*On a Class of Non-linear Elliptic Boundary Value Problems**

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1. Introduction. In [5], a theorem is proved which asserts the existence of a non-trivial solution to the problem

$$(1) \quad \Delta y + yF(y^2, x) = 0 \quad \text{in } \Omega, \quad y|_{\partial\Omega} = 0,$$

where Δ is the Laplace operator, Ω is a bounded region in R^N for which the Dirichlet problem is solvable, and F is a function locally Hölder continuous on $\bar{R}_+ \times \Omega$ satisfying, for some $\epsilon > 0$ and all $x \in \Omega$,

$$(2) \quad 0 < \eta_1^{-\epsilon} F(\eta_1, x) \leq \eta_2^{-\epsilon} F(\eta_2, x), \quad \text{for } 0 < \eta_1 < \eta_2 < \infty,$$

and also, for all $x \in \Omega$,

$$(3) \quad F(\eta, x) \leq c\eta^\gamma + \sigma, \quad 0 < \eta < \infty,$$

where c, σ and γ are positive constants, $(N - 2)\gamma < 2$. This result is the analogue of a result of [8] concerning a boundary value problem for a non-linear ordinary differential equation. The result of [5] concerning (1) was obtained by treating the integral equation equivalent to (1) by methods similar to those used in [9].

In this note we shall derive from the results of [5] an existence theorem for a boundary value problem of the form

$$(4) \quad \tau u = uF(u^2, x) \quad \text{in } \Omega, \quad D^\alpha u|_{\partial\Omega} = 0, \quad |\alpha| \leq m - 1,$$

where τ is an elliptic operator of order $2m$, $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$; $m \geq 1$, $N \geq 1$. The result obtained here was suggested by the main theorem of Berger's study, [4], of a nonlinear elliptic eigenvalue problem.

2. The differential operator. We shall assume throughout this section that Ω is a region of class C^{2m} (see Def. 9.2, [3]) and that the differential operator τ ,

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given in the form

$$(5) \quad \tau = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\alpha|} D^\alpha a_{\alpha\beta}(x) D^\beta,$$

has real coefficients satisfying

$$(6) \quad a_{\alpha\beta}(x) \in C^{\max(|\alpha|, |\beta|)}(\bar{\Omega}), \quad \text{all } \alpha, \beta.$$

In addition we assume that τ is uniformly elliptic in Ω and that there exists a positive constant c_0 such that

$$(7) \quad B[\varphi, \varphi] \geq c_0 \|\varphi\|_{m,2}^2, \quad \text{all } \varphi \in C_0^\infty(\Omega),$$

where

$$B[\varphi, \psi] = \sum_{|\alpha|, |\beta| \leq m} \int_{\Omega} a_{\alpha\beta}(x) D^\alpha \varphi(x) D^\beta \psi(x) dx.$$

We shall use standard notation. For $r > 1$, $\|\cdot\|_{m,r}$ is the Sobolev norm defined as follows,

$$\|\varphi\|_{m,r}^r = \sum_{|\alpha| \leq m} \int_{\Omega} |D^\alpha \varphi|^r dx,$$

for φ having strong L^r -derivatives of order up to m in Ω ; $W^{m,r}(\Omega)$ is the space of all such functions (because of the smoothness assumption concerning Ω this is equivalent to the more usual definition of $W^{m,r}(\Omega)$); $W^{m,r}(\Omega)$ is normed by $\|\cdot\|_{m,r}$. Finally, $W_0^{m,r}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,r}(\Omega)$. Without exception the function spaces considered here will be understood to consist of real valued functions.

By a standard result, see Theorem 8.2, [3], the generalized Dirichlet problem with zero boundary data

$$B[\varphi, u] = (\varphi, f), \quad \text{all } \varphi \in C_0^\infty(\Omega),$$

has a unique solution $u \in W_0^{m,2}(\Omega)$ for each $f \in L^2(\Omega)$. Actually the same is true for $f \in L^q(\Omega)$ provided

$$(8) \quad q > q_0 = \max(1, 2N/(N+2m)), \quad q < 2.$$

This follows from the fact that, because of the Sobolev Imbedding Theorem, $W^{m,2}(\Omega)$ is stronger than $L^p(\Omega)$ when $1/p + 1/q = 1$ and (8) holds. Thus for $f \in L^q(\Omega)$, $\varphi \in W_0^{m,2}(\Omega)$,

$$(9) \quad |(\varphi, f)| \leq \|\varphi\|_p \|f\|_q \leq \text{const.} \|\varphi\|_{m,2} \|f\|_q.$$

The same proof as in the case where $f \in L^2$ then works for $f \in L^q$.

We define an operator A , whose domain is $\bigcup_{q>q_0} L^q(\Omega)$ and whose range is contained in $W_0^{m,2}(\Omega)$, by

$$(10) \quad B[\varphi, Af] = (\varphi, f), \quad \text{all } \varphi \in W_0^{m,2}(\Omega),$$

upon taking $\varphi = Af$ in (10) it follows from (7) and (9) that A acts as a bounded operator from $L^q(\Omega)$ to $W_0^{m,2}(\Omega)$ for each $q > q_0$.

