The Hausdorff Dimension of the Range of a Process with Stationary Independent Increments

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Communicated by WILLIAM FELLER

1. Introduction. Blumenthal and Gettoor [1] defined upper and lower indices, β and β' , for a process with stationary and independent increments and related many properties of the sample paths to these indices. The particular property of interest in the present paper is the Hausdorff dimension of the range of X . Blumenthal and Gettoor showed (subject to certain additional assumptions on the indices) that

$$\beta' \leq \dim X[0, 1] \leq \beta \quad \text{a.s.}$$

(The restrictions on the indices were partially removed by Orey [5].) They also defined another index σ for subordinators and proved that for subordinators

$$\sigma \leq \dim X[0, 1] \leq \beta \quad \text{a.s.}$$

with the implication being that σ might be larger than β' . Recently Horowitz [4] has shown that in the case of a subordinator

$$\dim X[0, 1] = \sigma \quad \text{a.s.}$$

We define an index γ for any process with stationary independent increments by

$$(1.1) \quad \gamma = \sup \left\{ \alpha \geq 0 : \limsup_{a \rightarrow 0} a^{-\alpha} \int_0^1 P[|X(t)| \leq a] dt < \infty \right\}.$$

We will prove first that

$$\dim X[0, 1] = \gamma \quad \text{a.s.}$$

The proof is relatively easy and is based on two lemmas that appeared in [6]

¹ Research was supported in part by NSF Grant No. GP-7490.

as well as the density theorem of Rogers and Taylor [7]. Then we will obtain some relations between γ and the other indices. In particular, we will show that $\gamma = \sigma$ in the case of a subordinator (thereby giving an alternative proof of Horowitz' result), and that $\gamma = \min(\beta', N)$ for a symmetric process. We also give an example of a subordinator with $\beta' < \sigma$; since β' is left invariant under symmetrization, this provides an example of a process such that the dimension of the range is different for the process and its symmetrization.

2. Preliminaries. The N -dimensional characteristic function of a process with stationary and independent increments has the form $\exp[-t\psi(y)]$ where

$$\psi(y) = i(b, y) + \frac{1}{2}ySy' + \int \left[1 - e^{i(x, y)} + \frac{i(x, y)}{1 + |x|^2} \right] \nu(dx),$$

with $b \in \mathbf{R}^N$, S a non-negative definite symmetric N by N matrix, and ν a Borel measure on $\mathbf{R}^N$ satisfying

$$\int \frac{|x|^2}{1 + |x|^2} \nu(dx) < \infty.$$

We will assume that $S = 0$, i.e. there is no Gaussian part. Also if

$$\int_{|x| < 1} |x| \nu(dx) < \infty$$

we will, by subtracting a deterministic linear component from the process, write

$$\psi(y) = \int [1 - e^{i(x, y)}] \nu(dx).$$

We assume throughout that the process satisfies the conditions of Hunt's hypothesis (A). Thus the sample functions $X(t)$ will be right continuous and have left limits everywhere. The process will also have the strong Markov property. We will use $T(a, s)$ to denote the sojourn time of the process in a sphere up to time s , i.e.

$$T(a, s) = \int_0^s I_a(X(t)) dt,$$

where I_a is the indicator function of the closed sphere of radius a , center the origin.

For convenience we quote two lemmas from [6] which will be the basis for the proof of the main theorem. The first is the corollary to Lemma 3.1 and the second is Lemma 6.1.

Lemma 1. *There is a $\lambda_0 > 0$ such that*

$$P[T(a, s) \geq \lambda ET(2a, s)] \leq e^{-\lambda/2} \quad \text{for } \lambda \geq \lambda_0.$$

Lemma 2. *Let $\Lambda(a)$ be a collection of N -dimensional cubes of side a with the property that no point of $\mathbf{R}^N$ is contained in more than K of the cubes. If $M(a, s)$*

is the number of these cubes hit by the path $X(t)$ at some time $t \in [0, s]$, then there is a positive constant K' such that

$$EM(a, s) \leq K'[ET(a/3, s)]^{-1}.$$

3. The dimension. We first prove the main result that asserts that γ is the dimension of almost all sample paths. Then we will obtain an alternative characterization for γ and prove that $\gamma = \sigma$ for subordinators and $\gamma = \min(\beta', N)$ for symmetric processes. Some further information on γ will be obtained in other cases as well.

