*A Class of Wave Diffraction Problems Involving Geometrically-Induced Singularities*¹

G. C. SIH & J. F. LOEBER

Communicated by I. N. SNEDDON

Abstract. In this paper, a general method is presented for solving two-dimensional diffraction problems for cases where the region under consideration is interrupted by a line of discontinuity. The same basic method of solution also solves the problem of a penny-shaped surface of discontinuity.

For systems governed by a set of n wave potentials satisfying Helmholtz equations, use is made of suitable integral transforms and each problem is reduced to the solution of a series of coupled Fredholm integral equations. The mathematical analysis covers a class of problems in the propagation of elastic, acoustic and electromagnetic waves. A number of examples from the field of elastodynamics are given including those of longitudinal shear and flexural waves incident on a slit of finite width. Special attention is directed to obtaining the details of the singular character of the near-field solution which has been known to control the instability behavior of geometrically-induced singularities.

1. Introduction. The problem of the scattering of waves (elastic, acoustic, electromagnetic, etc.) about obstacles has long been of great interest to both physicists and mathematicians because of its practical importance and the considerable mathematical difficulty encountered. The great majority of work in this field, however, has been highly specialized in the sense that it has been largely restricted to geometrically simple obstacles whose boundaries conform to coordinate curves or surfaces [1, 2]. In such instances, solutions may be obtained using the classical separation of variables method.

An important class of problems that has received relatively little attention is the diffraction of waves by lines or surfaces of discontinuity; here, the method of separation of variables fails to apply. In particular, knowledge of the detailed structure of solutions in the vicinity of the discontinuities is lacking. In elastodynamics [3], the principal interest is in the near-field solution from which the

¹ This research work was supported by the U. S. Navy under Contract Nonr-610(06) with the Office of Naval Research in Washington, D. C.

instability condition may be established. This is due mainly to the mathematical complexity associated with diffraction problems involving geometrically-induced singularities.

The simplest geometry for a line of discontinuity is that of a semi-infinite cut. The solution to this diffraction problem has been obtained in closed form by use of the Wiener-Hopf technique [4] for the case of a single Helmholtz equation. Aside from the semi-infinite cut, the simplest nontrivial geometry is that of a cut or slit of finite width. Although this problem has been discussed by numerous authors whose publications are referenced in Noble's book [4], available solutions are (1) approximate in nature, (2) valid only at points far away from the slit, and (3) applicable only for specific ranges of the parameters under consideration. The major difficulty which arises in using the Wiener-Hopf technique lies in the fundamental factorization of certain functions into components regular in appropriate half-planes. By the application of a function-theoretic approach, Schmeltzer and Lewin [5] have also considered the problem of electromagnetic waves diffracted by a plane slit. Their results were expressed in terms of complicated integrals and no attempt was made to examine the detailed character of the near-field solution which, to the authors' knowledge, has not appeared in the literature.

The method developed here makes use of integral transforms to solve wave scattering problems governed by multiple Helmholtz equations. It yields solutions near to and far from the line of discontinuity when the disturbance consists of incident waves radiating periodically from a source. The qualitative character of the near-field solution is determined in closed form while the quantitative part is found numerically by solving a series of coupled Fredholm integral equations. To illustrate the method, several examples of basic interest in elastodynamics are given.

While the present analysis deals with the diffraction of waves around lines of discontinuities, the same treatment also applies to a penny-shaped surface of discontinuity excited by disturbances which possess axial symmetry. In such a case, the problem can be formulated in cylindrical coordinates.

2. Statement of the problem. Consider the propagation of waves in a two-dimensional region occupying the xy -plane. At an obstacle, some of the waves are deflected from their original course. The difference between the actual wave and the undisturbed wave which would exist if the obstacle were not present is called the scattered wave. Let $w_i^{(k)}$ ($k = 1, 2, \dots, n$) be the wave potentials of the total field. Then $w_i^{(k)}$ may be regarded as the sum of the incident and scattered fields, *i.e.*,

$$(2.1) \quad w_i^{(k)} = w_i^{(k)} + w_s^{(k)}, \quad k = 1, 2, \dots, p, \dots, n.$$

The incident wave $w_i^{(p)}$ may be represented by

$$(2.2) \quad w_i^{(p)} = W_0^{(p)} \exp \{ -i[\alpha_p(x \cos \vartheta_p + y \sin \vartheta_p) + \omega t] \}.$$

This is a plane wave with amplitude $W_0^{(p)}$ arriving from infinity $((x^2 + y^2)^{1/2} = \infty)$ for all time t and making an angle ϑ_p $(-\pi \leq \vartheta_p \leq \pi)$ with the positive x -axis. The frequency ω of the incoming wave defines the wave lengths $\beta_k = 2\pi c_k/\omega$, where c_k $(k = 1, 2, \dots, n)$ are the wave velocities. All components of the field are assumed to depend on time t in a purely periodic manner described by the factor $\exp(-i\omega t)$ which will henceforth be suppressed. The main objective of the diffraction problem solution becomes the determination of potentials $w_s^{(k)}$ $(k = 1, 2, \dots, n)$ of the scattered field.

Many physical problems involving wave motion in elasticity, acoustics, and electromagnetic theory are governed by a system of n Helmholtz equations

$$(2.3) \quad (\nabla^2 + \alpha_k^2)w_s^{(k)}(x, y) = 0, \quad k = 1, 2, \dots, n, \quad \text{in } S$$

where ∇^2 is the Laplace operator and $\alpha_k (= 2\pi/\beta_k)$ are the wave numbers; the wave potentials $w_s^{(k)}$ are coupled through the boundary conditions of the physical problem. In this study, the two-dimensional region S is assumed to be infinite in extent and contains a line of discontinuity along the x -axis from $-a$ to $+a$ which will be denoted by L while L' will represent the remaining portion of the x -axis. From symmetry conditions, the problem is separated into a symmetric part and an anti-symmetric part with respect to the variable y . This allows each part to be treated as a half-plane problem in, say $y > 0$, subject to mixed boundary conditions on $L + L'$. Suppose that $P^{(l)}$, $Q^{(l)}$, $M^{(l)}$, and $N^{(l)}$ as functions of $w_s^{(k)}$ are to be specified on L and L' . With $P^{(l)}$, $M^{(l)}$ even in y and $Q^{(l)}$, $N^{(l)}$ odd in y , the scattered problem may then be divided into two parts as follows:

A. Symmetric in y .

$$(2.4) \quad \left. \begin{aligned} P^{(l)}[w_s^{(k)}] &= -P^{(l)}[w_i^{(k)}] = -p_1^{(l)}(x), & l = 1, 2, \dots, m_1, \\ Q^{(l)}[w_s^{(k)}] &= 0, & l = m_1 + 1, \dots, n, \end{aligned} \right\} \quad \text{on } L$$

and

$$(2.5) \quad \left. \begin{aligned} M^{(l)}[w_s^{(k)}] &= 0, & l = 1, 2, \dots, m_1, \\ Q^{(l)}[w_s^{(k)}] &= 0, & l = m_1 + 1, \dots, n, \end{aligned} \right\} \quad \text{on } L'.$$

B. Anti-Symmetric in y .

$$(2.6) \quad \left. \begin{aligned} Q^{(l)}[w_s^{(k)}] &= -Q^{(l)}[w_i^{(k)}] = -p_2^{(l)}(x), & l = 1, 2, \dots, m_2, \\ P^{(l)}[w_s^{(k)}] &= 0, & l = m_2 + 1, \dots, n, \end{aligned} \right\} \quad \text{on } L$$

and

$$(2.7) \quad \left. \begin{aligned} N^{(l)}[w_s^{(k)}] &= 0, & l = 1, 2, \dots, m_2, \\ P^{(l)}[w_s^{(k)}] &= 0, & l = m_2 + 1, \dots, n, \end{aligned} \right\} \quad \text{on } L'.$$

The functions $p_1^{(l)}(x)$ and $p_2^{(l)}(x)$ of the symmetric and anti-symmetric parts of the problem are known; in general, they may be written as

$$(2.8) \quad p_i^{(l)}(x) = \sum_{j=1}^2 q_{ij}^{(l)}(x), \quad i = 1, 2 \quad \text{and} \quad l = 1, 2, \dots, m_i,$$

where the functions $q_{i1}^{(l)}(x)$ and $q_{i2}^{(l)}(x)$ are even and odd in x , respectively. In the following, the subscripts $i, j = 1, 2$ will be used to distinguish those functions corresponding to the four sub-problems which cover all the possible combinations of evenness and oddness in x and y .

