Steady Plane Magnetohydrodynamic Flow

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1. Introduction. A number of years ago, Martin [1] developed a theory of plane flow in which the pressure and the stream function were utilized as new independent variables replacing the usual spatial variables. In this theory, the solution of problems may be made equivalent to the problem of integrating a Monge–Ampère partial differential equation.

Later, Martin [2] showed that the determination of the plane steady flow of a fluid could be reduced to finding two functions, the Bernoulli function and the direction function, which together satisfy a quasi-linear partial differential equation. This equation could be used to determine one of these functions provided the other was prescribed. Applications of this method were given by Martin [3], [4] and Hansen and Martin [5].

In a recent paper [7], it was shown that Martin’s Monge–Ampère formulation [1] could be extended to the class of perfectly conducting compressible fluids subjected to magnetic fields whose induction vector had two components zero. Further, the three possible cases could be studied virtually simultaneously; structurally, they led to the same differential equation; the only difference was in the definition of one of the independent variables, *i.e.*, the total pressure distribution. In the limit of vanishing magnetic field, the equations reduced exactly to the equations studied by Martin.

It is the purpose of this paper to extend Martin’s other method [2] to the same special class of magnetohydrodynamic flows. In this way, it is possible to extend the idea of simple waves in rotational, iso-energetic flows to the magnetic case. As a very interesting by-product, it is found that the images of the characteristics in the hodograph plane form a fixed net of curves and do not depend on the solution under consideration, provided the flow is irrotational.

The analysis is based upon, and follows quite closely, the methods developed in [2]–[5]; generally speaking, the solutions of the present paper reduce exactly to the pertinent solutions in the cited references when the limit of vanishing magnetic field is taken.

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Finally, Naylor [9] used the Legendre contact transformation to develop a theory of simple waves in rotational gas flow, based on the concept of one-parameter velocity fields. In only one of the magnetic cases considered (the transverse field case) is it possible to extend Naylor's study to magnetic flows.

2. Summary of the Monge-Ampère formulation. The majority of the paper will be concerned with the special case $\mathbf{B} = (B_1, 0, 0)$, for which the equations governing the flow may be written as [7]

$$(2.1) \quad (\rho u)_x + (\rho v)_y = 0,$$

$$(2.2) \quad (p + \frac{B_1^2}{2\mu} + \rho u^2)_x + (\rho uv)_y = 0,$$

$$(2.3) \quad (p + \frac{B_1^2}{2\mu} + \rho v^2)_y + (\rho uv)_x = 0,$$

with the subsidiary conditions $B_1 = B_1(y)$, $v = v(y)$ only and

$$(2.4) \quad B_1 v = A_1, \quad \text{a constant.}$$

The symbols (B_1, B_2, B_3) , p , ρ , s , γ , $(u, v, 0)$, c , μ , $b_i^2 = B_i^2/\mu\rho$ ($i = 1, 2, 3$) represent, respectively, the induction vector, pressure, density, specific entropy, ratio of specific heats at constant pressure c_p and at constant volume c_v , velocity vector, local speed of sound, permeability and square of the Alfvén speed; all dependent variables are assumed to be at most functions of (x, y) , and the subscripts x and y denote partial differentiation with respect to these spatial variables.

Given any solution

$$(2.5) \quad u = u(x, y), \quad v = v(x, y), \quad p = p(x, y), \quad \rho = \rho(x, y) \neq 0$$

of the system of equations (2.1)–(2.3), which may be considered to be an undetermined system of three partial differential equations for four unknown functions, it is possible to introduce three functions ψ , ξ and η , defined by

$$(2.6) \quad d\psi(x, y) = -\rho v dx + \rho u dy,$$

$$(2.7) \quad d\xi(x, y) = -\rho uv dx + (P + \rho u^2) dy,$$

$$(2.8) \quad d\eta(x, y) = -(P + \rho v^2) dx + \rho uv dy,$$

where $P = p + B_1^2/2\mu$. Setting $\xi = \xi - yP$, $\eta = \eta + xP$, equations (2.7)–(2.8) may be replaced by

$$(2.9) \quad d\xi = u d\psi - y dP,$$

$$(2.10) \quad d\eta = v d\psi + x dP.$$

When conventional non-conducting flows are considered, p and ψ are used as new independent variables instead of x and y . For perfectly conducting flows

