An Example in Surface Area

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For a continuous mapping f on the unit square into n space, $n \geq 2$, if the partial derivatives of f exist on a measurable set $E \subset Q$, then the Lebesgue area $A(f)$ satisfies the inequality

$$(*) \quad A(f) \geq \int_E J,$$

where J is the square root of the sum of squares of the Jacobians of f .

We note that this inequality does not hold for mappings from m space, $m > 2$. This gives a striking distinction between 2 space and the higher dimensional spaces which seems to be new.

Theorem. *There is a continuous mapping f on the unit 3 cube Q into itself whose partial derivatives exist and are bounded on a measurable set E and is such that*

$$\int_E J > A(f).$$

The construction of such an example is a modification of constructions given by many mathematicians, notably Besicovitch [1] and Nöbeling [2], with some additional features. In particular, in the usual constructions of pathological mappings from 2 space, certain zero dimensional sets are exceptional. In our constructions of a mapping from 3 space, a one dimensional set is similarly exceptional.

A. We give such a construction. We start by defining a model set in the unit cube Q which is the union of parallelepipeds. Divide one of the faces of Q into k^2 non-overlapping equal closed squares. Shrink each of these squares by the same amount, so that we obtain a set R consisting of k^2 pair-wise disjoint closed squares, also disjoint with the boundary of the face of Q on which they lie. Consider the subset of Q which is the union of k^2 closed rectangular parallelepipeds of height 1 and whose base is R . Consider also the 2 subsets of Q which are similarly placed with regard to the other 2 coordinate directions. Take the union of these 3 sets and shrink slightly to obtain the subset S of the interior of Q . S is such that for parallelepipeds of S which are in different direc-

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tions, and which meet, the intersection is a cube whose side is equal to the side of the base of the parallelepipeds.

We consider a sequence $\{S_n\}$ of sets of this sort, where S_{n+1} is in the interior of S_n , $n = 1, 2, \dots$, and where the set

$$F = \bigcap_{n=1}^{\infty} S_n$$

has empty interior but has a subset $H \subset F$, of positive measure, each point of which is in the interior of 3 line segments which are contained in F and are in the coordinate directions.

Let $\epsilon_n > 0$, $n = 1, 2, \dots$, be such that $\sum_{n=1}^{\infty} \epsilon_n < \infty$. For each n , let T_n be a closed set which is a shrinkage of S_n , composed in the same way as S_n of rectangular parallelepipeds, such that S_{n+1} is in the interior of T_n and T_n is in the interior of S_n . We also choose each S_n so that the diagonals of the squares used in its construction are less than $\frac{1}{2}\epsilon_n$. Let $S_0 = T_0 = Q$.

We shall also consider a canonical mapping of the boundary of S_n onto that of T_n . This will be completely determined by a mapping between the vertices of the end squares of S_n and those of T_n . Each such square is displaced in a parallel way from one of the faces of Q , the one belonging to T_n being displaced by a slightly larger amount. Map each vertex of one of these squares for S_n onto the nearest vertex of the corresponding square for T_n . Extend the mapping linearly on each square. The boundary of S_n , and of T_n , may be partitioned into rectangles, some of which are end squares, the others of which lie on a lateral face of a parallelepiped and constitute that part of the face which is between consecutive parallelepipeds meeting the face on which they lie. The above one-one correspondence between end squares of S_n and T_n defines a one-one correspondence between parallelepipeds, which in turn determines one between the rectangles of the partition of the boundary of S_n and of T_n . Vertices of corresponding rectangles are mapped onto each other so that each vertex is taken onto the vertex nearest to it. The mapping is extended so as to be a linear homeomorphism between the rectangles. We thus have a quasilinear homeomorphism between the boundaries of S_n and of T_n . By appropriately restricting T_n the homeomorphism is such that it moves each point on the boundary of S_n by less than ϵ_n .

B. We consider 2 further sequences of figures $\{U_n\}$ and $\{V_n\}$. We first let $U_0 = V_0 = Q$. The relation which V_n bears to U_n , for $n > 1$, will not be the same as that which V_1 bears to U_1 . First $U_1 = V_1$. Moreover, for each n , U_n and V_n are composed of rectangular parallelepipeds with faces parallel to those of Q , just as are S_n and T_n . So U_n and V_n are determined by means of sets of squares in the faces of Q . We describe the relation between these sets of squares for S_n , T_n , U_n , V_n in one of the faces of Q .