Theorem 1. *If γ is defined by (1.1), then*

$$\dim X[0, 1] = \gamma \quad \text{a.s.}$$

Proof. We will first show that the dimension is at least γ . If $\gamma = 0$ there is nothing to prove. Otherwise let $0 < \alpha < \gamma$ and then choose $\beta \in (\alpha, \gamma)$. Let $a_k = e^{-k}$ and

$$f(a) = ET(2a, 1) = \int_0^1 P[|X(t)| \leq 2a] dt;$$

by Lemma 1,

$$P[T(a_k, 1) \geq 4 \log |\log a_k| f(a_k)] \leq k^{-2}$$

for k large. By Borel–Cantelli and the definition of γ ,

$$\limsup_{k \rightarrow \infty} \frac{T(a_k, 1)}{a_k^\beta \log |\log a_k|} \leq \limsup_{k \rightarrow \infty} \frac{T(a_k, 1)}{f(a_k) \log |\log a_k|} < \infty \quad \text{a.s.}$$

Then $\lim_{k \rightarrow \infty} T(a_k, 1)/a_k^\alpha = 0$ and since $T(a, 1)$ is monotone and a_k/a_{k+1} is bounded we have in fact that

$$\lim_{a \rightarrow 0} \frac{T(a, 1)}{a^\alpha} = 0 \quad \text{a.s.}$$

A standard application of the density theorem of Rogers and Taylor [7] then shows that the α -dimensional Hausdorff measure of $X[0, 1]$ is infinite with probability one. (This argument is in [3], p. 448.) We thus have $\dim X[0, 1] \geq \gamma$. For the other inequality let $\alpha > \gamma$; we know that $\limsup f(a)/a^\alpha = \infty$ so we choose $\{a_k\}$ such that $a_k \rightarrow 0$ and

$$f(a_k) \geq ka_k^\alpha.$$

Now cover $\mathbf{R}^N$ with abutting cubes of side $6a_k$. By Lemma 2, if M_k is the number hit by time one,

$$EM_k \leq K'[f(a_k)]^{-1} \leq K'k^{-1}a_k^{-\alpha}.$$

Therefore

$$P[M_k a_k^\alpha \geq \epsilon] \leq K'\epsilon^{-1}k^{-1}$$

so that $M_k a_k^\alpha \xrightarrow{P} 0$. We can then find a subsequence which will converge to zero a.s. If we use for our covering of $X[0, 1]$ only those cubes which the path $X(t)$ enters, we have a sequence of coverings such that the corresponding sums converge to zero. This means that the α -dimensional measure is zero for $\alpha > \gamma$ and completes the proof.

Theorem 2. *The index*

$$\gamma = \sup \left\{ \alpha \geq 0 : \int_0^1 E |X(t)|^{-\alpha} dt < \infty \right\}.$$

Proof. Let

$$G_t(u) = P[|X(t)| \leq u], \quad H(u) = \int_0^1 G_t(u) dt.$$

Then

$$\begin{aligned} (3.1) \quad \int_0^1 E |X(t)|^{-\alpha} dt &= \int_0^1 \int_0^\infty u^{-\alpha} dG_t(u) dt = \int_0^\infty u^{-\alpha} dH(u) \\ &= \alpha \int_0^\infty u^{-\alpha-1} H(u) du, \end{aligned}$$

for all $\alpha > 0$ in the sense that if one of the integrals is infinite they all will be. Now if $\alpha < \beta < \gamma$, then $H(u) \leq Mu^\beta$ for small u and (3.1) converges. If, on the other hand, (3.1) converges, then for any $a > 0$,

$$M \geq \alpha \int_a^\infty u^{-\alpha-1} H(u) du \geq H(a) \alpha \int_a^\infty u^{-\alpha-1} du = H(a) a^{-\alpha},$$

so that $\alpha \leq \gamma$.

We now have Horowitz' result as a corollary.

Theorem 3. *If X is a subordinator, then*

$$\dim X[0, 1] = \sigma \quad a.s.$$

Proof. Blumenthal and Gettoor prove ([1], p. 503) that for all $\alpha > 0$

$$\int_0^1 E |X(t)|^{-\alpha} dt < \infty \quad \text{if and only if} \quad \int_1^\infty \frac{r^{\alpha-1}}{g(r)} dr < \infty$$

where g is the subordinator exponent of X , i.e.

$$g(r) = \int_0^\infty (1 - e^{-ur}) \nu(du).$$

This is almost the proof since

$$\sigma = \sup \left\{ \alpha \leq 1 : \int_1^\infty \frac{r^{\alpha-1}}{g(r)} dr < \infty \right\};$$

we must simply show that the condition $\alpha \leq 1$ may be omitted here. But for a subordinator $\int_0^1 u \nu(du) < \infty$ which implies that $g(r) \leq c_1 r + c_2$; thus the integral which occurs in the definition of σ must automatically diverge for $\alpha \geq 1$.