of the special type under consideration it is clear that (P, ψ) provide a natural choice for new independent variables to be used instead of (x, y) ; this choice of independent variables leads to equations structurally identical to the equations studied by Martin, and, in the limit of vanishing magnetic field, the former reduce exactly to the latter. Thus, considering the curves of constant total pressure and the streamlines as a system of curvilinear coordinates in the appropriate region of the (x, y) -plane, it is assumed that

$$(2.11) \quad 0 < |J^{-1}| < \infty, \quad J^{-1} = \frac{\partial(P, \psi)}{\partial(x, y)}$$

throughout the region. This condition excludes flows for which $P = P(\psi)$, i.e., flows for which the streamlines are curves of constant total pressure; however, these special flows may be studied directly [7]. When (P, ψ) are introduced as new independent variables, equations (2.6), (2.9), (2.10) lead to the following underdetermined system of four partial differential equations for five unknown functions, x, y, u, v, ρ :

$$(2.12) \quad \begin{aligned} u_P &= -y_\psi, \quad v_P = x_\psi, \\ uy_P - vx_P &= 0, \quad uy_\psi - vx_\psi = \rho^{-1}. \end{aligned}$$

Starting with a solution, equation (2.5), of the original system, equation (2.6) may be used to calculate $\psi = \psi(x, y)$. Because of equation (2.11), the equations $P = P(x, y)$, $\psi = \psi(x, y)$ may be inverted; thus, given any solution, equation (2.5), of the original underdetermined system, there follows a solution of the underdetermined system of equations (2.12) and conversely, provided $0 < |J| < \infty$.

From equations (2.12), it follows that

$$(2.13) \quad \begin{aligned} uu_P + vv_P &= -\rho^{-1}, \\ x_P &= \rho u J, \quad y_P = \rho v J, \end{aligned}$$

$$(2.14) \quad \frac{dP}{dz} = (\rho q J)^{-1}, \quad q^2 = u^2 + v^2,$$

with z arc length measured along a streamline from a fixed point. This leads to an analogy of Martin's Theorem 1; namely,

Theorem 1. *The solution of the underdetermined system of equations (2.1)–(2.3) and the solution of the underdetermined system of equations (2.12), containing the arbitrary function $\rho = \rho(P, \psi)$, are equivalent problems provided that the streamlines are not curves of constant P , the directional derivative of the (total) pressure along the streamlines remains finite, the density does not vanish and the flow has no stagnation points.*

Once the density function, $\rho = \rho(P, \psi)$, has been specified, equations (2.12) become a determinate system, a solution

$$(2.15) \quad x = x(P, \psi), \quad y = y(P, \psi), \quad u = u(P, \psi), \quad v = v(P, \psi)$$

of which provides a parametric representation of the streamlines and their hodographs in terms of P as parameter.

3. The Bernoulli function. Equation (2.13) may be integrated to give a generalized Bernoulli Theorem

$$(3.1) \quad q^2 = q_0^2 - 2 \int_{P_0}^P \frac{dP}{\rho},$$

$$P_0 = P_0(\psi), \quad q_0 = q_0(\psi),$$

where P_0 is some reference (total) pressure, which may vary from streamline to streamline, and $P_0(\psi)$, $q_0(\psi)$ are arbitrary functions. If q_0 is constant, the flow is *iso-energetic*.

If the density function and the arbitrary functions P_0 and q_0 are specified, then from equation (3.1)

$$(3.2) \quad q = q(P, \psi)$$

is a known function called the *Bernoulli function*. Conversely, if the Bernoulli function is specified, the density function is given by

$$(3.3) \quad \rho = -\frac{1}{qq_P}.$$

Since the vorticity $\omega = u_y - v_x$ is given by

$$(3.4) \quad \omega = \rho qq_\psi = -q_\psi/q_P,$$

the flow will be irrotational if, and only if, the Bernoulli function is independent of ψ .

In terms of the functions (ξ, η) the problem has been reduced to considering the system of equations

$$(3.5) \quad \xi_\psi^2 + \eta_\psi^2 = q^2,$$

$$(3.6) \quad \xi_\psi \xi_{PP} + \eta_\psi \eta_{PP} = 0,$$

where $q = q(P, \psi)$ is given by equation (3.1). It is easily seen that ξ and η are solutions of the same Monge-Ampère partial differential equation [7].