The squares corresponding to $U_1 = V_1$ lie one in each of the squares corresponding to T_1 , but are much smaller. In Fig. 1, we have a typical square

FIGURE 1

corresponding to S_1 together with the associated squares corresponding to T_1 , U_1 and V_1 . These are labeled S_1 , T_1 , U_1 , V_1 . The squares contained in this one for S_1 which correspond to S_2 are also indicated (we take their number to be 4 for convenience).

For our construction, the figures U_n and V_n will be quasilinear homeomorphisms of S_n and will be composed of parallelepipeds in the same way. The parallelepipeds will be much thinner. We describe the square in a face of Q corresponding to S_2 , T_2 , U_2 , V_2 . Each square corresponding to V_2 lies in the associated square corresponding to T_2 but is much smaller. The squares corresponding to U_2 lie in those corresponding to V_1 in the same manner as those corresponding to S lie in those corresponding to T_1 . Because of the smallness of the square corresponding to V_1 , those of U_2 cannot lie, as we would like them to do, in the squares corresponding to S_2 . However, we may place squares corresponding to V_2 in those of T_2 , in a one-one manner so that they are in one-one correspondence with those of U_2 by the natural mapping between the squares of V_2 and those of T_2 followed by the contained in relation. See Fig. 2.

Vertices of corresponding squares of S_2 , T_2 , U_2 , V_2 are paired according to their nearness to a vertex of the face of Q on which they lie.

We proceed by induction to obtain figures U_n and V_n , for every $n \geq 1$, so that

- a) Each square of V_n lies in the corresponding square of T_n but is much smaller.
- b) The squares of U_{n+1} lie in those of V_n in the same manner as those of S_{n+1} lie in those of T_n .
- c) Because of the smallness of the squares of V_n , those of U_{n+1} cannot lie in the corresponding squares of S_{n+1} .
- d) However, there is a figure U_{n+1} so that its squares do lie in those of T_{n+1} .

C. The figures S_n , T_n , U_n , V_n are constructed from the end squares and appropriate shrinkings, and canonical quasilinear homeomorphisms are determined completely from the mappings between vertices of these squares described above in terms of partitions into rectangles as for S_n . The shrinking

FIGURE 2

is such that V_n is in the interior of T_n and U_{n+1} , $n > 1$, in the interior of V_n . However, the shrinkage is so small under the natural homeomorphisms described above for S_n , T_n , U_n , V_n that no point is moved by more than ϵ_n . In the partitions of the boundaries of U_n and V_n into rectangles, except for the end squares, each rectangle is on a face of a parallelepiped. Its height is the distance between the consecutive parallelepipeds which it separates and its width is the side of the base of the parallelepiped on which it lies. We make the parallelepipeds of U_n and V_n so thin that for each partitioning rectangle, except for end squares, the height exceeds the width. Moreover, if k_n is the number of rectangles in the partition of U_n , or of V_n , we make sure that the sides of the end squares are less than $\epsilon_n/9k_n$. Finally, the area of V_n is less than ϵ_n .

D. We need the following elementary lemma in our discussion of the desired mapping f in relation to the U_n and V_n . Let R_1 and R_2 be rectangles with sides parallel to the coordinate axes:

$$R_1 = [a_1, a_2] \times [b_1, b_2], \quad R_2 = [c_1, c_2] \times [d_1, d_2].$$

Let φ_0 be the linear mapping of the unit square σ onto R_1 which takes $(0, 0)$ onto (a_1, b_1) , $(1, 0)$ onto (a_2, b_1) , $(0, 1)$ onto (a_1, b_2) , and $(1, 1)$ onto (a_2, b_2) , and let φ_1 be the linear mapping of σ onto R_2 , which takes $(0, 0)$ onto (c_1, d_1) , $(1, 0)$ onto (c_2, d_1) , $(0, 1)$ onto (c_1, d_2) , and $(1, 1)$ onto (c_2, d_2) . Consider the homotopy mapping φ_t , $0 \leq t \leq 1$ for which $\varphi_t(p) = t\varphi_0(p) + (1-t)\varphi_1(p)$. Let A_t be the area of $\varphi_t(\sigma)$.