3. Method of solution. The geometry of the diffraction problem suggests the use of the Fourier sine and cosine transform pairs [6], namely

$$(3.1) \quad \begin{aligned} \check{f}(s) &= \int_0^\infty f(x) \cos(sx) dx, & f(x) &= \frac{2}{\pi} \int_0^\infty \check{f}(s) \cos(sx) ds; \\ \check{g}(s) &= \int_0^\infty g(x) \sin(sx) dx, & g(x) &= \frac{2}{\pi} \int_0^\infty \check{g}(s) \sin(sx) ds. \end{aligned}$$

The choice of the particular transform to be used for each wave potential is dictated, for each sub-problem, by the even and odd behavior in x . Using a bar to denote either transform of (3.1), the governing equations (2.3) become

$$\left[\frac{d^2}{dy^2} - (s^2 - \alpha_k^2) \right] \bar{w}_s^{(k)}(s, y) = 0, \quad k = 1, 2, \dots, n.$$

For $y > 0$, the solution bounded as $y \rightarrow \infty$ is

$$(3.2) \quad \bar{w}_s^{(k)}(s, y) = a_{ij}^{(k)}(s) \exp(-\gamma_k y)$$

where

$$\gamma_k = (s^2 - \alpha_k^2)^{1/2} = -i(\alpha_k^2 - s^2)^{1/2}$$

the branch cuts of which are discussed in [4]. Inverting (3.2), the solutions become

$$(3.3) \quad w_s^{(k)}(x, y) = \frac{2}{\pi} \int_0^\infty a_{ij}^{(k)}(s) \begin{bmatrix} \cos(sx) \\ \sin(sx) \end{bmatrix} \exp(-\gamma_k y) ds.$$

Keeping in mind that the subscripts i and j are used to distinguish the evenness and oddness of the boundary conditions on L , the scattered wave potentials $w_s^{(k)}$ may be separated into two groups. The first $k = 1, 2, \dots, q$ potentials are even in x for $i = j = 1$; $i = j = 2$ and odd in x for $i = 1, j = 2$; $i = 2, j = 1$. The second group of $k = q + 1, \dots, n$ potentials are odd in x if $i = j = 1$; $i = j = 2$ and even in x if $i = 1, j = 2$; $i = 2, j = 1$. Now, inserting (3.3) into the boundary conditions (2.4) to (2.7) inclusive yields

$$(3.4) \quad \frac{2}{\pi} \int_0^\infty F_{ij}^{(l)}(s) \begin{bmatrix} \cos(sx) \\ \sin(sx) \end{bmatrix} ds = 0, \quad l = 1, 2, \dots, m_i, \quad \text{on } L',$$

$$(3.5) \quad \frac{2}{\pi} \int_0^\infty G_{ij}^{(l)}(s) \begin{bmatrix} \cos(sx) \\ \sin(sx) \end{bmatrix} ds = -q_{ij}^{(l)}(x), \quad l = 1, 2, \dots, m_i, \quad \text{on } L,$$

and

$$(3.6) \quad \frac{2}{\pi} \int_0^\infty H_{ij}^{(l)}(s) \begin{bmatrix} \sin(sx) \\ \cos(sx) \end{bmatrix} ds = 0, \quad l = m_i + 1, \dots, n, \quad \text{on } L + L'.$$

The integrands $F_{ij}^{(l)}(s)$, $G_{ij}^{(l)}(s)$ and $H_{ij}^{(l)}(s)$ are linear in the $a_{ij}^{(k)}(s)$. By multiplying (3.4) and (3.5) by appropriate operators $C_{ij}^{(l)}$ and $D_{ij}^{(l)}$, which stand for differentiation, integration, or equality, the following dual integral equations are obtained:

$$(3.7) \quad \frac{2}{\pi} \int_0^\infty A_{ij}^{(l)}(s) \cos(sx) ds = 0, \quad l = 1, 2, \dots, m_i, \quad \text{on } L',$$

$$(3.8) \quad \frac{2}{\pi} \int_0^\infty B_{ij}^{(l)}(s) \sin(sx) ds = -f_{ij}^{(l)}(x), \quad l = 1, 2, \dots, m_i, \quad \text{on } L.$$

The objective of these operations on the variable x , aside from obtaining the indicated sine or cosine kernel, is to generate physical quantities in (3.7) and (3.8) with the same order in their highest derivatives.

From the Fourier inversion theorem [6], (3.6) gives

$$(3.9) \quad H_{ij}^{(l)}(s) = 0, \quad l = m_i + 1, \dots, n.$$

With the aid of (3.9), the dual integral equations (3.7) and (3.8) may be written as

$$(3.10) \quad \frac{2}{\pi} \int_0^\infty A_{ij}^{(l)}(s) \cos(sx) ds = 0, \quad \text{on } L',$$

$$(3.11) \quad \frac{2}{\pi} \int_0^\infty \sum_{k=1}^{m_i} b_{ijk}^{(l)}(s) A_{ij}^{(k)}(s) \sin(sx) ds = -f_{ij}^{(l)}(x), \quad \text{on } L,$$

where $l = 1, 2, \dots, m_i$. The coefficient functions $b_{ijk}^{(l)}(s)$ are generally given in the physical problem and by the manner in which they are constructed,

$$(3.12) \quad b_{ijk}^{(l)}(s) \rightarrow \begin{cases} O(1), & k = l, \\ O(s^{-2}), & k \neq l, \end{cases} \quad \text{as } s \rightarrow \infty.$$

Denoting the integrals (3.10) as $g_{ij}^{(l)}(x)$, the Fourier inversion theorem leads to

$$(3.13) \quad A_{ij}^{(l)}(s) = \int_0^a g_{ij}^{(l)}(x) \cos(sx) dx$$

which can be substituted into (3.11) to render

$$(3.14) \quad -f_{ij}^{(l)}(x) = \frac{2}{\pi} \int_0^\infty \sum_{k=1}^{m_i} b_{ijk}^{(l)} \left[\int_0^a g_{ij}^{(k)}(z) \cos(sz) dz \right] \sin(sx) ds, \quad \text{on } L.$$

At this point, it is necessary to make an assumption on the behavior of the quantities $g_{ij}^{(l)}(x)$ as $x \rightarrow a$, the ends of the line of discontinuity. This decision can be somewhat guided by the structure of the solution for the limiting case

$\omega = 0$. Depending upon whether the $g_{ii}^{(l)}(x)$ behave as $r_1^{1/2}$ or $r_1^{-1/2}$, r_1 being the radial distance measured from one of the endpoints of L , say $x = a$, the following representations of $g_{ii}^{(l)}(x)$ are admitted:

$$(3.15) \quad g^{(l)}(x) = \begin{cases} \psi^{(l)}(a)(a^2 - x^2)^{1/2} - \int_x^a \frac{d\psi^{(l)}(\tau)}{d\tau} (\tau^2 - x^2)^{1/2} d\tau, & \text{on } L, \\ 0, & \text{on } L', \end{cases}$$

or

$$(3.16) \quad g^{(l)}(x) = \begin{cases} \frac{a\sigma^{(l)}}{(a^2 - x^2)^{1/2}} + \int_x^a \phi^{(l)}(\tau) \frac{d\tau}{(\tau^2 - x^2)^{1/2}}, & \text{on } L, \\ 0, & \text{on } L', \end{cases}$$

where $l = 1, 2, \dots, m_i$. (The subscripts i, j used to define the four sub-problems are dropped for the sake of conciseness. In addition, the numerical sequence $l = 1, 2, \dots, m_i$ applying to the sets of integral equations is suppressed.) The two different representations of $g^{(l)}(x)$ are easily distinguished through $\psi^{(l)}(\tau)$ in (3.15) and $\phi^{(l)}(\tau)$ in (3.16). These functions are assumed to be continuous on $[0, a]$ and the constants $\sigma^{(l)}$ can be determined from

$$(3.17) \quad a\sigma^{(l)} = \frac{2}{\pi} \int_0^a g^{(l)}(x) dx - \int_0^a \phi^{(l)}(\tau) d\tau.$$

Putting (3.15) or (3.16) into (3.14) and noting the identity [7]

$$\frac{2}{\pi} \int_0^x \frac{\cos(yz)}{(x^2 - z^2)^{1/2}} dz = J_0(xy)$$

where J_0 is the zero order Bessel function of the first kind, it is found that

$$(3.18) \quad -f^{(l)}(x) = \int_0^\infty \left[\sum_{k=1}^{m_i} b_k^{(l)}(s) \int_0^a J_0(s\tau) \psi^{(k)}(\tau) d\tau \right] \sin(sx) ds, \quad \text{on } L,$$

or

$$(3.19) \quad -f^{(l)}(x) = \int_0^\infty \left\{ \sum_{k=1}^{m_i} b_k^{(l)}(s) \left[a\sigma^{(k)} J_0(sa) + \int_0^a J_0(s\tau) \phi^{(k)}(\tau) d\tau \right] \right\} \sin(sx) ds, \quad \text{on } L.$$