4. The quasi-linear equation. Integration of the determinate system of equations (2.12) may be made equivalent to the integration of a quasi-linear partial differential equation. Suppose equations (2.15) give a solution, and let $q = q(P, \psi)$ be the associated Bernoulli function. Let the angle of inclination θ of the streamlines be introduced by setting

$$(4.1) \quad u = q \cos \theta, \quad v = q \sin \theta.$$

Then, noting that $(u, v) = (\xi_\psi, \eta_\psi)$ from equations (2.9)–(2.10), it follows that equation (3.5) is identically satisfied by the substitution, equation (4.1). Differentiating equation (3.6) twice with respect to ψ gives three equations from which (ξ, η) may be eliminated to give the equation

$$(4.2) \quad (q_{PP} - q\theta_P^2)\theta_{\psi\psi} + 2q\theta_P\theta_\psi\theta_{P\psi} - q\theta_\psi^2\theta_{PP} \\ = [q_{PP\psi} - q_\psi\theta_P^2 + 2q_P\theta_P\theta_\psi]\theta_\psi,$$

which may be written alternatively as

$$(4.3) \quad \left[\frac{q_{PP} - q\theta_P^2}{\theta_\psi} \right]_\psi + \frac{(q^2\theta_P)_P}{q} = 0.$$

Conversely, if θ is a solution of equation (4.2) for which $\theta_\psi \neq 0$ for a prescribed Bernoulli function $q = q(P, \psi)$, it is possible to construct a solution of equations (2.12).

For the excluded special case $\theta_\psi \equiv 0$, it is clear that $\theta = \theta(P)$ only, so that the velocity vectors are parallel along a curve of constant total pressure. It is easily seen that the pertinent equation reduces to

$$(4.4) \quad q_{PP} - q\theta_P^2 = 0,$$

for which solutions may be constructed.

5. The special case $\theta = \theta(P)$. Equation (4.4) may be considered to be an equation for θ or for q according as the Bernoulli function q or the function θ is prescribed.

First, consider the case for which $q = q(P)$ is preassigned; then, equation (4.4) may be solved to give

$$(5.1) \quad \theta(P) = \int_{P_0}^P \left[\frac{q_{PP}}{q} \right]^{1/2} dP + \theta_0,$$

where θ_0 is a constant, and it is assumed that $q_{PP} \geq 0$. These values for q and θ may be utilized to give a parametric representation of the solution. In view of equation (3.4), the flow will be irrotational, and it clearly has a one-dimensional hodograph; in the hodograph plane, the curves of constant total pressure are circles concentric with the origin. The streamlines in the physical plane may be constructed as indicated by Martin [2].

Following the designations of conventional gas dynamics, let flows for which $\theta = \theta(P)$ be called *isoclinic* and flows with a one-dimensional hodograph *Prandtl-Meyer Flows (PM)*. In this terminology, it has been shown that:

Theorem 2. *Any irrotational, isoclinic flow is necessarily a Prandtl-Meyer flow for which the curves of constant total pressure are straight lines along which the velocity vectors are parallel and equal in length.*

If the flow is rotational, there exist isoclinic flows which are not *PM*-flows.

In order to study them, let $\theta = \theta(P)$ be chosen arbitrarily, so that equation (4.4) becomes an equation for q . Let q_1 and q_2 be two linearly independent solutions of equation (4.4); then, the general solution may be written as

$$(5.2) \quad q = a_1(\psi)q_1 + a_2(\psi)q_2,$$

in which a_1 and a_2 are arbitrary differentiable functions of ψ .

Specification of the arbitrary functions in equation (5.2) provides a definite Bernoulli function, $q(P, \psi)$, which, together with the previously prescribed θ , provides a solution of equation (4.4). The hodograph curves are a one-parameter family given by

$$(5.3) \quad \begin{aligned} u &= a_1(\psi)u_1 + a_2(\psi)u_2, \\ v &= a_1(\psi)v_1 + a_2(\psi)v_2, \end{aligned}$$

where

$$\begin{aligned} u_1 &= q_1 \cos \theta, & u_2 &= q_2 \cos \theta, \\ v_1 &= q_1 \sin \theta, & v_2 &= q_2 \sin \theta \end{aligned}$$

represent two "base curves" in the hodograph plane.