We shall require the following results from [1] and [2].

(*) If $f \in L^r(\Omega)$, $r > q_0$ then $Af \in W^{2m,r}(\Omega)$, and there exists a constant k_r such that

$$(11) \quad \|Af\|_{2m,r} \leq k_r(\|f\|_r + \|Af\|_r).$$

(**) If $f \in L^\infty(\Omega)$ and $u = Af$, then (after modification on a set of measure zero) $u \in C^{2m-1}(\bar{\Omega})$ and u satisfies the boundary conditions

$$(12) \quad D^\alpha u = 0, \quad \text{on } \partial\Omega, |\alpha| < m.$$

(***) If $f \in C^{0,\mu}(\Omega)$ and if

$$(13) \quad a_{\alpha\beta}(x) \in C^{\max(|\alpha|, |\beta|), \mu}(\Omega), \quad \text{all } \alpha, \beta,$$

for some $\mu: 0 < \mu < 1$, then $u = Af \in C^{2m}(\Omega)$ and u satisfies (12).

$C^{k,\mu}(\Omega)$ is the space consisting of those functions in $C^k(\Omega)$ whose k^{th} order derivatives are uniformly Hölder continuous of order μ in Ω .

The first assertion, (*), follows from Theorem 8.2, [2] (Actually this is not so unless (6) is strengthened. An adaptation of the arguments in [2] to the case where the operator is given in divergence form gives a result implying (*) under condition (6).); the proof of (**) is also in [2]; (***) follows from Theorem A5.1, Appendix 5, [1]. Although we do not use the fact here it is interesting to note that the assertion (**) is equivalent to the assertion that for $r > q_0$, a function u belongs to $W_0^{m,2}(\Omega) \cap W^{2m,r}(\Omega)$ if and only if it is the limit in $W^{2m,r}(\Omega)$ of a sequence $\{u_n\}$, where, for each n , $u_n \in C^{2m-1}(\bar{\Omega}) \cap W^{2m,r}(\Omega)$ and $u = u_n$ satisfies (12).

If $r > 0$ then τu can be defined for $u \in W^{2m,r}(\Omega)$, if $r > q_0$ then $W^{2m,r}$ can be imbedded in $W^{m,r}$, we shall denote by $\mathfrak{D}_r$, for $r > q_0$, the operator in $L^2(\Omega)$ whose domain is $M^r = W^{2m,r}(\Omega) \cap W_0^{m,2}(\Omega)$ and which sends $u \rightarrow \tau u$. Since

$$B[\varphi, \psi] = (\varphi, \tau\psi) = (\varphi, \mathfrak{D}_r\psi), \quad \text{for } \varphi \in C_0^\infty(\Omega), \psi \in M^r,$$

it follows that

$$(14) \quad A\mathfrak{D}_r u = u, \quad \text{for } u \in M.$$

On the other hand, by (*), A maps $L^r(\Omega)$ into M^r so

$$B[\varphi, Af] = (\varphi, \mathfrak{D}_r Af), \quad \text{for } \varphi \in C_0^\infty(\Omega), f \in L^r(\Omega),$$

thus

$$(15) \quad \mathfrak{D}_r Af = f, \quad \text{for } f \in L^r(\Omega).$$

We put $A_r = A|L^r(\Omega)$.

Lemma 1. For each $r > q_0$,

$$A_r : L^r(\Omega) \rightarrow M^r$$

is a bijection. Moreover there are positive constants c_r , c'_r such that for $f \in L^r(\Omega)$

$$(16) \quad c_r \|A_r f\|_{2m,r} \leq \|f\|_r \leq c'_r \|A_r f\|_{2m,r}.$$

Proof. We have shown that A_r and $\mathfrak{D}_r$ are inverses of one another. It readily follows from (6) that $\|\mathfrak{D}_r u\|_r \leq \text{const.} \|u\|_{2m,r}$, which implies the second inequality of (16). M^r is a subspace of $W^{2m,r}(\Omega)$ so the first inequality of (16) follows from the Open Mapping Theorem.

Lemma 2. Let $r > q_0$. If $2mr \leq N$ then A maps $L^r(\Omega)$ compactly into $L^s(\Omega)$ for

$$(17) \quad 1 < s < Nr/(N - 2mr);$$

if $2mr > N$ then A can be regarded as a compact mapping of $L^r(\Omega)$ into $C(\bar{\Omega})$.

Proof. Let $q_0 < r$, then by Lemma 1, A maps $L^r(\Omega)$ into $W^{2m,r}(\Omega)$. If $2mr \leq N$ then, by Sobolev's Theorem, $W^{2m,r}(\Omega)$ can be imbedded compactly in $L^s(\Omega)$ for any s satisfying (17). If $2mr > N$ then $W^{2m,r}(\Omega)$ can be imbedded compactly in $C(\bar{\Omega})$.

The following lemma will simplify matters by making the results of [5] applicable, as they stand, to the problem considered here.