Next, the functions $b_k^{(l)}(s)$ are expanded in series for large values of s . Recall (3.12) and define $h_k^{(l)}(s)$ according to

$$(3.20) \quad h_k^{(l)}(s) = \begin{cases} b_k^{(l)}(s) - E^{(l)} & \text{for } l = k \\ b_k^{(l)}(s) & \text{for } l \neq k \end{cases}$$

such that as $s \rightarrow \infty$, $h_k^{(l)}(s) \rightarrow O(s^{-2})$ and $b_k^{(l)}(s) \rightarrow E^{(l)}$ for $l = k$. By virtue of the identity [7]

$$\int_0^\infty J_0(yz) \sin(xz) dz = \begin{cases} 0, & 0 \leq x < y, \\ \frac{1}{(x^2 - y^2)^{1/2}}, & 0 < y < x, \end{cases}$$

the integral equations (3.18) and (3.19) become

$$(3.21) \quad \int_0^x \psi^{(l)}(\tau) \frac{\tau d\tau}{(x^2 - \tau^2)^{1/2}} = -[E^{(l)}]^{-1} \left[f^{(l)}(x) + \int_0^\infty \int_0^a \sum_{k=1}^{m_i} h_k^{(l)}(s) \sin(sx) J_0(s\tau) \tau \psi^{(k)}(\tau) d\tau ds \right]$$

and

$$(3.22) \quad \int_0^x \phi^{(l)}(\tau) \frac{d\tau}{(x^2 - \tau^2)^{1/2}} = -[E^{(l)}]^{-1} \left\{ f^{(l)}(x) + \int_0^\infty \sum_{k=1}^{m_i} h_k^{(l)}(s) \sin(sx) \cdot \left[a\sigma^{(k)} J_0(sa) + \int_0^a J_0(s\tau) \phi^{(k)}(\tau) d\tau \right] ds \right\}$$

which are Abel integral equations of the form

$$\int_0^z \frac{U(\tau) d\tau}{(z^2 - \tau^2)^{1/2}} = V(z), \quad 0 \leq z \leq a.$$

Such an equation may be inverted to read as [7]

$$(3.23) \quad U(\tau) = \frac{2\tau}{\pi} \int_0^\tau \frac{dV(z)}{dz} \frac{dz}{(\tau^2 - z^2)^{1/2}}, \quad 0 < t \leq a.$$

Before recording results by (3.23), it is convenient to express the derivative of $f^{(l)}(x)$ as

$$(3.24) \quad \frac{d}{dx} f^{(l)}(x) = (i)^{\lambda_l} R^{(l)} \cos(\alpha_p x \cos \vartheta_p), \quad l = 1, 2, \dots, m_i,$$

where $\lambda_l = 0, 1, 2$, or 3 are chosen so that the quantities $-R^{(l)}/E^{(l)}$ are positive. In view of (3.23), the following integral equations are obtained:

$$(3.25) \quad \psi^{(l)}(\tau) = -[E^{(l)}]^{-1} \left[(i)^{\lambda_l} R^{(l)} J_0(\alpha_p \cos \vartheta_p \tau) + \int_0^\infty \int_0^a \sum_{k=1}^{m_i} s h_k^{(l)}(s) J_0(s\tau) J_0(st) t \psi^{(k)}(t) dt ds \right], \quad 0 < \tau \leq a,$$

and

$$(3.26) \quad \phi^{(l)}(\tau) = -\tau [E^{(l)}]^{-1} \left\{ (i)^{\lambda_l} R^{(l)} J_0(\alpha_p \cos \vartheta_p \tau) + \int_0^\infty \sum_{k=1}^{m_i} s h_k^{(l)}(s) \left[a\sigma^{(k)} J_0(sa) + \int_0^a J_0(st) \phi^{(k)}(t) dt \right] J_0(s\tau) ds \right\}, \quad 0 < \tau \leq a.$$

Further, set $\xi = \tau/a$, $\eta = t/a$ and adopt the dimensionless notation

$$\begin{aligned}\psi^{(l)}(\tau) &= -[R^{(l)}/E^{(l)}] \frac{\Psi^{(l)}(\xi)}{\xi^{1/2}}, \quad \phi^{(l)}(\tau) = -[\alpha_p^{-1} R^{(l)}/E^{(l)}] \xi^{1/2} \Phi^{(l)}(\xi), \\ \sigma^{(l)} &= -[\alpha_p^{-1} R^{(l)}/E^{(l)}] \Lambda^{(l)}.\end{aligned}$$

Under these considerations, (3.25) and (3.26) become

$$(3.27) \quad \Psi^{(l)}(\xi) = (i)^{\lambda_l} \xi^{1/2} J_0(\alpha_p a \xi \cos \vartheta_p) + \int_0^1 \sum_{k=1}^{m_i} \Gamma_k^{(l)}(\xi, \eta) \Psi^{(k)}(\eta) d\eta, \quad 0 < \xi \leq 1,$$

and

$$(3.28) \quad \begin{aligned}\Phi^{(l)}(\xi) &= (i)^{\lambda_l} \alpha_p a \xi^{1/2} J_0(\alpha_p a \xi \cos \vartheta_p) + \int_0^1 \sum_{k=1}^{m_i} \Gamma_k^{(l)}(\xi, \eta) \Phi^{(k)}(\eta) d\eta \\ &\quad + \sum_{k=1}^{m_i} \Lambda^{(k)} \Gamma_k^{(l)}(\xi, 1), \quad 0 < \xi \leq 1,\end{aligned}$$

where

$$\Lambda^{(l)} = - \int_0^1 \xi^{1/2} \Phi^{(l)}(\xi) d\xi.$$

(In most physical problems, the conditions

$$\int_0^a g^{(l)}(x) dx = 0, \quad l = 1, 2, \dots, m_i,$$

are satisfied.) The kernels $\Gamma_k^{(l)}(\xi, \eta)$ in (3.27) and (3.28) are symmetric in ξ and η and they are given by

$$(3.29) \quad \Gamma_k^{(l)}(\xi, \eta) = -(\xi\eta)^{1/2} [d_k^{(l)}/E^{(l)}] \int_0^\infty s h_k^{(l)}\left(\frac{s}{a}\right) J_0(s\xi) J_0(s\eta) ds$$

in which

$$d_k^{(l)} = E^{(l)} R^{(k)} / E^{(k)} R^{(l)}.$$

Finally, because all the kernels in (3.29) as well as the original boundary conditions are complex, the functions $\Phi^{(l)}(\xi)$ and $\Psi^{(l)}(\xi)$ must also be complex, *i.e.*, $\Phi^{(l)}(\xi) = \Phi_1^{(l)}(\xi) + i\Phi_2^{(l)}(\xi)$ and $\Psi^{(l)}(\xi) = \Psi_1^{(l)}(\xi) + i\Psi_2^{(l)}(\xi)$. In fact, it is inherent in the manner in which diffraction problems are formulated that a complex solution to the spatial part of the problem will be obtained. Upon separating (3.27) and (3.28) into real and imaginary parts, the results are

$$(3.30) \quad \begin{aligned}\Psi_1^{(l)}(\xi) &= \epsilon_1^{(l)} \xi^{1/2} J_0(\alpha_p a \xi \cos \vartheta_p) \\ &\quad + \int_0^1 \sum_{k=1}^{m_i} \{[\Gamma_k^{(l)}(\xi, \eta)]_1 \Psi_1^{(k)}(\eta) - [\Gamma_k^{(l)}(\xi, \eta)]_2 \Psi_2^{(k)}(\eta)\} d\eta, \\ \Psi_2^{(l)}(\xi) &= \epsilon_2^{(l)} \xi^{1/2} J_0(\alpha_p a \xi \cos \vartheta_p) \\ &\quad + \int_0^1 \sum_{k=1}^{m_i} \{[\Gamma_k^{(l)}(\xi, \eta)]_2 \Psi_1^{(k)}(\eta) + [\Gamma_k^{(l)}(\xi, \eta)]_1 \Psi_2^{(k)}(\eta)\} d\eta\end{aligned}$$

and

$$\begin{aligned}
 \Phi_1^{(l)}(\xi) &= \epsilon_1^{(l)}(\alpha_p a) \xi^{1/2} J_0(\alpha_p a \xi \cos \vartheta_p) \\
 &+ \int_0^1 \sum_{k=1}^{m_i} \{ [\Gamma_k^{(l)}(\xi, \eta)]_1 \Phi_1^{(k)}(\eta) - [\Gamma_k^{(l)}(\xi, \eta)]_2 \Phi_2^{(k)}(\eta) \} d\eta \\
 &- \int_0^1 \sum_{k=1}^{m_i} \{ [\Gamma_k^{(l)}(\xi, 1)]_1 \Phi_1^{(k)}(\eta) - [\Gamma_k^{(l)}(\xi, 1)]_2 \Phi_2^{(k)}(\eta) \} \eta^{1/2} d\eta, \\
 \Phi_2^{(l)}(\xi) &= \epsilon_2^{(l)}(\alpha_p a) \xi^{1/2} J_0(\alpha_p a \xi \cos \vartheta_p) \\
 &+ \int_0^1 \sum_{k=1}^{m_i} \{ [\Gamma_k^{(l)}(\xi, \eta)]_2 \Phi_1^{(k)}(\eta) + [\Gamma_k^{(l)}(\xi, \eta)]_1 \Phi_2^{(k)}(\eta) \} d\eta \\
 &- \int_0^1 \sum_{k=1}^{m_i} \{ [\Gamma_k^{(l)}(\xi, 1)]_2 \Phi_1^{(k)}(\eta) + [\Gamma_k^{(l)}(\xi, 1)]_1 \Phi_2^{(k)}(\eta) \} \eta^{1/2} d\eta
 \end{aligned}
 \tag{3.31}$$

in which $0 < \xi \leq 1$ and the notations

$$(i)^{\lambda_l} = \epsilon_1^{(l)} + i\epsilon_2^{(l)},$$

$$\Gamma_k^{(l)}(\xi, \eta) = [\Gamma_k^{(l)}(\xi, \eta)]_1 + i[\Gamma_k^{(l)}(\xi, \eta)]_2$$

have been adopted.