Before we prove that the dimension of the range is $\min(\beta', N)$ for a symmetric process, we need the following

Lemma 3. *Let $p_{\alpha N}(t, x)$ denote the symmetric stable density of index α in $\mathbb{R}^N$. Then for $\alpha > 0$,*

$$(3.2) \quad \int_0^1 E |X(s)|^{-\alpha} ds = \int_0^\infty \int p_{\alpha N}(t, x) \frac{1 - e^{-\psi(x)}}{\psi(x)} dx dt.$$

Proof. First note that

$$(3.3) \quad \begin{aligned} \int e^{-|u|^{\alpha t}} dF_s(u) &= \iint e^{i(x, u)} p_{\alpha N}(t, x) dx dF_s(u) \\ &= \int p_{\alpha N}(t, x) e^{-s\psi(x)} dx. \end{aligned}$$

Then we have

$$\begin{aligned} \int_0^1 E |X(s)|^{-\alpha} ds &= \int_0^1 \int |u|^{-\alpha} dF_s(u) ds \\ &= \int_0^1 \int \int_0^\infty e^{-|u|^{\alpha t}} dt dF_s(u) ds \\ &= \int_0^\infty \int_0^1 \int e^{-|u|^{\alpha t}} dF_s(u) ds dt \\ &= \int_0^\infty \int_0^1 \int p_{\alpha N}(t, x) e^{-s\psi(x)} dx ds dt \\ &= \int_0^\infty \int p_{\alpha N}(t, x) \frac{1 - e^{-\psi(x)}}{\psi(x)} dx dt, \end{aligned}$$

where (3.3) has been used.

Theorem 4. *If X is symmetric, then*

$$\dim X[0, 1] = \min(\beta', N) \quad a.s.$$

Proof. Since X is symmetric, $\psi(x)$ is positive and we can interchange order of integration in (3.2). Using the well known formula for the potential kernel of the symmetric stable process, we obtain if $\alpha < N$

$$\int_0^1 E |X(t)|^{-\alpha} dt = c \int |x|^{\alpha-N} \frac{1 - e^{-\psi(x)}}{\psi(x)} dx.$$

If $\alpha \geq N$, the integral diverges. Since

$$\beta' = \sup \left\{ \alpha \geq 0 : \int |x|^{\alpha-N} \frac{1 - e^{-\operatorname{Re} \psi(x)}}{\operatorname{Re} \psi(x)} dx < \infty \right\},$$

this completes the proof.

We would like to be able to define γ in terms of ψ in general; actually we have done this via Lemma 3, but we would like a simpler definition. If we could justify the interchange of order of integration in (3.2) in general, we would have such a definition, but we have been unable to do this. This interchange is valid in many cases, however, and we state the results we have as

Theorem 5. *For any process with stationary independent increments, $\min(\beta', N) \leq \gamma \leq \beta$. Moreover, for any $R > 0$,*

$$(3.4) \quad \gamma \geq \sup \left\{ \alpha < N : \int_{|x| > R} |x|^{\alpha-N} |\psi(x)|^{-1} dx < \infty \right\}.$$

If $\operatorname{Re} \psi(x) \geq 2 \log |x|$ for $|x|$ large, then

$$(3.5) \quad \gamma = \sup \left\{ \alpha < N : \int |x|^{\alpha-N} \operatorname{Re} \frac{1 - e^{-\psi(x)}}{\psi(x)} dx < \infty \right\}.$$

(Note that (3.5) is valid whenever Blumenthal and Gettoor's index β'' is positive.)

Proof. The two lower bounds for γ follow from Lemma 3 by observing that

$$\left| \frac{1 - e^{-\psi(x)}}{\psi(x)} \right| \leq \frac{1 - e^{-\operatorname{Re} \psi(x)}}{\operatorname{Re} \psi(x)},$$

and also

$$\left| \frac{1 - e^{-\psi(x)}}{\psi(x)} \right| \leq \min(1, 2 |\psi(x)|^{-1}),$$

and then making the interchange of order of integration. The proof of (3.5) is similar since

$$(3.6) \quad \operatorname{Re} \frac{1 - e^{-\psi(x)}}{\psi(x)} = \frac{\operatorname{Re} \psi(x) - e^{-\operatorname{Re} \psi(x)} \operatorname{Re} [e^{-i \operatorname{Im} \psi(x)} \overline{\psi(x)}]}{|\psi(x)|^2},$$

and then

$$\begin{aligned} |e^{-\operatorname{Re} \psi(x)} \operatorname{Re} [e^{-i \operatorname{Im} \psi(x)} \overline{\psi(x)}]| &\leq e^{-\operatorname{Re} \psi(x)} |\psi(x)| \\ &\leq |x|^{-2} |\psi(x)|. \end{aligned}$$