In view of equation (3.4), this isoclinic flow will be irrotational if, and only if, a_1 and a_2 are constant, *i.e.*, if, and only if, equation (5.3) consists of only a single curve. Thus, an isoclinic flow will be irrotational if, and only if, it is a *PM*-flow.

According to the Munk-Prim Substitution Principle [6], which is valid for the special class of hydromagnetic flows under consideration [7], if u and v are the velocity components of a given flow, the flow with velocity components $\lambda_1 u$, $\lambda_1 v$ has the same streamlines and curves of constant total pressure as the given flow, provided $\lambda_1 = \lambda_1(\psi)$. Following conventional usage, let two such flows be termed *equivalent*; then, it follows that all flows equivalent to a given isoclinic flow are isoclinic. Further, since an isoclinic flow will be equivalent to an irrotational flow if, and only if, the functions a_1 and a_2 in equation (5.3) are linearly dependent, it follows that

Theorem 3. *The necessary and sufficient condition for an isoclinic flow to be equivalent to an irrotational PM-flow is that a_1 and a_2 be linearly dependent.*

With equation (5.2) defining the Bernoulli function for isoclinic flows, equation (3.3) shows that the density function must necessarily have the form

$$(5.4) \quad \rho = -[a_1 q_1 + a_2 q_2]^{-1} [a_1 q'_1 + a_2 q'_2]$$

in terms of the quantities defined previously; the prime denotes differentiation with respect to the argument P . Conversely, starting with the density function given by equation (5.4), an isoclinic flow may be constructed. Thus,

Theorem 4. *For a fluid to permit isoclinic flows for which $\theta = \theta(P)$ under the hypotheses of Theorem 1, it is necessary and sufficient that the density function be given by equation (5.4).*

The simplest possible case would be $\theta = \text{constant}$, so that the streamlines are parallel straight lines. The functions (q_1, q_2) may be taken to be $(q_1, q_2) = (-P, 1)$, so that the Bernoulli function is given by $q = a_2 - a_1 P$, and the density function is given by

$$\rho^{-1} = a_1(a_2 - a_1 P).$$

6. The case $\theta = \theta(\psi)$. Equation (4.2) may also be considered to be an equation for q or θ according as θ or q is prescribed as a known function of (P, ψ) . Consider the former case with $\theta = \theta(\psi)$ prescribed under the condition $\theta_\psi \neq 0$, so that the streamlines lie on a family of non-parallel straight lines. For these flows, termed *rectilinear*, equation (4.3) reduces to $[q_{PP}/\theta_\psi]_\psi = 0$, the general solution of which may be written as

$$(6.1) \quad q = [R_1(P) + R_2(\psi)P + R_3(\psi)]\theta_\psi$$

in terms of three arbitrary, differentiable functions R_1, R_2 and R_3 . If these functions are specified, a definite Bernoulli function is known. The corresponding density function is given by

$$(6.2) \quad -\rho^{-1} = \theta_\psi^2[R_1' + R_2][R_1 + R_2P + R_3].$$

Conversely, given this density function, a flow with the desired properties may be constructed. Thus,

Theorem 5. *For a fluid to permit rectilinear flows under the hypotheses of Theorem 1, it is necessary and sufficient that the density function be of the form given by equation (6.2) in which R_1 is any nonlinear function of the total pressure P .*

The following results may be established easily:

Theorem 6. *Under the hypotheses of Theorem 1, the necessary and sufficient condition for the curves of constant total pressure to be orthogonal trajectories of the streamlines is that the streamlines lie on straight lines.*

Theorem 7. *The streamlines in the rectilinear flow of a gas having the density function given by equation (6.2), with prescribed functions θ_ψ and R_3 , may be made to lie on straight lines enveloping any given curve in the physical plane with a continuous radius of curvature by properly choosing the arbitrary function R_2 . In particular, the straight lines will be concurrent if, and only if, R_2 is constant.*

7. Generalized Prandtl–Meyer flows. In Section 5, a Prandtl–Meyer flow was defined to be a flow in the physical plane whose hodograph was a single curve in the hodograph plane. Further, isoclinic flows which were not *PM*-flows were studied.