Prior to concluding the general development of the present method, it is expedient to improve the convergence of the infinite integrals for $\Gamma_k^{(l)}(\xi, \eta)$ in (3.29). Assuming that $h_k^{(l)}(s) \rightarrow O(s^{-2})$ for large s , use is made of the identity [8]

$$\int_0^\infty \frac{s}{s^2 + n^2} J_0(s\xi) J_0(s\eta) ds = I_0(\xi\eta) K_0(\eta n), \quad 0 < \xi \leq \eta,$$

where $n^2 > 0$ and I_0 and K_0 are modified Bessel functions of zero order. Thus, let

$$e_k^{(l)}(s) = -\frac{d_k^{(l)}}{E^{(l)}} h_k^{(l)}(s) - \frac{u_k^{(l)}}{s^2 + [n_k^{(l)}]^2}, \quad [n_k^{(l)}]^2 > 0,$$

and expand $e_k^{(l)}(s)$ in a series for large s obtaining

$$e_k^{(l)}(s) \rightarrow \frac{[e_k^{(l)}]^{(2)}}{s^2} - \frac{u_k^{(l)}}{s^4} + \frac{[e_k^{(l)}]^{(4)} + u_k^{(l)}[n_k^{(l)}]^2}{s^4} + \frac{[e_k^{(l)}]^{(6)} - u_k^{(l)}[n_k^{(l)}]^4}{s^6} + \dots$$

where the $[e_k^{(l)}]^{(2p)}$ are the coefficients of s^{-2p} in the expansion of $-[d_k^{(l)}/E^{(l)}] h_k^{(l)}(s)$. By choosing

$$(3.32) \quad u_k^{(l)} = [e_k^{(l)}]^{(2)}, \quad [n_k^{(l)}]^2 = \left| -\frac{[e_k^{(l)}]^{(4)}}{[e_k^{(l)}]^{(2)}} \right|$$

the functions $e_k^{(l)}(s)$ will be either $O(s^{-4})$ or $O(s^{-6})$ depending upon whether $-[e_k^{(l)}]^{(4)}/[e_k^{(l)}]^{(2)}$ is negative or positive. Thus, the kernels (3.29) become

$$(3.33) \quad \Gamma_k^{(l)}(\xi, \eta) = (\xi\eta)^{1/2} \left[a^2 u_k^{(l)} I_0(a\xi n_k^{(l)}) K_0(a\eta n_k^{(l)}) + \int_0^\infty s e_k^{(l)} \left(\frac{s}{a} \right) J_0(s\xi) J_0(s\eta) ds \right].$$

This completes the mathematical formulation of the wave diffraction problem with lines of discontinuities. The problem has been reduced to the solution of the simultaneous integral equations (3.30) or (3.31) depending upon whether $g^{(i)}(x)$ takes the form of (3.15) or (3.16). These integral equations can be solved numerically on an electronic computer once the parameters in the physical problem under consideration are known. The advantage which the present formulation possesses is that the singular portion of the solution may be easily extracted from the total solution in an exact manner.

4. Singularities of the solution. In elastodynamic problems of wave propagation [3], a knowledge of the near-field solution is essential to establishing the critical condition under which a line of discontinuity becomes unstable. Such information can be directly incorporated into the modern theories of fracture mechanics [9].

As stated earlier, one of the inherent features of this method is that the behavior of the solution near the ends of the line of discontinuity can be found immediately. Specifically, both the strength of the singularities as well as the angular variation of those quantities which are singular at $|x| = a$ can be isolated. Let these m singular quantities be given by

$$(4.1) \quad S_{ij}^{(r)}[w_s^{(k)}] = \frac{2}{\pi} \int_0^\infty \sum_{k=1}^{m_i} v_{ijk}^{(r)}(s) A_{ij}^{(k)}(s) \begin{bmatrix} \cos(sx) \\ \sin(sx) \end{bmatrix} \exp(-\gamma_k y) ds, \\ r = 1, 2, \dots, m,$$

where the $v_{ijk}^{(r)}(s)$ are known functions and $A_{ij}^{(i)}(s)$ are given by (3.13) as

$$(4.2) \quad A_{ij}^{(i)}(s) = -\frac{\pi a R^{(i)}}{2sE^{(i)}} \left\{ \Psi^{(i)}(1) J_1(sa) - \int_0^1 \xi J_1(sa\xi) \frac{d}{d\xi} \left[\frac{\Psi^{(i)}(\xi)}{\xi^{1/2}} \right] d\xi \right\}$$

or

$$(4.3) \quad \frac{A_{ij}^{(i)}(s)}{s} = -\frac{\pi a R^{(i)}}{2s\alpha_p E^{(i)}} \left[-J_0(sa) \int_0^1 \xi^{1/2} \Phi^{(i)}(\xi) d\xi + \int_0^1 \xi J_0(sa\xi) \frac{\Phi^{(i)}(\xi)}{\xi^{1/2}} d\xi \right].$$

It is not difficult to show that only the non-integral parts of (4.2) or (4.3) yield singularities at $|x| = a$. The singular character of (4.1) is found by expanding the $v_{ijk}^{(r)}(s)$ for large s and noting

$$\exp(-\gamma_k y) \rightarrow \exp(-sy) \left[1 + \frac{y\alpha_k^2}{2s} + O(s^{-2}) \right], \quad \text{as } s \rightarrow \infty.$$

Employing the well known identities [8]

$$\int_0^\infty s J_0(sa) \begin{bmatrix} \sin(sx) \\ \cos(sx) \end{bmatrix} \exp(-sy) ds = -\frac{r_0}{[r_1 r_2]^{3/2}} \begin{bmatrix} \sin[\theta_0 - \frac{3}{2}(\theta_1 + \theta_2)] \\ \cos[\theta_0 - \frac{3}{2}(\theta_1 + \theta_2)] \end{bmatrix}, \\ \int_0^\infty J_0(sa) \begin{bmatrix} \sin(sx) \\ \cos(sx) \end{bmatrix} \exp(-sy) ds = \frac{1}{[r_1 r_2]^{1/2}} \begin{bmatrix} \cos[\frac{1}{2}(\theta_1 + \theta_2)] \\ \sin[\frac{1}{2}(\theta_1 + \theta_2)] \end{bmatrix},$$

$$\int_0^\infty s J_1(sa) \begin{bmatrix} \sin(sx) \\ \cos(sx) \end{bmatrix} \exp(-sy) ds = -\frac{a}{[r_1 r_2]^{3/2}} \begin{bmatrix} \cos[\frac{3}{2}(\theta_1 + \theta_2)] \\ \sin[\frac{3}{2}(\theta_1 + \theta_2)] \end{bmatrix},$$

$$\int_0^\infty J_1(sa) \begin{bmatrix} \sin(sx) \\ \cos(sx) \end{bmatrix} \exp(-sy) ds = \begin{bmatrix} 0 \\ \frac{1}{a} \end{bmatrix} - \frac{r_0}{a[r_1 r_2]^{1/2}} \begin{bmatrix} \sin[\theta_0 - \frac{1}{2}(\theta_1 + \theta_2)] \\ \cos[\theta_0 - \frac{1}{2}(\theta_1 + \theta_2)] \end{bmatrix},$$

where

$$x = r_0 \cos \theta_0 = r_1 \cos \theta_1 + a = r_2 \cos \theta_2 - a,$$

$$y = r_0 \sin \theta_0 = r_1 \sin \theta_1 = r_2 \sin \theta_2,$$

the singular parts of the solution can be evaluated in closed elementary form.