Lemma. For every $t \in [0, 1]$,

$$A_t \leq \{\max(a_2 - a_1, c_2 - c_1)\} \{\max(b_2 - b_1, d_2 - d_1)\}.$$

Proof. The images of the vertices of σ under φ_t are $(ta_1 + (1-t)c_1, tb_1 + (1-t)d_1)$, $(ta_2 + (1-t)c_2, tb_1 + (1-t)d_1)$, $(ta_1 + (1-t)c_1, tb_2 + (1-t)d_2)$, and $(ta_2 + (1-t)c_2, tb_2 + (1-t)d_2)$. The area of $\varphi_t(\sigma)$ is then

$$A_t = \{t(a_2 - a_1) + (1-t)(c_2 - c_1)\} \{t(b_2 - b_1) + (1-t)(d_2 - d_1)\}$$

and the lemma is obvious.

E. We may now define the mapping f . We use the canonical quasilinear mappings between figures which were defined above. Whenever mappings are designated as canonical they refer to these mappings. We first define f as canonical on ∂S_1 to ∂U_1 and on ∂Q to ∂Q , as the identity. f is extended to a quasilinear homeomorphism on $Q \sim S_1$ to $Q \sim U_1$. We next define f as canonical on ∂T_1 to ∂V_1 . Then each point on ∂T_1 is moved by less than ϵ_1 by f . We extend f so as to be quasilinear on $(Q \sim T_1) \cup \partial T_1$.

We define f as canonical on $T_1 \sim S_2$ to $V_1 \sim U_2$. Then for every $x \in T_1 \sim S_2$, the distance between x and $f(x)$ is less than ϵ_1 . Define f as canonical between ∂T_2 and ∂V_2 ; the distance between x and $f(x)$ is then less than ϵ_2 . Extend f so that it is quasilinear on $(Q \sim T_2) \cup \partial T_2$. The 3 dimensional area of $f(S_2 \sim T_2)$ is less than $9k_2\epsilon_2/9k_2 = \epsilon_2$, with the use of the lemma.

We continue in this way and define the mapping f on $Q \sim F$ so that it has the following properties:

- (i) For every n , f is quasilinear on $Q \sim T_n$.
- (ii) The Lebesgue area of f on $Q \sim T_n$ is less than

$$A(f(S_1 \sim T_1)) + \sum_1^{n-1} \epsilon_k + \sum_2^{n-1} \epsilon_k,$$

where $A(f(S_1 \sim T_1))$ is the Lebesgue area.

- (iii) For every $x \in T_n \sim F$, the distance between x and $f(x)$ is less than ϵ_n .

It follows that f may be extended to a continuous mapping on Q into Q so that for every $x \in F$, $f(x) = x$.

F. The Jacobian of f exists on $H \cup Q$. Let $J = J(x)$ be its magnitude. We note that the Lebesgue area of the mapping f on Q satisfies

$$A(f) \leq \liminf_n \left(\int_{Q \sim T_n} J + \epsilon_n \right) = \int_{Q \sim F} J.$$

We accomplish this by defining the quasilinear f_n , $n = 1, 2, \dots$, as $f_n(x) = f(x)$, for $x \in (Q \sim T_n) \cup \partial T_n$, and on T_n as a quasilinear homeomorphism between T_n and V_n . The sequence $\{f_n\}$ converges uniformly to f and satisfies

$$A(f_n) \leq \int_{Q \sim T_n} J + \epsilon_n.$$

By lower semicontinuity of Lebesgue area, we have

$$A(f) \leq \liminf_n \left(\int_{Q \sim T_n} J + \epsilon_n \right) = \int_{Q \sim F} J.$$

Let $E = (Q \sim F) \cup H$. Since f is the identity on F , and H has positive measure, we have

$$\int_E J > \int_{Q \sim F} J = A(f).$$

We may shrink E so that the partial derivatives of f are bounded (call the smaller set E) and the same inequality holds.

Remark. Whether or not such an example exists for which the partial derivatives exist almost everywhere and are in appropriate L_p classes remains open.

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