In this section, the concept of a *PM*-flow, in which the flow is irrotational is generalized to rotational flows. A *Generalized Prandtl–Meyer Flow (GPM)* will be defined to be a flow whose hodograph curve h may be described by

$$\begin{aligned} h: \theta &= \theta(q) \neq \text{constant}, \\ q &\neq \text{constant}, \end{aligned}$$

where (q, θ) denote polar coordinates in the hodograph plane. Under the same restrictions as to the classes of admissible flows permitted by Martin [3], these flows will necessarily be rotational.

As shown in Section 4, the problem may be reduced to finding two quantities, namely, the Bernoulli function, $q(P, \psi)$, and the direction function, $\theta(P, \psi)$, which satisfy equation (4.2). It is assumed that $q(P, \psi)$ is prescribed subject to the conditions $q_\psi \neq 0$ (rotational flow) and $q_{PP} \neq 0$, which excludes flows considered in Section 5 for which q was a linear function of P . Under these conditions, equation (4.2) becomes a quasi-linear partial differential equation for the determination of θ . As noted by Martin, it is necessary and sufficient for a GPM-flow that equation (4.2) permit solutions of the form $\theta = \theta(q)$. When this substitution is made in the present case, it is seen that the analysis closely parallels Martin's analysis [3] which may be consulted for details.

The following theorem is easily established:

Theorem 8. *If the Bernoulli function satisfies the equation $q_{PP} = Hq_P^2$, $H = H(q)$, the isovels in a GPM-flow are straight lines.*

8. Characteristics and hodograph curves. With the definitions

$$\begin{aligned} T_1 &= q\theta_\psi^2, \quad T_2 = -q\theta_P\theta_\psi, \quad T_3 = q\theta_P^2 - q_{PP}, \\ T_4 &= [q_{PP}\psi - q_\psi\theta_P^2 + 2q_P\theta_P\theta_\psi]\theta_\psi, \end{aligned}$$

equation (4.2) may be written as

$$(8.1) \quad T_1\theta_{PP} + 2T_2\theta_{P\psi} + T_3\theta_{\psi\psi} + T_4 = 0.$$

For a general flow, $q \neq 0$, $\theta_\psi \neq 0$ imply $T_1 \neq 0$ and $T_3 \neq 0$. The characteristic strips of equation (8.1) satisfy

$$(8.2a) \quad T_1 d\psi^2 - 2T_2 dP d\psi + T_3 dP^2 = 0,$$

$$(8.2b) \quad T_1 d\psi d\theta_P + T_3 dP d\theta_\psi + T_4 dP d\psi = 0.$$

Substitution of the definitions of the T 's into equation (8.2a) gives

$$(8.3) \quad [\theta_P \pm (q_{PP}/q)^{1/2}] \frac{dP}{d\psi} + \theta_\psi = 0.$$

Let $q(P, \psi)$ be the prescribed Bernoulli function and $\theta(P, \psi)$ be a solution of equation (8.1). A region in the (P, ψ) -plane for which $q_{PP} > 0$ is carried into a corresponding region in the physical plane. The solutions of equation (8.3) form the usual network of characteristic base curves in the (P, ψ) -plane. These characteristic base curves are carried by equation (4.1), i.e., by

$$u = q(P, \psi) \cos \theta(P, \psi),$$

$$v = q(P, \psi) \sin \theta(P, \psi)$$

into the network of hodograph characteristics.

Usually, the characteristics will depend on the solution $\theta = \theta(P, \psi)$ under consideration, but, for irrotational flow ($q_\psi = 0$), $q = q(P)$ and the hodograph characteristics form a fixed net independent of the flow in the physical plane. This may be easily seen by writing equation (8.3) as $d\theta = \pm (q_{PP}/q)^{1/2} dP$. For $q = q(P)$, this equation may be integrated to give the following parametric representation of the hodograph characteristics

$$q = q(P), \quad \theta = \pm \int (q_{PP}/q)^{1/2} dP + \theta_0.$$

The conclusion that the hodograph characteristics form a fixed net of curves for the special class of hydromagnetic flows under consideration is a quite remarkable result. It provides the basis for the development of numerical procedures entirely similar to those used in conventional gas dynamics.