Other physical quantities of importance such as (3.15) or the integrals of (3.16) may also be computed at the line of discontinuity. Without going into details, it can be shown that for $x/a \geq 0$

$$(4.4) \quad g_{ii}^{(l)}(x) = -\frac{R^{(l)}}{E^{(l)}} a \int_{x/a}^1 \frac{\xi}{\left(\xi^2 - \left(\frac{x}{a}\right)^2\right)^{1/2}} \frac{\Psi^{(l)}(\xi)}{\xi^{1/2}} d\xi$$

or

$$(4.5) \quad \int_0^x g_{ii}^{(l)}(z) dz = -\frac{aR^{(l)}}{\alpha_p E^{(l)}} \left[-\sin^{-1}\left(\frac{x}{a}\right) \int_0^1 \xi^{1/2} \Phi^{(l)}(\xi) d\xi \right. \\ \left. + \frac{\pi}{2} \int_0^{x/a} \xi^{1/2} \Phi^{(l)}(\xi) d\xi + \int_{x/a}^1 \sin^{-1}\left(\frac{x}{a\xi}\right) \xi^{1/2} \Phi^{(l)}(\xi) d\xi \right].$$

Although the problem treated herein is that of a line of discontinuity excited by input waves (2.2), the solution is readily generalized to problems involving any arbitrary periodic disturbance. This can be accomplished by expressing the boundary conditions $q_{ii}^{(l)}(x)$ as complex Fourier series on L , i.e.,

$$(4.6) \quad q_{ii}^{(l)}(x) = \sum_{n=0}^{\infty} \delta_n \left\{ [a_n^{(l)}]_{ii} \cos \frac{n\pi x}{a} + [b_n^{(l)}]_{ii} \sin \frac{n\pi x}{a} \right\}$$

where $[b_0^{(l)}]_{ii} = 0$ and

$$(4.7) \quad \begin{bmatrix} [a_n^{(l)}]_{ii} \\ [b_n^{(l)}]_{ii} \end{bmatrix} = \frac{1}{2a} \int_{-a}^a q_{ii}^{(l)}(x) \begin{bmatrix} \cos\left(\frac{n\pi x}{a}\right) \\ \sin\left(\frac{n\pi x}{a}\right) \end{bmatrix} dx.$$

The symbol δ_n stands for

$$\delta_n = \begin{cases} 1, & n = 0, \\ 2, & n \geq 1. \end{cases}$$

The separation of $q_i^{(1)}(x)$ into even and odd sub-problems in x ($j = 1, 2$) is immediate; subsequent developments are similar to those following (3.11) and will not be dealt with here.

It is now more pertinent to illustrate the method of solution by solving some key elastodynamic problems of wave propagation. Attention will be focused on finding the physical parameters which govern the stability behavior of a line of discontinuity disturbed by various incident waves.

5. Horizontally-polarized secondary waves. Consider an infinite region S occupying the xy -plane with a slit of discontinuity L of width $2a$ along the x -axis. The slit extends indefinitely in the z -direction. A plane harmonic shear wave, which extends uniformly along the entire z -axis, is assumed to impinge on the slit at an angle ϑ_1 with the x -axis. The displacement vector is assumed to lie in the plane of the wave front parallel to the z -axis such that its magnitude and phase are independent of z . Such waves are known in seismology as horizontally-polarized secondary waves or simply as *SH*-waves. The only nonzero stress components τ_{xz} and τ_{yz} can be computed from the displacement $w_i^{(1)}$ according to

$$(5.1) \quad \tau_{xz} = \mu \frac{\partial w_i^{(1)}}{\partial x}, \quad \tau_{yz} = \mu \frac{\partial w_i^{(1)}}{\partial y}$$

where μ is the shear modulus of elasticity. With the understanding that the time factor in the problem is $\exp(-i\omega t)$ and that the equations of motion in the classical theory of elasticity are satisfied, a Helmholtz equation for $w_i^{(1)}$ is obtained:

$$(5.2) \quad [\nabla^2 + \alpha_1^2]w_i^{(1)}(x, y) = 0, \quad \text{in } S,$$

in which $\alpha_1 = \omega/c_1$ and $c_1 = (\mu/\rho)^{1/2}$ is the shear wave velocity with ρ being the mass density of the elastic solid.

The problem is anti-symmetric in y and hence from the input wave (2.2) the quantities in (2.8) are given by

$$(5.3) \quad p_2^{(1)}(x) = -i\tau_0 \exp(-i\alpha_1 x \cos \vartheta_1)$$

and

$$(5.4) \quad \begin{aligned} q_{21}^{(1)}(x) &= -i\tau_0 \cos(\alpha_1 x \cos \vartheta_1), \\ q_{22}^{(1)}(x) &= -\tau_0 \sin(\alpha_1 x \cos \vartheta_1) \end{aligned}$$

in which $\tau_0 = W_0^{(1)}\mu\alpha_1 \sin \vartheta_1$ is the magnitude of the incident stress on the crack, assumed to remain finite as α_1 approaches zero. It follows that the conditions of a stress free slit surface and of the displacement being continuous across L' , the portion of the x -axis excluding L , require $w_s^{(1)}$, the scattered wave, to satisfy

$$(5.5) \quad \begin{aligned} Q^{(1)}[w_s^{(1)}] &= \mu \frac{\partial w_s^{(1)}}{\partial y} = -\mu \frac{\partial w_i^{(1)}}{\partial y} = i\tau_0 \exp(-i\alpha_1 x \cos \vartheta_1), \quad \text{on } L, \\ N^{(1)}[w_s^{(1)}] &= w_s^{(1)} = 0, \quad \text{on } L'. \end{aligned}$$

From the scattered waves

$$w_s^{(1)}(x, y) = \frac{2}{\pi} \int_0^\infty \begin{bmatrix} a_{21}^{(1)}(s) \cos(sx) \\ a_{22}^{(2)}(s) \sin(sx) \end{bmatrix} \exp(-\gamma_1 y) ds$$

the conditions (3.4) and (3.5) become

$$(5.6) \quad \frac{2}{\pi} \int_0^\infty \begin{bmatrix} a_{21}^{(1)}(s) \cos(sx) \\ a_{22}^{(1)}(s) \sin(sx) \end{bmatrix} ds = 0, \quad \text{on } L',$$

$$(5.7) \quad \frac{2}{\pi} \int_0^\infty (-\gamma_1) \begin{bmatrix} a_{21}^{(1)}(s) \cos(sx) \\ a_{22}^{(1)}(s) \sin(sx) \end{bmatrix} ds = \frac{\tau_0}{\mu} \begin{bmatrix} i \cos(\alpha_1 x \cos \vartheta_1) \\ \sin(\alpha_1 x \cos \vartheta_1) \end{bmatrix}, \quad \text{on } L.$$

With a view toward reducing the integral equations to standard forms, the upper expression in (5.7) is integrated once in x while the lower expression in (5.6) is differentiated once in x . Further, it is apparent from (5.4) that the outgoing waves in the scattered field consist of two parts, one being even in x and the other odd in x . Hence, (3.15) can be identified with

$$(5.8) \quad g_{21}^{(1)}(x) = w_s^{(1)}(x, 0)$$

for $j = 1$ and (3.16) with

$$(5.9) \quad g_{22}^{(1)}(x) = \frac{d}{dx} w_s^{(1)}(x, 0)$$

for $j = 2$.

For this problem $E^{(1)} = -1$ and the quantities to be normalized are

$$(5.10) \quad \psi^{(1)}(\tau) = \frac{\tau_0}{\mu} \frac{\Psi^{(1)}(\xi)}{\xi^{1/2}}, \quad \phi^{(1)}(\tau) = \frac{\tau_0}{\mu} \cos \vartheta_1 \xi^{1/2} \Phi^{(1)}(\xi),$$

$$\sigma^{(1)} = \frac{\tau_0}{\mu} \cos \vartheta_1 \Lambda^{(1)}.$$

The governing integral equations for $j = 1$ and $j = 2$ are respectively

$$(5.11) \quad \Psi^{(1)}(\xi) = -i\xi^{1/2} J_0(\alpha_1 a \xi \cos \vartheta_1) + \int_0^1 \Gamma_1^{(1)}(\xi, \eta) \Psi^{(1)}(\eta) d\eta$$

and

$$(5.12) \quad \Phi^{(1)}(\xi) = -(\alpha_1 a) \xi^{1/2} J_0(\alpha_1 a \xi \cos \vartheta_1) + \int_0^1 \Gamma_1^{(1)}(\xi, \eta) \Phi^{(1)}(\eta) d\eta \\ - \Gamma_1^{(1)}(\xi, 1) \int_0^1 \eta^{1/2} \Phi^{(1)}(\eta) d\eta$$

for $0 < \xi \leq 1$. The kernel function $\Gamma_1^{(1)}(\xi, \eta)$ in (5.11) and (5.12) is

$$(5.13) \quad \Gamma_1^{(1)}(\xi, \eta) = (\xi\eta)^{1/2} \left\{ \frac{1}{2}(\alpha_1 a)^2 I_0\left(\frac{1}{2}\alpha_1 a\xi\right) K_0\left(\frac{1}{2}\alpha_1 a\eta\right) \right. \\ \left. + \int_0^\infty s \left[1 - \left(1 - \left(\frac{\alpha_1 a}{s} \right)^2 \right)^{1/2} - \frac{1}{2} \frac{1}{(s/\alpha_1 a)^2 + \frac{1}{4}} \right] J_0(s\xi) J_0(s\eta) ds \right\}.$$

Following the method of solution developed earlier, the desired singular stresses for $j = 1$ and $j = 2$ may be combined to yield

$$(5.14) \quad \begin{bmatrix} \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \frac{k_3}{(2r_1)^{1/2}} \begin{bmatrix} -\sin(\theta_1/2) \\ \cos(\theta_1/2) \end{bmatrix}.$$

(The singular character of the total stress field is the same as that of the scattered stress field since the incident stresses are non-singular.) Those terms which are bounded at the endpoints of L have been neglected. The polar coordinates r_1 and θ_1 are measured from the point $x = a$ and k_3 is

$$(5.15) \quad k_3 = \tau_0 a^{1/2} \left[\Psi^{(1)}(1) + \cos \vartheta_1 \int_0^1 \xi^{1/2} \Phi^{(1)}(\xi) d\xi \right],$$

a parameter controlling the stability or instability behavior of the slit [9].

