by

$$(6.23) \quad \tau_{11} = \frac{c}{r^\gamma} + \frac{\alpha}{\gamma}$$

in which c is the constant of integration.

It now follows from (6.23) and the boundary conditions (6.1) that

$$(6.24) \quad \frac{c}{a^\gamma} + \frac{\alpha}{\gamma} = -p; \quad \frac{c}{b^\gamma} + \frac{\alpha}{\gamma} = 0.$$

Eliminating the constant c between the two equations (6.24) we thus obtain

$$(6.25) \quad p = \frac{\alpha}{\gamma} \left[\left(\frac{b}{a} \right)^\gamma - 1 \right] = \frac{K}{k} \left[\left(\frac{b}{a} \right)^\gamma - 1 \right]$$

when use is made of the equations (6.20) and (6.22) giving the values of the quantities α and γ , respectively. The following result has now been established. *The critical pressure p is given by the equation (6.25) in which a and b are the inner and outer radii of the circular pipe.*

A definitive expression for the stress component τ_{11} is obtained when we replace the integration constant c in the equation (6.23) by the value given by the second equation (6.24). Thus we find that

$$\tau_{11} = \frac{K}{k} \left[1 - \left(\frac{b}{r} \right)^\gamma \right].$$

Substituting this expression for τ_{11} into the equation (6.19), we obtain the definitive expression for τ_{22}/r^2 and the corresponding expression for the component τ_{33} is then obtained from (6.11) by the obvious substitutions.

Remark. Putting $m = b/a$ let us consider the power series expansion of the function

$$f(\gamma) = m^\gamma$$

about the value $\gamma = 0$; but we have

$$\frac{df}{d\gamma} = (\log m)m^\gamma; \quad \frac{d^2f}{d\gamma^2} = (\log m)^2 m^\gamma; \dots$$

and hence

$$f(\gamma) = 1 + (\log m)\gamma + \frac{1}{2} (\log m)^2 \gamma^2 + \frac{1}{3!} (\log m)^3 \gamma^3 + \dots$$

Substituting this series for $(b/a)^\gamma$ into the first bracket expression in (6.25) it is seen that the equation for the critical pressure p can be written

$$(6.26) \quad p = \alpha \sum_{n=1}^{\infty} \frac{1}{n!} \left(\log \frac{b}{a} \right)^n \gamma^{n-1}.$$

The case of the thin pipe is of particular interest from the practical standpoint. But for such pipes the quantity $\log b/a$ is small and since the modulus γ is also small, the critical pressure p can be represented with sufficient accuracy by the first term in the power series expansion in (6.26). Replacing α by its value from (6.20) we therefore have the following result. *A thin pipe will fracture when the internal pressure p reaches the value given by the equation*

$$(6.27) \quad p = 2[(1 + k^2)^{1/2} - k]K \log \frac{b}{a}.$$

In the case of pipes of such thickness that the simple equation (6.27) is not

applicable, one can avail himself of the equation (6.25) or (6.26) for the determination of the critical pressure.

7. Hollow sphere under hydrostatic pressure. The problem treated in this section is of some practical interest because of its relation to the problem of the determination of the depth beyond which a so called bathysphere cannot be submerged in the ocean without danger of fracture or collapse due to the hydrostatic pressure. A system of spherical coordinates r, θ, φ with origin at the center of the sphere will be used in the discussion of this problem. If we also denote the coordinates r, θ, φ by x^1, x^2, x^3 respectively, the covariant components g_{ij} and the contravariant components g^{ij} of the metric tensor will be given by the matrices

$$||g_{ij}|| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{vmatrix}; \quad ||g^{ij}|| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2 \theta} \end{vmatrix}.$$

In addition the formulas

$$\begin{aligned} \Gamma_{22}^1 &= -r; & \Gamma_{33}^1 &= -r \sin^2 \theta; & \Gamma_{12}^2 &= \frac{1}{r}, \\ \Gamma_{33}^2 &= -\sin \theta \cos \theta; & \Gamma_{13}^3 &= \frac{1}{r}; & \Gamma_{23}^3 &= \cot \theta, \end{aligned}$$

for the nonvanishing Christoffel symbols will be needed in the following discussion.

It is evident that the components u_i of the plastic deformation will be given by equations of the form

$$(7.1) \quad u_1 = u_1(r); \quad u_2 = 0; \quad u_3 = 0$$

from the symmetry of the problem. Hence we find that

$$\begin{aligned} u_{1;1} &= \frac{du_1}{dr}; & u_{1;2} &= 0; & u_{1;3} &= 0, \\ u_{2;1} &= 0; & u_{2;2} &= ru_1; & u_{2;3} &= 0, \\ u_{3;1} &= 0; & u_{3;2} &= 0; & u_{3;3} &= r \sin^2 \theta u_1, \end{aligned}$$

when use is made of the above equations for the nonvanishing Christoffel symbols; consequently we have

$$(7.2) \quad \begin{cases} e_{11} = \frac{du_1}{dr}; & e_{22} = ru_1; & e_{33} = r \sin^2 \theta u_1 \\ e_{12} = 0; & e_{13} = 0; & e_{23} = 0 \end{cases}$$

from the equations (1.6) defining the components e_{ij} of the plastic strain tensor.