Now $|x|^{-2} |\psi(x)| \rightarrow 0$ as $|x| \rightarrow \infty$ ([2], p. 67) so that (3.6) is positive for large $|x|$ and the integrations in (3.2) may be performed in the opposite order. We must still show that $\gamma \leq \beta$. For this we use Theorem 3.1 of [1] which asserts that if $\alpha > \beta$, then $t^{-1/\alpha} X(t) \rightarrow 0$ a.s. as $t \rightarrow 0$. Looking at a fixed path, $|X(t)| \leq t^{1/\alpha}$ for all t sufficiently small. But then if a is small $|X(t)| \leq a$ for $t \leq a^\alpha$ so that $T(a, 1) \geq a^\alpha$. In the proof of Theorem 1, we proved that for $\alpha < \gamma$, $a^{-\alpha} T(a, 1) \rightarrow 0$ a.s. In the present case, then, $\alpha \geq \gamma$ which completes the proof.

4. **A subordinator with $\beta'' < \beta' < \sigma$.** Let the Lévy measure ν be atomic with mass $p_k = k^{-2}2^{2k}$ at $x_k = 2^{-2k} \cdot 2\pi$. Then

$$\operatorname{Re} \psi(y) = \sum_{k=1}^{\infty} (1 - \cos x_k y) p_k.$$

Let $y_i = 2^{2i}$; then for $k \leq j$, $x_k y_i$ is a multiple of 2π so that

$$(4.1) \quad \begin{aligned} \operatorname{Re} \psi(y_i) &= \sum_{k=j+1}^{\infty} (1 - \cos x_k y_i) p_k \leq y_i^2 \sum_{k=j+1}^{\infty} x_k^2 p_k \\ &\leq c j^{-2} 2^{-2j+1} y_i^2 = c j^{-2}, \end{aligned}$$

and $\operatorname{Re} \psi(y_i) \rightarrow 0$. This implies that the index β'' which Blumenthal and Gettoor defined by

$$\beta'' = \sup \{ \alpha \geq 0 : |y|^{-\alpha} \operatorname{Re} \psi(y) \rightarrow \infty \text{ as } |y| \rightarrow \infty \}$$

is zero in the present case. To compute σ we use the formula obtained by Horowitz [4]

$$\sigma = \sup \left\{ \alpha \leq 1 : x^{\alpha-1} \int_0^x \nu(y, \infty) dy \rightarrow \infty \text{ as } x \rightarrow 0 \right\}.$$

If $x_{k+1} \leq x < x_k$ and $\alpha < 1$, then

$$x^{\alpha-1} \int_0^x \nu(y, \infty) dy \geq x^{\alpha-1} (x_{k+1} - x_{k+2}) p_{k+1} \geq k^{-2} x_k^{\alpha-1} \rightarrow \infty,$$

so that $\sigma = 1$. It turns out to be the case that $\beta' = \frac{3}{2}$ in this example but the computations are somewhat involved so we will simply show that $\frac{1}{2} \leq \beta' \leq \frac{2}{3}$. This will serve our purpose since the point was to provide an example with β'' , β' , and $\sigma (= \gamma)$ all different. First, we have

$$\begin{aligned} \operatorname{Re} \psi(y) &\leq \sum_{k=1}^j 2p_k + \sum_{k=j+1}^{\infty} y^2 x_k^2 p_k \\ &\leq c_1 2^{2j} + c_2 y^2 2^{-2j+1}. \end{aligned}$$

Also since $1 - \cos \theta \geq c_3 \theta^2$ for $|\theta| \leq \pi$, we have for $y \leq \frac{1}{2} 2^{2j+1}$,

$$(4.2) \quad \operatorname{Re} \psi(y) \geq c_3 y^2 x_{j+1}^2 p_{j+1} \geq c_3 j^{-2} 2^{-2j+1} y^2.$$