Of significant note is that the detailed structure of the dynamic stresses in r_1 and θ_1 is the same as in the static case. The intensity of the stress field governed by (5.15), however, is altered and depends upon the circular frequency of the input wave, the crack length and the angle of incidence. Plots of the absolute value of the normalized factor $|k_3/\tau_0(a)^{1/2}|$ against the incident angle ϑ_1 are shown

FIGURE 1. Normalized k_3 -factor versus incident angle ϑ_1 for SH -waves.

in Fig. 1 for various different values of the non-dimensional wave number $\alpha_1 a$. The horizontal line $\alpha_1 a = 0$ represents the static solution. As $\alpha_1 a$ increases, the curves start to become wavy and to increase in amplitude. For values of $\alpha_1 a$ between 0.75 and 1.00, the curves begin to intersect the static line. This implies that the magnitude of the local dynamic stresses can be either larger or smaller than the static value depending upon the angle of incidence. For $\alpha_1 a = 1.00$, the peak value of $|k_3/\tau_0(a)^{1/2}|$ can be 36 percent higher than the static one; this occurs at $\vartheta_1 \approx 130^\circ$.

Numerical calculations for the displacement of the slit

$$(5.16) \quad \frac{\mu}{a\tau_0} w_i^{(1)}(x, 0) = \left(1 - \left(\frac{x}{a}\right)^2\right)^{1/2} \Psi^{(1)}(1) - \int_{x/a}^1 \left(\xi^2 - \left(\frac{x}{a}\right)^2\right)^{1/2} \frac{d}{d\xi} \left[\frac{\Psi^{(1)}(\xi)}{\xi^{1/2}} \right] d\xi \\ + \cos \vartheta_1 \left[\frac{\pi}{2} \int_0^{x/a} \xi^{1/2} \Phi^{(1)}(\xi) d\xi + \int_{x/a}^1 \sin^{-1} \left(\frac{x}{a\xi} \right) \xi^{1/2} \Phi^{(1)}(\xi) d\xi \right. \\ \left. - \sin^{-1} \left(\frac{x}{a} \right) \int_0^1 \xi^{1/2} \Phi^{(1)}(\xi) d\xi \right]$$

may be carried out in the same way but the final results will not be given here.

6. Flexural waves. Let a plate containing a cut L be set into motion by a propagating plane flexural wave. When this wave passes through L , there result three types of scattered waves which, according to their velocities and modes, are known as slow flexural, fast flexural, and thickness shear waves [10]. Referring to the Cartesian coordinates x , y and z , the displacements of the plate in steady state flexural motion are assumed to be

$$(6.1) \quad [u_x, u_y, u_z] = [zv_x(x, y), zv_y(x, y), v_z(x, y)] \exp(-i\omega t)$$

where the z -axis is chosen to be normal to the plate surface. In (6.1), v_z is the normal displacement and v_x, v_y are the rotations of the normals about the y and x axes. These quantities can be expressed in terms of three displacement potentials $w_i^{(k)}$ ($k = 1, 2, 3$) by

$$(6.2) \quad v_x = (S_1 - 1) \frac{\partial w_i^{(1)}}{\partial x} + (S_2 - 1) \frac{\partial w_i^{(2)}}{\partial x} + \frac{\partial w_i^{(3)}}{\partial y}, \\ v_y = (S_1 - 1) \frac{\partial w_i^{(1)}}{\partial y} + (S_2 - 1) \frac{\partial w_i^{(2)}}{\partial y} - \frac{\partial w_i^{(3)}}{\partial x}, \\ v_z = w_i^{(1)} + w_i^{(2)}$$

so that each potential satisfies a Helmholtz equation

$$(6.3) \quad [\nabla^2 + \alpha_k^2] w_i^{(k)}(x, y) = 0, \quad \text{in } S,$$

where $k = 1, 2, 3$ and the region S coincides with the middle plane of the plate. The potential $w_i^{(1)}$ generates the slow flexural wave, $w_i^{(2)}$ the fast flexural wave, and $w_i^{(3)}$ the thickness shear waves. In the foregoing equations, the parameters

S_1 , S_2 and α_k ($k = 1, 2, 3$) stand for

$$[S_1, S_2] = [\alpha_1^{-2}, \alpha_2^{-2}] R_2 \alpha_0^4,$$

$$\alpha_0^4 = \frac{\rho \omega^2 h}{D},$$

$$[\alpha_1^2, \alpha_2^2] = \frac{1}{2} \alpha_0^4 [R_2 + R_1 \pm ((R_2 - R_1)^2 + 4\alpha_0^{-4})^{1/2}],$$

$$(1 - \nu) \alpha_3^2 = 2[R_1 \alpha_0^4 - R_2^{-1}]$$

and

$$R_1 = \frac{h^2}{12}, \quad R_2 = \frac{12 D}{\pi^2 \mu h}.$$

The flexural rigidity of the plate is $D = Eh^3/12(1 - \nu^2)$ where h is the plate thickness, E the Young's modulus and ν the Poisson's ratio. The moments and shears everywhere in the plate can be related to the potentials $w_i^{(k)}$ through v_x , v_y and v_z as

$$\begin{aligned} M_x &= D \left(\frac{\partial v_x}{\partial x} + \nu \frac{\partial v_y}{\partial y} \right), \\ M_y &= D \left(\frac{\partial v_y}{\partial y} + \nu \frac{\partial v_x}{\partial x} \right), \\ M_{xy} &= \frac{1 - \nu}{2} D \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \end{aligned} \quad (6.4)$$

and

$$Q_x = \frac{\pi^2 \mu h}{12} \left(\frac{\partial v_z}{\partial x} + v_x \right), \quad Q_y = \frac{\pi^2 \mu h}{12} \left(\frac{\partial v_z}{\partial y} + v_y \right). \quad (6.5)$$

The plate is assumed to be vibrating symmetrically with respect to the line $L + L'$ coinciding with the x -axis. Such a motion can be generated from the incident waves

$$(6.6) \quad w_i^{(1)} = \frac{1}{2} W_0^{(1)} \sum_{n=0}^3 \exp \left\{ -i \left[\alpha_1 x \cos \frac{\pi}{4} (2n+1) + \alpha_1 y \sin \frac{\pi}{4} (2n+1) + \omega t \right] \right\}.$$

The symmetry conditions require the vanishing of the normal rotation v_y on L' as well as the vanishing of the twisting moment M_{xy} and shear resultant Q_y on $L + L'$. To be prescribed on the cut L is the bending moment M_y . From (6.6), it is found that

$$(6.7) \quad p_1^{(1)}(x) = q_{11}^{(1)}(x) = -M_0 \cos \left(\alpha_1 x \cos \frac{\pi}{4} \right)$$

where

$$M_0 = (1 + \nu) D (S_1 - 1) \alpha_1^2 W_0^{(1)}.$$

Therefore, the conditions (2.4) and (2.5) read as

$$(6.8) \quad P^{(1)}[w_s^{(k)}] = D \sum_{k=1}^2 (S_k - 1) \left[\frac{\partial^2 w_s^{(k)}}{\partial y^2} + \nu \frac{\partial^2 w_s^{(k)}}{\partial x^2} \right] - (1 - \nu) \frac{\partial^2 w_s^{(3)}}{\partial x \partial y} \\ = -p_1^{(1)}(x), \quad \text{on } L,$$

$$(6.9) \quad M^{(1)}[w_s^{(k)}] = \sum_{k=1}^2 (S_k - 1) \frac{\partial w_s^{(k)}}{\partial y} - \frac{\partial w_s^{(3)}}{\partial x} = 0, \quad \text{on } L',$$

and

$$(6.10) \quad Q^{(2)}[w_s^{(k)}] = \frac{1 - \nu}{2} D \left[\sum_{k=1}^2 2(S_k - 1) \frac{\partial^2 w_s^{(k)}}{\partial x \partial y} + \frac{\partial^2 w_s^{(3)}}{\partial y^2} - \frac{\partial^2 w_s^{(3)}}{\partial x^2} \right] = 0, \\ \text{on } L + L',$$

$$(6.11) \quad Q^{(3)}[w_s^{(k)}] = \frac{\pi^2 \mu h}{12} \left[\sum_{k=1}^2 S_k \frac{\partial w_s^{(k)}}{\partial y} - \frac{\partial w_s^{(3)}}{\partial x} \right] = 0, \quad \text{on } L + L'.$$

The scattered waves $w_s^{(k)}$ ($k = 1, 2, 3$), which satisfy the Helmholtz equations, are given by

$$w_s^{(k)} = \frac{2}{\pi} \int_0^\infty a_{11}^{(k)}(s) \cos(sx) \exp(-\gamma_k y) ds, \quad k = 1, 2,$$

$$w_s^{(3)} = \frac{2}{\pi} \int_0^\infty a_{12}^{(3)}(s) \sin(sx) \exp(-\gamma_3 y) ds.$$