9. Some geometric properties. Martin and Hansen [5] have made a detailed study of the geometric configuration of five families of curves in the physical plane; namely, the streamlines ($\psi = \text{constant}$), the isobars ($p = \text{constant}$), the isovels ($q = \text{constant}$), isoclines ($\theta = \text{constant}$), the isopycnics ($\rho = \text{constant}$). Their study is quite general and is valid for a wide class of so-called abstract fluids which includes polytropic gases as a special case. It is the purpose of this section to extend their study, in part, to the special class of magnetic flows under consideration. In the present discussion, the curves of constant total pressure will be introduced rather than the usual isobars.

The first result is immediately apparent from equations (2.12), *i.e.*, from $x_\psi = v_P$, $u_P = -y_\psi$:

Theorem 9. *The tangents drawn to a curve of constant total pressure in the physical plane and to a streamline in the hodograph plane at corresponding points are orthogonal to each other.*

As shown previously [7], the Mach number $M = q/c$ is given by

$$(9.1) \quad M^2 = \frac{1 + qq_{PP}/q_P^2}{1 + A_1^2 v_P/\mu v^3}$$

so that a flow will be supersonic if

$$(9.2) \quad q_{PP} > \frac{A_1^2 v_P q_P^2}{\mu q v^3}$$

and subsonic if the inequality is reversed; in the nonmagnetic case the right-hand side is zero. While there is a natural demarcation of regions in the non-magnetic case given by subsonic ($q_{PP} < 0$) and supersonic ($q_{PP} > 0$), separated by a sonic curve along which $q_{PP} = 0$, the same situation is not exactly true in the present

case. From equation (8.3), real characteristics are obtained as long as $q_{PP} > 0$ and while the regions of elliptic and hyperbolic flow are indeed separated by the curve along which $q_{PP} = 0$, this separation does not delineate regions of subsonic and supersonic flow. Throughout this discussion, it will be assumed that q is prescribed (equation 3.1) so that equation (4.2) is the pertinent equation for the determination of θ .

In addition to the five families of curves mentioned above, two additional families of curves, provided $q_{PP} > 0$, may be defined by equation (8.3) for a given solution $\theta(P, \psi)$ of equation (4.2). By the use of the appropriate transformation, these characteristic base curves are carried over into a region of the physical plane; extending the usual terminology, these curves in the physical plane will be called *Generalized Mach Lines*. From equation (8.3), the parametric equations for these curves are

$$(9.3) \quad \begin{aligned} x &= \int \{ \pm (qq_{PP})^{1/2} \cos \theta + q_P \sin \theta \} d\psi, \\ y &= \int \{ \pm (qq_{PP})^{1/2} \sin \theta - q_P \cos \theta \} d\psi. \end{aligned}$$

Let θ^* denote the angle between a streamline and a generalized Mach line measured positively counterclockwise from the direction of flow; then

$$(9.4) \quad \cot^2 \theta^* = qq_{PP}/q_P^2.$$

Since generalized Mach lines may also be obtained from the equation

$$(9.5) \quad d\theta = \pm (q_{PP}/q)^{1/2} dP,$$

substitution of equation (9.4) into equation (9.5) gives

$$(9.6) \quad d\theta = \pm \frac{\cot \theta^* dP}{\rho q^2}.$$

For the special case for which only the third component of the induction vector is not zero, equation (9.5) reduces to [8]

$$(9.7) \quad d\theta = \pm \frac{\sin 2\theta_1^*}{2\rho a^2} dP,$$

where $\sin \theta_1^* = 1/N$, and the magnetic Mach number $N = q/a$; $a = (b_s^2 + c^2)^{1/2}$ is the true speed of sound.

From equation (9.4), it follows that

Theorem 10. *The streamlines and generalized Mach lines intersect at a constant angle along an isovel, provided the Bernoulli function satisfies the partial differential equation $q_{PP} = H(q)q_P^2$.*

Returning now to the five families of curves mentioned in the first paragraph of this section, let the tangents drawn to these curves at a given point define

the principal directions at the point. In general, these families of curves are distinct, but in certain special flows, two or more may coincide. Let a direction be fixed by the angle ϵ measured positively counterclockwise from the tangent to the streamline at a point, measured in the direction of flow. For a direction at the point tangent to the transform of a curve $F(P, \psi) = \text{constant}$ in the (P, ψ) -plane, it is true that

$$(9.8) \quad -\cot \epsilon = \frac{q\theta_P}{q_P} + \frac{(q_{PP} - q\theta_P^2)F_\psi}{\theta_\psi q_P F_P}.$$

Let the directions at a point of the curve of constant total pressure, isocline, isovel and isopycnic be denoted by $\epsilon = \beta, \alpha, \zeta, \delta$. Further, let the principal directions themselves be denoted by the letters ψ, P, θ, q, ρ ; finally, let the generalized Mach directions be denoted by m_1 and m_2 , or simply by m .