Thus, letting $\xi_j = 2^{2j(3/2)}$, $\eta_j = \frac{1}{2} 2^{2j+1}$,

$$\begin{aligned} \int_{\xi_j}^{\eta_j} y^{\alpha-1} \frac{1 - e^{-\operatorname{Re} \psi(y)}}{\operatorname{Re} \psi(y)} dy &\geq c_4 2^{2j+1} \int_{\xi_j}^{\eta_j} y^{\alpha-3} dy \\ &\geq c_5 2^{2j(3\alpha/2-1)}. \end{aligned}$$

Summing on j we see that the integral in the definition of β' will diverge if $\alpha > \frac{2}{3}$ and thus $\beta' \leq \frac{2}{3}$. For the other bound we must be a little more careful. Let

$$y_\nu = \nu 2^{2^i}, \quad \nu = 1, 2, \dots, 2^{2^i}.$$

Then $1 - \cos x_i y$ is periodic with period 2^{2^i} and vanishes at y_v . If we let $y = y_v + z$ for $y \in [y_v, u_v]$, where $u_v = (y_v + y_{v+1})/2$, then

$$(4.3) \quad \operatorname{Re} \psi(y) \geq (1 - \cos x_i y) p_i \geq c_3 x_i^2 z^2 p_i \geq c_3 j^{-2} 2^{-2^i} z^2.$$

Now let $t_v = y_v(1 + 2^{-2^{i-1}})$ and use the estimate (4.2) on the interval $[y_v, t_v]$ and the estimate (4.3) on the interval $[t_v, u_v]$. We obtain

$$\begin{aligned} \int_{y_v}^{u_v} y^{\alpha-1} \frac{1 - e^{-\operatorname{Re} \psi(y)}}{\operatorname{Re} \psi(y)} dy &\leq c_6 y_v^{\alpha-3} j^2 2^{2^i+1} y_v 2^{-2^{i-1}} + c_7 y_v^{\alpha-1} j^2 2^{2^i} (y_v 2^{-2^{i-1}})^{-1} \\ &= c_8 v^{\alpha-2} j^2 2^{2^i(\alpha-1/2)}. \end{aligned}$$

Since $1 - \cos x_i y$ is symmetric about u_v and $y^{\alpha-1}$ and $(1 - \cos x_{i+1} y)^{-1}$ are smaller on $[u_v, y_{v+1}]$, the same bound is valid for that interval. Summing on v , we have

$$\int_{y_1}^{y_{1^2}} y^{\alpha-1} \frac{1 - e^{-\operatorname{Re} \psi(y)}}{\operatorname{Re} \psi(y)} dy \leq c_9 j^2 2^{2^i(\alpha-1/2)}.$$

Finally, this series will converge when summed on j if $\alpha < \frac{1}{2}$ which implies that $\beta' \geq \frac{1}{2}$.

We should point out that the right hand sides of (3.4) and (3.5) in Theorem 5 give the right value for γ here (even though the hypothesis for (3.5) is not satisfied). It would be interesting to know if (3.5) is true in general or if there is even equality in (3.4).

As a final remark we recall that it is a consequence of Theorems 3 and 4 that the range of the process under consideration has dimension one almost surely, but if we symmetrize the process the dimension of the range will then be $\frac{3}{2}$ almost surely.

REFERENCES

- [1] R. M. BLUMENTHAL & R. K. GETTOOR, Sample functions of stochastic processes with stationary independent increments, *J. Math. Mech.*, **10** (1961) 493–516.
- [2] S. BOCHNER, *Harmonic Analysis and the Theory of Probability*, Univ. Calif. Press, Berkeley and Los Angeles, 1955.
- [3] Z. CIESIELSKI & S. J. TAYLOR, First passage times and sojourn times for Brownian motion in space and the exact Hausdorff measure of the sample path, *Trans. Amer. Math. Soc.*, **103** (1962) 434–450.
- [4] J. HOROWITZ, The Hausdorff dimension of the sample path of a subordinator, *Israel J. Math.*, **6** (1968) 176–182.
- [5] S. OREY, Polar sets for processes with stationary independent increments, pp. 117–126, *Markov Processes and Potential Theory*, edited by J. Chover, Wiley, New York, 1967.
- [6] W. E. PRUITT & S. J. TAYLOR, Sample path properties of processes with stable components, *Z. Wahrscheinlichkeitstheorie und verw. Geb.*, **12** (1969) 267–289.
- [7] C. A. ROGERS & S. J. TAYLOR, Functions continuous and singular with respect to a Hausdorff measure, *Mathematika*, **8** (1961) 1–31.

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Date communicated: APRIL 25, 1969