Substitution of these expressions into (6.8) through (6.11) yields

$$(6.12) \quad \frac{2}{\pi} \int_0^\infty \left[\sum_{k=1}^2 (S_k - 1)(\gamma_k^2 - \nu s^2) a_{11}^{(k)}(s) + (1 - \nu) \gamma_3 s a_{12}^{(3)}(s) \right] \cos(sx) ds \\ = \frac{M_0}{D} \cos\left(\alpha_1 x \cos \frac{\pi}{4}\right), \quad \text{on } L,$$

$$(6.13) \quad \frac{2}{\pi} \int_0^\infty \left[\sum_{k=1}^2 (S_k - 1) \gamma_k a_{11}^{(k)}(s) + s a_{12}^{(3)}(s) \right] \cos(sx) ds = 0, \quad \text{on } L',$$

and

$$(6.14) \quad \frac{2}{\pi} \int_0^\infty \left[\sum_{k=1}^2 2(S_k - 1) \gamma_k s a_{11}^{(k)}(s) + (\gamma_3^2 + s^2) a_{12}^{(3)}(s) \right] \sin(sx) ds = 0, \\ \text{on } L + L',$$

$$(6.15) \quad \frac{2}{\pi} \int_0^\infty \left[\sum_{k=1}^2 S_k \gamma_k a_{11}^{(k)}(s) + s a_{12}^{(3)}(s) \right] s \sin(sx) ds = 0, \quad \text{on } L + L',$$

where the condition (6.11) has been differentiated once in x . The proper form of the solution is then achieved by further integrating (6.12) once in x . Note that

$$(6.16) \quad g_{11}^{(1)}(x) = v_\nu(x, 0)$$

takes the form of (3.15).

In this case $E^{(1)} = -2/(1 - \nu^2)$ and use of the normalization

$$(6.17) \quad \psi^{(1)}(\tau) = -\frac{2M_0}{D(1 - \nu^2)} \frac{\Psi^{(1)}(\xi)}{\xi^{1/2}}$$

leads to the Fredholm integral equation

$$(6.18) \quad \Psi^{(1)}(\xi) = -\xi^{1/2} J_0 \left(\alpha_1 a \xi \cos \frac{\pi}{4} \right) + \int_0^1 \Gamma_1^{(1)}(\xi, \eta) \Psi^{(1)}(\eta) d\eta, \quad 0 < \xi \leq 1.$$

The kernel is

$$(6.19) \quad \Gamma_1^{(1)}(\xi, \eta) = (\xi\eta)^{1/2} \left\langle u_1^{(1)}(\alpha_1 a)^2 I_0(\alpha_1 a \xi n_1^{(1)}) K_0(\alpha_1 a \eta n_1^{(1)}) \right. \\ \left. + \int_0^\infty \left\{ \frac{2}{1 - \nu^2} (s/\alpha_1 a)^2 [1 - (\alpha_2/\alpha_1)^2]^{-1} [-H_1(s) + H_2(s) \right. \right. \\ \left. \left. - (1 - \nu^2)(S_2 - S_1)(1 - (\alpha_3 a/s)^2)^{1/2}] + 1 - \frac{u_1^{(1)}}{(s/\alpha_1 a)^2 + [n_1^{(1)}]^2} \right\} \right. \\ \left. \cdot s J_0(s\xi) J_0(s\eta) ds \right\rangle$$

in which

$$H_k(s) = (S_k - 1) \frac{[(1 - \nu) - (\alpha_k a/s)^2]^{1/2}}{(1 - (\alpha_k a/s)^2)^{1/2}}, \quad k = 1, 2,$$

and

$$u_1^{(1)} = \frac{1}{4(1 - \nu^2)} [1 - (\alpha_2/\alpha_1)^2]^{-1} \\ \cdot \{ -(3\nu^2 + 2\nu + 3)[S_1 - 1 - (\alpha_2/\alpha_1)^4(S_2 - 1)] + (1 - \nu)^2(\alpha_3/\alpha_1)^4(S_2 - S_1) \}, \\ [n_1^{(1)}]^2 \\ = \frac{1}{2} \left| \frac{(1 + 2\nu + 5\nu^2)[S_1 - 1 - (\alpha_2/\alpha_1)^6(S_2 - 1)] - (1 - \nu)^2(\alpha_3/\alpha_1)^6(S_2 - S_1)}{-(3 + 2\nu + 3\nu^2)[S_1 - 1 - (\alpha_2/\alpha_1)^4(S_2 - 1)] + (1 - \nu)^2(\alpha_3/\alpha_1)^4(S_2 - S_1)} \right|.$$

Of chief interest is the singular character of the moments close to the endpoints $x = \pm a$ of L . Without going into details, the terms whose contributions to the moments grow beyond bounds are

$$(6.20) \quad M_x = \frac{K_1}{(2r_1)^{1/2}} \cos \frac{\theta_1}{2} \left(1 - \sin \frac{\theta_1}{2} \sin \frac{3\theta_1}{2} \right), \\ M_y = \frac{K_1}{(2r_1)^{1/2}} \cos \frac{\theta_1}{2} \left(1 + \sin \frac{\theta_1}{2} \sin \frac{3\theta_1}{2} \right), \\ M_{xy} = \frac{K_1}{(2r_1)^{1/2}} \cos \frac{\theta_1}{2} \sin \frac{\theta_1}{2} \cos \frac{3\theta_1}{2}.$$

The shears Q_x , Q_y remain bounded at the endpoints of the cut and are of the order $r_1^{1/2}$ as $r_1 \rightarrow 0$.

The moment field parameter K_1 defined as

$$(6.21) \quad K_1 = \Psi^{(1)}(1)M_0(a)^{1/2}$$

is useful for collating fracture data [9] of plates weakened by mechanical imperfections. The function $\Psi^{(1)}(1)$ representing the ratio of the dynamic to static value of K_1 may be evaluated numerically from (6.18). Figure 2 displays the variation of $|K_1/M_0(a)^{1/2}|$ with the normalized frequency $\omega h/\pi c_2$ for a/h ratios of $\frac{1}{3}$, $\frac{1}{2}$ and 1. Within the frequency range considered, the dynamic values of K_1 are less than the static one. They decrease with increasing frequencies for a fixed a/h ratio and increase with decreasing a/h ratios at a particular frequency.

As in the previous example, only the analytical expression of the normal displacement of L will be recorded:

$$(6.22) \quad \frac{Eh^3}{6M_0a} \frac{u_y(x, 0)}{a} = \frac{z}{a} \left\{ \left(1 - \left(\frac{x}{a} \right)^2 \right)^{1/2} \Psi^{(1)}(1) - \int_{x/a}^1 \left(\xi^2 - \left(\frac{x}{a} \right)^2 \right)^{1/2} \frac{d}{d\xi} \left[\frac{\Psi^{(1)}(\xi)}{\xi^{1/2}} \right] d\xi \right\}.$$

7. Concluding remarks. A method is developed for solving a class of diffraction problems in which the region is infinite in extent and contains a cut. The region may be viewed as two half-planes joined along a straight line except for a finite segment which represents a line of discontinuity or cut. By means of

FIGURE 2. Normalized K_1 -factor versus dimensionless frequency for flexural waves.

symmetry conditions, the general problem has been reduced to the solution of a system of n partial differential equations of the Helmholtz type subject to any linear mixed boundary conditions on a half-plane. Indeed the only effect of involved boundary conditions arising from any given physical problem is to increase the algebraic complexity of the kernel in the governing integral equation.

A principal feature, which could not be predicted by existing methods of analysis, is that the near-field solution can be readily deduced. This throws light on the stability or instability behavior of a line of discontinuity excited by elastic stress waves. The particular examples considered are those of horizontally-polarized shear and flexural waves scattering about a cut of finite width. Other types of input waves such as plane harmonic compressional waves (P -waves) and vertically-polarized shear waves (SV -waves) may be treated in the same way. Without going into details, it can be shown that the singular in-plane stresses corresponding to incident P - and SV - waves are given by

$$\begin{aligned}
 \sigma_x &= \frac{k_1}{(2r_1)^{1/2}} \cos \frac{\theta_1}{2} \left(1 - \sin \frac{\theta_1}{2} \sin \frac{3\theta_1}{2} \right) \\
 &\quad - \frac{k_2}{(2r_1)^{1/2}} \sin \frac{\theta_1}{2} \left(2 + \cos \frac{\theta_1}{2} \cos \frac{3\theta_1}{2} \right), \\
 \sigma_y &= \frac{k_1}{(2r_1)^{1/2}} \cos \frac{\theta_1}{2} \left(1 + \sin \frac{\theta_1}{2} \sin \frac{3\theta_1}{2} \right) + \frac{k_2}{(2r_1)^{1/2}} \sin \frac{\theta_1}{2} \cos \frac{\theta_1}{2} \cos \frac{3\theta_1}{2}, \\
 \tau_{xy} &= \frac{k_1}{(2r_1)^{1/2}} \cos \frac{\theta_1}{2} \sin \frac{\theta_1}{2} \cos \frac{3\theta_1}{2} + \frac{k_2}{(2r_1)^{1/2}} \cos \frac{\theta_1}{2} \left(1 - \sin \frac{\theta_1}{2} \sin \frac{3\theta_1}{2} \right).
 \end{aligned}
 \tag{7.1}$$

At normal incidence, the parameters k_1 and k_2 take the forms

$$k_1 = \Psi^{(1)}(1)\sigma_1(a)^{1/2}, \quad k_2 = \Psi^{(2)}(1)\sigma_2(a)^{1/2}
 \tag{7.2}$$

in which $\sigma_1 = -\mu W_0^{(1)}\alpha_2^2$ is the normal stress acting at the P -wave front while $\sigma_2 = -\mu W_0^{(2)}\alpha_2^2$ is the shear stress acting at the SV -wave front. Here, α_1 and α_2 are the wave numbers of compressional (dilatational) and shear (equivoluminal) waves, respectively. Numerical values of (7.2) are shown in Fig. 3 which displays the variation of $|k_l/\sigma_l(a)^{1/2}|$ ($l = 1, 2$) with the normalized wave number $\alpha_2 a$ for a Poisson's ratio of $\nu = 0.20$ under the condition of plane strain. As $\alpha_2 a$ increases, the curve rise to a maximum and then start to decrease in magnitude.