Also

$$(7.3) \quad \tau_{11}^* = \frac{2}{3} \tau_{11} - \frac{1}{3} \left(\frac{\tau_{22}}{r^2} + \frac{\tau_{33}}{r^2 \sin^2 \theta} \right),$$

$$(7.4) \quad \frac{\tau_{22}^*}{r^2} = \frac{2}{3} \frac{\tau_{22}}{r^2} - \frac{1}{3} \left(\tau_{11} + \frac{\tau_{33}}{r^2 \sin^2 \theta} \right),$$

$$(7.5) \quad \frac{\tau_{33}^*}{r^2 \sin^2 \theta} = \frac{2}{3} \frac{\tau_{33}}{r^2 \sin^2 \theta} - \frac{1}{3} \left(\tau_{11} + \frac{\tau_{22}}{r^2} \right),$$

$$(7.6) \quad \tau_{12}^* = \tau_{12}; \quad \tau_{13}^* = \tau_{13}; \quad \tau_{23}^* = \tau_{23},$$

where the τ_{ij}^* are the components of the stress deviator defined by the equations (1.6). Using (7.2) and the equations (7.3), $\dots$, (7.6) it is now seen that the constitutive equations (1.5) lead to the following relations, namely

$$(7.7) \quad 2\tau_{11} - \left(\frac{\tau_{22}}{r^2} + \frac{\tau_{33}}{r^2 \sin^2 \theta} \right) = 3\psi \frac{du_1}{dr},$$

$$(7.8) \quad 2 \frac{\tau_{22}}{r^2} - \left(\tau_{11} - \frac{\tau_{33}}{r^2 \sin^2 \theta} \right) = 3\psi \frac{u_1}{r},$$

$$(7.9) \quad 2 \frac{\tau_{33}}{r^2 \sin^2 \theta} - \left(\tau_{11} + \frac{\tau_{22}}{r^2} \right) = 3\psi \frac{u_1}{r},$$

$$(7.10) \quad \tau_{12} = \tau_{13} = \tau_{23} = 0.$$

Subtracting corresponding members of the equations (7.8) and (7.9) we obtain

$$(7.11) \quad \frac{\tau_{22}}{r^2} = \frac{\tau_{33}}{r^2 \sin^2 \theta}.$$

The above equations (7.7), (7.8) and (7.9) are now seen to reduce to the following two equations

$$(7.12) \quad 2\tau_{11} - 2 \frac{\tau_{22}}{r^2} = 3\psi \frac{du_1}{dr},$$

$$(7.13) \quad \frac{\tau_{22}}{r^2} - \tau_{11} = 3\psi \frac{u_1}{r},$$

when we eliminate the component τ_{33} by means of (7.11). Combining the equations (7.12) and (7.13) we find that

$$\frac{du_1}{dr} + 2 \frac{u_1}{r} = 0$$

and this differential equation admits the solution

$$(7.14) \quad u_1 = -\frac{\zeta}{r^2}, \quad \zeta > 0,$$

in which the integration constant ζ is positive, as indicated, from the nature of

the applied load, *i.e.* the imposed hydrostatic pressure. Finally the equations (7.12) and (7.13) reduce to the single equation

$$(7.15) \quad \tau_{11} - \frac{\tau_{22}}{r^2} = \frac{3\psi\zeta}{r^3}$$

as a consequence of the equation (7.14). But $\psi > 0$ by the assumption in Section 1 and hence the equation (7.15) implies the important inequality

$$(7.16) \quad \tau_{11} > \frac{\tau_{22}}{r^2}.$$

The principal stresses are given as the solutions s of the equation

$$|\tau_{ii} - sg_{ii}| = \begin{vmatrix} \tau_{11} - s & 0 & 0 \\ 0 & \tau_{22} - sr^2 & 0 \\ 0 & 0 & \tau_{33} - sr^2 \sin^2 \theta \end{vmatrix} = 0$$

when account is taken of the equations (7.10). Hence it is seen that

$$(7.17) \quad s_1 = \tau_{11}; \quad s_2 = s_3 = \frac{\tau_{22}}{r^2} = \frac{\tau_{33}}{r^2 \sin^2 \theta}$$

in view of the equation (7.11) and the inequality (7.16). Substituting τ_{11} and τ_{22}/r^2 for s_1 and s_3 in (1.3) and solving the resulting equation for τ_{22}/r^2 we are led to the equation

$$(7.18) \quad \frac{\tau_{22}}{r^2} = [(1 + k^2)^{1/2} + k]^2 \tau_{11} - 2[(1 + k^2)^{1/2} + k]K$$

which will be needed in the following discussion.