First, let $F = P, \theta$ in equation (9.8); this gives

$$(9.9) \quad \cot \beta = -\frac{q\theta_P}{q_P}, \quad \cot \alpha = -\frac{q_{PP}}{q_P \theta_P},$$

$$(9.10) \quad \cot \beta \cot \alpha = \frac{qq_{PP}}{q_P^2}, \quad \cot \beta - \cot \alpha = \frac{q_{PP} - q\theta_P^2}{q_P \theta_P},$$

and, if $q_{PP} > 0$,

$$(9.11) \quad \cot \beta \cot \alpha = \cot^2 \theta^*.$$

As a first consequence of these formulas:

Theorem 11. *In general plane flow, the isoclines and curves of constant total pressure form an orthogonal net if, and only if, the flow is incompressible, i.e., $\rho = \rho(\psi)$.*

The proof is trivial; from equations (9.10), the necessary and sufficient condition for an orthogonal net is $qq_{PP} + q_P^2 = (qq_P)_P = 0$, i.e., $qq_P = f(\psi)$; in view of equation (3.3), this proves the theorem.

If $\theta_P = 0$ at the point under consideration, equations (9.9) show that the principal directions ψ, θ coincide and the principal direction P is perpendicular to them, provided $q_{PP} \neq 0$. If q_{PP} vanishes, the principal directions ψ, θ are still perpendicular, but θ is not defined by equations (9.9). When $\theta_P = \infty$, the principal directions ψ, P coincide, and the principal direction θ is perpendicular to them.

At a point for which q_{PP} vanishes, both generalized Mach directions m_1, m_2 are perpendicular to the principal direction ψ , as is the principal direction θ , provided $\theta_P \neq 0$.

These conclusions may be summarized as

Theorem 12. *For finite $\theta_P \neq 0$, the principal directions P, θ at a point lie in the same or in different quadrants according as $q_{PP} > 0$ or $q_{PP} < 0$ at the point;*

in the former case, they are separated by the generalized Mach directions in accordance with equation (9.11), coinciding with one of them if $q_{PP} - q\theta_P^2 = 0$.

In conventional gas dynamics, it is of interest to determine the locus Γ , the line of branch points, of those points in the physical plane at which the principal directions p , θ coincide. For the present discussion, the following theorem follows directly from Martin's proof:

Theorem 13. *In general plane flow, the locus of points Γ at which the principal directions P , θ coincide is tangent to a generalized Mach line at any point of Γ at which $q_\psi = q_{PP\psi} = 0$, provided the curvature of the streamline is finite at the point. In particular, for irrotational flow, Γ is a generalized Mach line.*

10. One-parameter velocity fields. In some of the previous sections, it has been shown that the concept of a simple wave flow (*PM*-flow) could be extended to rotational flows, so that the idea of a *GPM*-flow could be developed. An alternative method for extending the concept of a *PM*-flow to rotational flows has been developed by Naylor [9], who used the Legendre contact transformation, applied to Crocco's stream function, to study the properties of flows whose velocity vector was a function of only one independent variable. The extension of these ideas to magnetic flows is the purpose of this section.

As will be seen directly, it seems possible to extend the concept only in the transverse field case $\mathbf{B} = (0, 0, B_3)$. Limiting the discussion to the special class of hydromagnetic flows under consideration, the equations of motion may be written as:

$$(10.1) \quad \operatorname{div}(\rho \mathbf{q}) = 0,$$

$$(10.2) \quad \rho(\mathbf{q} \cdot \operatorname{grad})\mathbf{q} + \operatorname{grad} p = \mu^{-1} \operatorname{curl} \mathbf{B} \times \mathbf{B},$$

$$(10.3) \quad \mathbf{q} \cdot \operatorname{grad} s = 0.$$

The induction equations need not be written out explicitly, since they merely provide integrals in these special cases.