The present approach can also handle the axisymmetric diffraction problem of a penny-shaped plane of discontinuity using the properties of the Hankel transform [6]. For axisymmetric problems, the displacement potentials $w_i^{(k)}$ ($k = 1, 2$) are governed by

$$[\nabla_k^2 + \alpha_k^2]w_i^{(k)}(r, z) = 0
 \tag{7.3}$$

where

$$\nabla_k^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) - \frac{k-1}{r^2} + \frac{\partial^2}{\partial z^2}, \quad k = 1, 2.$$

FIGURE 3. Normalized k -factors versus wave numbers $\alpha_2 a$ for P - and SV -waves.

For a transverse compressional wave impinging normally on the penny-shaped surface of discontinuity described by $r \leq a$, $z = 0$, the input wave becomes

$$(7.4) \quad w_i^{(1)} = W_0^{(1)} \exp(-i\alpha_1 z).$$

The shear stress τ_{rz} vanishes everywhere on the plane $z = 0$ in which the penny-shaped cut is situated, *i.e.*,

$$(7.5) \quad \tau_{rz} = 2\mu \left[\frac{\partial^2 w_i^{(1)}}{\partial r \partial z} - \frac{\partial^2 w_t^{(2)}}{\partial z^2} - \frac{1}{2} \alpha_2^2 w_t^{(2)} \right] = 0 \quad \text{for all } r.$$

The vanishing of normal stress σ_z on the cut and continuity of normal displacement w off the cut give

$$(7.6) \quad \frac{\partial^2 w_s^{(1)}}{\partial z^2} + \alpha_1^2 [1 - \frac{1}{2}(\alpha_2/\alpha_1)^2] w_s^{(1)} + \frac{\partial^2 w_s^{(2)}}{\partial r \partial z} + \frac{1}{r} \frac{\partial w_s^{(2)}}{\partial z} = \alpha_1^2 W_0^{(1)}, \quad r < a,$$

and

$$(7.7) \quad \frac{\partial w_s^{(1)}}{\partial z} + \frac{\partial w_s^{(2)}}{\partial r} + \frac{1}{r} w_s^{(2)} = 0, \quad r > a.$$

Following a derivation similar to that for a line cut yields the Fredholm integral equation

$$(7.8) \quad \Psi^{(1)}(\xi) = -\xi + \int_0^1 \Gamma_1^{(1)}(\xi, \eta) \Psi^{(1)}(\eta) d\eta, \quad 0 < \xi \leq 1,$$

in which

$$\begin{aligned}
 (7.9) \quad \Gamma_1^{(1)}(\xi, \eta) = & \frac{2}{\pi} \left\langle \alpha_2 a [u_1^{(1)} / n_1^{(1)}] \sinh(\alpha_2 a \xi n_1^{(1)}) \exp(-\alpha_2 a \eta n_1^{(1)}) \right. \\
 & + \int_0^\infty \left\{ 2(s/\alpha_2 a)^2 [1 - (\alpha_1/\alpha_2)^2]^{-1} \left[\frac{[1 - \frac{1}{2}(\alpha_2 a/s)^2]^2}{(1 - (\alpha_1 s/a)^2)^{1/2}} - (1 - (\alpha_2 a/s)^2)^{1/2} \right] \right. \\
 & \left. \left. + 1 - \frac{u_1^{(1)}}{(s/\alpha_2 a)^2 + [n_1^{(1)}]^2} \right\} \sin(s\xi) \sin(s\eta) ds \right\rangle.
 \end{aligned}$$

The constants $u_1^{(1)}$ and $n_1^{(1)}$ stand for

$$\begin{aligned}
 u_1^{(1)} &= \frac{1}{4} [1 - (\alpha_1/\alpha_2)^2]^{-1} [3(\alpha_1/\alpha_2)^4 - 4(\alpha_1/\alpha_2)^2 + 3], \\
 [n_1^{(1)}]^2 &= \frac{1}{2} [3(\alpha_1/\alpha_2)^4 - 4(\alpha_1/\alpha_2)^2 + 3]^{-1} [5(\alpha_1/\alpha_2)^6 - 6(\alpha_1/\alpha_2)^4 + 2(\alpha_1/\alpha_2)^2 + 1].
 \end{aligned}$$

The singular stresses in a toroidal region about the periphery of the penny-shaped cut are

$$(7.10) \quad \begin{bmatrix} \sigma_r \\ \sigma_z \\ \tau_{rz} \end{bmatrix} = \frac{k_1}{(2r_1)^{1/2}} \cos \frac{\theta_1}{2} \begin{bmatrix} 1 - \sin \frac{\theta_1}{2} \sin \frac{3\theta_1}{2} \\ 1 + \sin \frac{\theta_1}{2} \sin \frac{3\theta_1}{2} \\ \sin \frac{\theta_1}{2} \cos \frac{3\theta_1}{2} \end{bmatrix}$$

with

$$(7.11) \quad k_1 = \frac{2}{\pi} \Psi^{(1)}(1) \sigma_0 a^{1/2}$$

FIGURE 4. Normalized k_1 -factor versus wave number $\alpha_1 a$ for compressional waves incident on penny-shaped cut.

and $\sigma_0 = 2\mu\alpha_1^2 W_0^{(1)}$. The variation of the absolute value of $|\pi k_1/2\sigma_0(a)^{1/2}|$ with $\alpha_1 a$ is illustrated in Fig. 4. For $\nu = 0.25$, the maximum value of k_1 is about 29% higher than the static value of $k_1 = (2/\pi)\sigma_0(a)^{1/2}$ and this occurs at $\alpha_1 a \approx 0.84$.

To complete the discussion, the normal surface displacement of the penny-shaped cut is found to be

$$(7.12) \quad \frac{\pi(1-\nu^2)}{4a\sigma_0 E} w(r, 0) = \left(1 - \left(\frac{r}{a}\right)^2\right)^{1/2} \Psi^{(1)}(1) - \int_{r/a}^1 \left(\xi^2 - \left(\frac{r}{a}\right)^2\right)^{1/2} \frac{d}{d\xi} \left[\frac{\Psi^{(1)}(\xi)}{\xi} \right] d\xi.$$

It is clear that the newly developed method is effective in dealing with diffraction problems of the infinite medium containing line or surface discontinuities. Depending upon the shape of the geometric discontinuity, a standard transform representation may be used to suit the particular problem at hand.

REFERENCES

- [1] P. M. MORSE & H. FESHBACH, *Methods of Theoretical Physics*, Parts 1 and 2, McGraw-Hill, New York, 1953.
- [2] V. A. FOCK, *Electromagnetic Diffraction and Propagation Problems*, Pergamon Press, New York, 1965.
- [3] G. C. SIH, Some elastodynamic problems of cracks, *International Journal of Fracture Mechanics*, **4** (1968) 51-68.
- [4] B. NOBLE, *Methods Based on the Wiener-Hopf Technique*, Pergamon Press, New York, 1958.
- [5] R. A. SCHMELTZER & M. LEWIN, Function-theoretic solution to a class of dual integral equations and an application to diffraction theory, *Quarterly of Applied Mathematics*, **21** (1964) 269-283.
- [6] I. N. SNEDDON, *Fourier Transforms*, McGraw-Hill, New York, 1951.
- [7] E. T. WHITTAKER & G. N. WATSON, *Modern Analysis*, Cambridge Press, London, 1962.
- [8] G. N. WATSON, *Theory of Bessel Functions*, Cambridge Press, London, 1958.
- [9] G. C. SIH & H. LIEBOWITZ, Mathematical theories of brittle fracture, *Mathematical Fundamentals of Fracture*, Vol. 2, Academic Press, New York, 1968, pp. 67-190.
- [10] R. D. MINDLIN, Influence of rotatory inertia and shear on flexural motions of isotropic elastic plates, *Journal of Applied Mechanics*, **18** (1951) 31-38.

Lehigh University
Knolls Atomic Power Laboratory
Schenectady, New York
Date communicated: MARCH 28, 1969