Further information can be obtained from the equilibrium equations (1.4). But before we make use of these equations it will be helpful to determine the general functional character of the components τ_{11} , τ_{22} and τ_{33} of the stress tensor. In this connection let S be a surface element which is perpendicular to the radial direction, the θ direction or the φ direction and let ν^i be the contravariant components of the unit vector ν normal to the element S . The quantity $\tau_{ij}\nu^i\nu^j$, which gives the normal force per unit area on the surface element S , can evidently depend only on the radial coordinate r from the symmetry of the problem. But it is easily seen that the vector ν has components

$$(1, 0, 0); \quad (0, 1/r, 0); \quad (0, 0, 1/r \sin \theta)$$

according as S is perpendicular to the radial direction, the θ direction or the φ direction, respectively. Hence we have

$$(7.19) \quad \tau_{ij}\nu^i\nu^j = \tau_{11}, \quad \text{for } \nu = (1, 0, 0),$$

$$(7.20) \quad \tau_{ij}\nu^i\nu^j = \frac{\tau_{22}}{r^2}, \quad \text{for } \nu = (0, 1/r, 0),$$

$$(7.21) \quad \tau_{i;\nu^i} \nu^i = \frac{\tau_{33}}{r^2 \sin^2 \theta}, \quad \text{for } \nu = (0, 0, 1/r \sin \theta).$$

It appears from (7.19) and (7.20) that the components τ_{11} and τ_{22} can depend only on the radial coordinate. Hence τ_{33} must be the product of $\sin^2 \theta$ and a function of the radial coordinate from (7.11); this result is also implied by the above equation (7.21). With these facts in mind and also the fact that the components τ_{12} , τ_{13} and τ_{23} vanish in accordance with (7.10), let us now consider the equilibrium equations (1.4). Putting $i = 1$ in (1.4) and expanding the resulting equation, we find that

$$(7.22) \quad r \frac{d\tau_{11}}{dr} + 2\tau_{11} - 2 \frac{\tau_{22}}{r^2} = 0$$

when account is taken of the equation (7.11). If we put $i = 2$ in (1.4) we obtain the relation (7.11) which was previously derived. Also no additional information is obtained by taking $i = 3$ in (1.4) since the resulting equation is found to be satisfied identically.

The above equation (7.22) becomes

$$(7.23) \quad r \frac{d\tau_{11}}{dr} - \omega \tau_{11} + \eta = 0$$

when we eliminate τ_{22}/r^2 by means of (7.18); the quantities ω and η in (7.23) are given by the equations

$$(7.24) \quad \omega = 4k[(1 + k^2)^{1/2} + k]; \quad \eta = 4[(1 + k^2)^{1/2} + k]K.$$

Solving the differential equation (7.23), we obtain

$$(7.25) \quad \tau_{11} = cr^\omega + \frac{\eta}{\omega}$$

in which c is a constant. But

$$(7.26) \quad \tau_{11} = 0, \quad \text{for } r = a; \quad \tau_{11} = -p, \quad \text{for } r = b,$$

where a and b are the inner and outer radii of the sphere, respectively. Hence it follows from (7.25) and (7.26) that

$$(7.27) \quad ca^\omega + \frac{\eta}{\omega} = 0; \quad cb^\omega + \frac{\eta}{\omega} = -p.$$

When we substitute the value of c from the first equation (7.27) into the second of these equations, we now obtain

$$(7.28) \quad p = \frac{\eta}{\omega} \left[\left(\frac{b}{a} \right)^\omega - 1 \right] = \frac{K}{k} \left[\left(\frac{b}{a} \right)^\omega - 1 \right]$$

and the following result can be stated. *The critical value of the hydrostatic pressure p is given by the equation (7.28) in which a and b are the inner and outer radii of the sphere. It is also clear from the Remark in Section 6 that an alternative formula for the critical pressure p is given by the series*

$$(7.29) \quad p = \eta \sum_{n=1}^{\infty} \frac{1}{n} \left(\log \frac{b}{a} \right)^n \omega^{n-1}.$$

In the case of a thin sphere, *i.e.* a sphere for which the outer radius is not appreciably larger than the inner radius, the formula (7.29) can be approximated by the simple equation

$$(7.30) \quad p = 4[(1 + k^2)^{1/2} + k]K \log \left(\frac{b}{a} \right).$$

Remark 1. The *thickness* of the sphere can be defined as the difference $b - a$ of its outer and inner radii. However, it is not the thickness as so defined but the *relative thickness* defined by the ratio $(b - a)/a$ which is of primary importance due to the fact that the critical pressure depends directly on the quantity b/a in this fracture problem. A thin sphere should therefore be regarded as a sphere whose relative thickness is extremely small. Similar remarks can, of course, be made in the case of the pipe discussed in Section 6.

Remark 2. Analogous results are obtained if the internal pressure is replaced by a hydrostatic pressure p acting on the outer surface of the pipe or hollow cylinder considered in Section 6. In the case of a thin hollow cylinder the critical hydrostatic pressure p can be shown to be given by the equation

$$(7.31) \quad p = 2[(1 + k^2)^{1/2} + k]K \log \frac{b}{a}$$

where a and b are the inner and outer radii of the cylinder. Comparison of the formulas (7.30) and (7.31) enables us to state the following result. *The hydrostatic pressure required to fracture a thin hollow sphere will be twice as large as that required to fracture a thin hollow cylinder of the same relative thickness.* It is of course assumed in making this statement that the values of the slip moduli k and K are the same for the cylinder and the sphere.

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