Lemma 3. Let $a_0 = N/(N - 2m)$ or let $a_0 = \infty$ according as $N > 2m$ or $N \leq 2m$. There exists a measurable function $G(x, t)$ on $\Omega \times \Omega$ such that the mapping

$$x \rightarrow G(x, \cdot)$$

is uniformly continuous from Ω to $L^a(\Omega)$, with respect to the weak topology on $L^a(\Omega)$, for $1 < a < a_0$, and

$$(18) \quad \text{ess sup}_{x \in \Omega} \int_{\Omega} |G(x, t)|^a dt < \infty, \quad \text{ess sup}_{t \in \Omega} \int_{\Omega} |G(x, t)|^a dx < \infty,$$

for $1 < a < a_0$. For $f \in L^r(\Omega)$, $r > q_0$,

$$(19) \quad [Af](x) = \int_{\Omega} G(x, t)f(t) dt, \quad \text{a.e. in } \Omega.$$

Proof. Let $2mr > N$, so that A can be regarded as a map of $L^r(\Omega)$ into $C(\bar{\Omega})$. By a well known representation theorem (Theorem VI.7.1, [6]), there is a weakly continuous map $x \rightarrow G(x, \cdot)$ of $\bar{\Omega}$ into $L^a(\Omega)$, where $1/r + 1/a = 1$, such that (19) holds everywhere in Ω for $f \in L^r(\Omega)$. It is easily seen that the function $G(x, t)$ is independent of the particular choice of r . Let

$$(20) \quad \tau^* = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\beta|} D^{\beta} a_{\alpha\beta}(x) D^{\alpha},$$

(notice the symmetry of (6)), and let A^* be defined by

$$(21) \quad B[A^*f, \psi] = (f, \psi), \quad \psi \in W_0^{m,2}(\Omega), f \in L^q(\Omega), q > q_0.$$

It follows that for $f, g \in L^q(\Omega)$, $q > q_0$, we have

$$(22) \quad (A^*g, f) = (g, Af).$$

Because of the symmetry of (6), A^* has an integral representation analogous to (19) for $f \in L^r(\Omega)$, $2mr > N$; let $G^*(x, t)$ denote the corresponding kernel. It follows then from (22) that

$$\int_{\Omega} \int_{\Omega} G^*(t, x) f(x) g(t) dx dt = \int_{\Omega} \int_{\Omega} G(x, t) f(x) g(t) dx dt,$$

for $f, g \in L^r(\Omega)$, $2mr > N$. Thus we have

$$G^*(t, x) = G(x, t), \quad \text{a.e. in } \Omega \times \Omega.$$

The inequalities (18) follow directly from the representation theorem cited above.

It remains to show that (19) is valid for $f \in L^r(\Omega)$ when $r > q_0$, and $2mr \leq N$. In this case however it follows from (18) and Theorem 9.5.6, [7], that the right hand side of (19) defines a compact mapping from $L^r(\Omega)$ to $L^s(\Omega)$ for

$$1 < s < Nr/(N - 2mr).$$

Thus since (19) is valid for f in a dense subset of $L^r(\Omega)$, (namely for $f \in L^{r'}(\Omega)$, $2mr_1 > N$), it follows from Lemma 2 that it is valid for $f \in L^r(\Omega)$.

3. The non-linear problem. Let Ω and τ be as in Section 2. The main result of this note is the following.

Theorem. Suppose that (13) holds for some positive μ less than 1 and that F is uniformly Hölder continuous on $\bar{R}_+ \times \Omega$. Suppose also that F satisfies (2) for some $\epsilon > 0$ and that (3) holds, for all $x \in \Omega$, with positive constants, c , σ and γ where

$$(23) \quad 0 \leq \gamma, \quad \gamma(N - 2m) < 2m.$$

Finally assume that τ is formally self-adjoint. Then there exists a function $u \in C^{2m}(\Omega) \cap C^{m-1}(\bar{\Omega})$ which is not identically zero and satisfies (4) (in the ordinary sense).

Proof. We consider the operator equation

$$(24) \quad u = AuF(u^2, x),$$

where A has the same meaning as in Section 2. By Lemma 3 this is equivalent to an integral equation

$$(25) \quad u(x) = \int_{\Omega} G(x, t) u(t) F(u^2(t), t) dt,$$

where, since τ is formally self-adjoint, $G(x, t)$ is symmetric; (18) holds for $1 < a$, $a(N - 2m) < N$. It readily follows from (7) that A , regarded as an operator in $L^2(\Omega)$, is positive definite; the range of A contains $C_0^\infty(\Omega)$ and therefore

has dense intersection with $L^p(\Omega)$ for any $p > 1$. Now from Theorems 1 and 3 of [5] it follows that (25) has a non-trivial essentially bounded solution u ; see also the remarks following the statement of Theorem 2 of [5]. From the equivalence of (24) and (25) it follows that u satisfies (24). By (3), $u(x)F(u^2(x), x)$ is essentially bounded and thus, by (**), $u \in C^{2m-1}(\bar{\Omega})$ and u satisfies (12). From the differentiability of u and the hypothesis concerning F it follows that $u(x)F(u^2(x), x)$ is uniformly Hölder continuous in Ω . Finally by (***) we conclude that $u \in C^{2m}(\Omega)$ and is an ordinary solution of (4). This completes the proof.

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