By introduction of the sound speed, equation (10.1) may be written as

$$(10.4) \quad \operatorname{div}[\mathbf{q}c^{2/(\gamma-1)}] = 0.$$

Also, equation (10.2) may be written as

$$(10.5) \quad \operatorname{grad} H_1 - \frac{c^2}{\gamma(\gamma-1)c_*} \operatorname{grad} s = \mathbf{q} \times \mathbf{w},$$

where $\mathbf{w} = \operatorname{curl} \mathbf{q}$ and

$$(10.6) \quad H_1 = \frac{q^2}{2} + \frac{c^2}{\gamma-1} + \int b_i^2 \frac{dB_i}{B_i}$$

and $i = 1, 2, 3$ corresponds to which of the special cases is being considered. Clearly $H_1 = H_1(\psi)$ in steady plane flow, but only iso-energetic flows, for which

H_1 is a constant throughout the flow, will be studied. Thus

$$(10.7) \quad \frac{q^2}{2} + \frac{c^2}{\gamma - 1} + \int b_i^2 \frac{dB_i}{B_i} = \text{constant}.$$

Consequently, equation (10.5) reduces to

$$(10.8) \quad c^2 \text{grad } s = \gamma(\gamma - 1)c_* \mathbf{w} \times \mathbf{q},$$

which implies

$$(10.9) \quad \text{curl} [\mathbf{w} \times \mathbf{q}/c^2] = 0.$$

In summary, for the three special cases for which the induction vector has two components zero, the general equations governing steady plane, rotational, iso-energetic flow are equations (10.4), (10.7), (10.9).

The continuity equation (10.4) is identically satisfied by the introduction of Crocco's stream function ψ_1 defined by

$$(10.10) \quad \text{grad } \psi_1 = Q\mathbf{n}$$

where $Q = qc^{2/(\gamma-1)}$ and $\mathbf{n}$ is the unit vector normal to the streamline; it is much more convenient to use this stream function rather than the usual stream function ψ defined by equation (2.6) as

$$(10.11) \quad \text{grad } \psi = \rho q \mathbf{n}.$$

The stream function ψ_1 is a simple function of q , but the same is not true of ψ because ρ cannot be expressed in this manner without introducing either the entropy or ψ itself.

Letting $\mathbf{e}$ denote a unit vector perpendicular to the plane of motion, it follows that $\mathbf{w} = w\mathbf{e}$; substitution of equation (10.10) into equation (10.9) shows that

$$(10.12) \quad \text{grad } \psi_1 \times \text{grad} [wc^{-2\gamma/(\gamma-1)}] = 0,$$

i.e., the vectors are collinear, so that there exists a relation of the form

$$wc^{-2\gamma/(\gamma-1)} = G(\psi_1),$$

where G is some function of ψ_1 alone. Finally, equation (10.8) may be integrated to give the entropy distribution

$$(10.13) \quad s - s_0 = \gamma(\gamma - 1)c_* g(\psi_1),$$

where

$$G(\psi_1) = g'(\psi_1).$$

Suppose the velocity vector is a function of only one (unspecified) parameter $\lambda = \lambda(\mathbf{r})$, then from equations (10.7) and (10.10), it is necessary to determine when $\text{grad } \psi_1$ and c may be expressible in terms of λ only. Success is not forthcoming for case 1 (or the symmetric case 2) because the integral in equation

(10.7) may be written as

$$-\frac{A_1^2}{\mu} \int \frac{dv}{\rho v}$$

by the use of equation (2.4); since ρ appears in the integral, the method fails. For case 3, however, the usual frozen-flux relation holds for the type of flow being investigated, so that equation (10.7) may be written as

$$(10.14) \quad \frac{q^2}{2} + \frac{c^2}{\gamma - 1} + b_s^2 = \frac{q^2}{\gamma - 1} + \frac{c^2}{\gamma - 1} + kc^{2/(\gamma-1)} = \text{constant},$$

where k is constant throughout the flow [7], [8]. Clearly c may be expressed as a simple function of q by the use of equation (10.14) and $\text{grad } \psi_1$, so expressed by the use of equation (10.10). Consequently, Naylor's method is applicable only to the transverse field case. For this case, however, Naylor's analysis may be extended directly with but minor modifications.

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